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A Note on the q-Gamma and q-Beta Functions

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In this note we obtain a generalization of an identity of Jacobi by employing Ramanujan's ${}_1\Psi_1$ summation formula and from it we deduce series representations for $B_q^2(x,y)$ and $\Gamma_q^2(x)$. We also obtain an interesting definite integral with value $\pi^2/2$.

1. Introduction

F.H. Jackson [6] defined the q-analogue of the gamma function by

(1) $$\Gamma_q(x) = \frac{(q)_\infty}{(q^x)_\infty}(1-q)^{1-x}, \quad 0 < q < 1$$

where

$$(a)_\infty \equiv (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

Jackson [6] also defined a q-integral by

$$\int_0^a f(t)d_qt = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n$$

and

$$\int_0^\infty f(t)d_qt = (1-q)\sum_{-\infty}^{\infty} f(q^n)q^n.$$

In his interesting paper [3] on the q-gamma and q-beta functions, R. Askey has obtained analogues of several classical results about the gamma function including the Bohr-Mollerup theorem, the duplication formula and an asymptotic formula for large x. Using the q-binomial theorem

$$\frac{(at)_\infty}{(a)_\infty} = \sum_{n=0}^{\infty} \frac{(t)_n}{(q)_n} a^n$$

where

$$a_n \equiv (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

and the definition of the q-gamma function, Askey [3] observed that

$$\Gamma_q(x) = (q)_\infty (1-q)^{1-x} \sum_{n=0}^{\infty} \frac{q^{nx}}{(q)_n} \tag{2}$$

and

$$\frac{1}{\Gamma_q(x)} = \frac{(1-q)^{x-1}}{(q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} q^{nx}}{(q)_n}. \tag{3}$$

Further he has shown that the natural choice for the q-beta function is

$$B_q(x,y) = (1-q) \sum_{n=0}^{\infty} q^{nx} \frac{(q^{n+1})_\infty}{(q^{n+y})_\infty} \tag{4}$$

by showing that

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. \tag{5}$$

When ever a series of hypergeometric or q-hypergeometric or related type can be explicitly evaluated in terms of quotients of products of (q)-gamma functions or ($q-$) shifted factorials, it is very useful for many different applications. Sometimes such formulas get additional honour because they give representations of π or π^2 or of some power of the gamma or beta functions.

The purpose of this note is to prove the following two theorems and point out an interesting definite integral with value $\pi^2/2$ (equation (19) of Section 2)):

Theorem 1.

$$B_q^2(x,y) = \frac{(1-q)^2(q^{x+y})_\infty}{2(q)_\infty} \sum_{-\infty}^{\infty} \frac{(2n+1)(q^{1-x})_n q^{nx}}{(q^y)_{n+1}} \tag{6}$$

and

Theorem 2.

$$\Gamma_q^2(x) = \frac{(1-q)^{2(1-x)}}{2(q)_\infty} \sum_{-\infty}^{\infty} (2n+1)(q^{1-x})_n q^{nx}. \tag{7}$$

We also show that Theorem 1 is a q-integral extension of

$$B^2(x,y) = J(x,y) + J(y,x)$$

where

$$J(x,y) = \frac{\Gamma(y)}{\Gamma(1-x)\Gamma(x+y)} \int_0^1 \frac{\log(1/t)t^{x-1}}{(1-t)^{x+y}} dt, \quad 0 < x < 1.$$

For proving these theorems we make use of Ramanujan's ${}_1\Psi_1$ sum [8]:

$$\frac{(-qz;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty(\alpha\beta q^2;q^2)_\infty}{(-\alpha qz;q^2)_\infty(-\beta q/z;q^2)_\infty(\alpha q^2;q^2)_\infty(\beta q^2;q^2)_\infty} \tag{8}$$

$$= \sum_{-\infty}^{\infty} \frac{(1/\alpha;q^2)_n(-\alpha qz)^n}{(\beta q^2;q^2)_n}$$

Here $|q| < 1$ and $|\beta q| < |z| < 1/|\alpha q|$. Simple proofs of (8) can be found in [1], [2] and [5].

2. Proofs of the main theorems

To prove Theorems 1 and 2 we first establish the following theorem:

Theorem 3. *If $|q| < 1$, $|\alpha| < 1$ and $|\beta| < 1$, then*

$$\frac{(q)_\infty^3(\alpha\beta)_\infty}{(\alpha)_\infty^2(\beta)_\infty^2} = \frac{1}{2} \sum_{-\infty}^{\infty} \frac{(2n+1)(q/\alpha)_n}{(\beta)_{n+1}} \alpha^n. \tag{9}$$

Proof. Changing q to $q^{1/2}$, z to $-q^{1/2}z^2$ in (8) and multiplying the resulting identity by z we get

$$(z - z^{-1})\frac{(qz^2)_\infty(q/z^2)_\infty(q)_\infty(\alpha\beta q)_\infty}{(\alpha qz^2)_\infty(\beta/z^2)_\infty(\alpha q)_\infty(\beta q)_\infty} \tag{10}$$

$$= \sum_{-\infty}^{\infty} \frac{(1/\alpha)_n(\alpha q)^n z^{2n+1}}{(\beta q)_n}.$$

Differentiating (10) with respect to z and putting $z = 1$ we obtain

$$\frac{2(q)^3_\infty(\alpha\beta q)_\infty}{(\alpha q)^2_\infty(\beta)_\infty(\beta q)_\infty} = \sum_{-\infty}^{\infty} \frac{(2n+1)(1/\alpha)_n(\alpha q)^n}{(\beta q)_n}.$$

Dividing the above identity by $2(1 - \beta)$ and then changing α to α/q the required result follows.

Note that (9) is a generalization of a celebrated identity of Jacobi [7, p.237, (5)]:

$$(q)^3_\infty = \sum_{n=0}^{\infty}(2n+1)(-1)^n q^{n(n+1)/2}.$$

A slight variant of (9) may be found in [4].

Proof of Theorem 1.

Putting $\alpha = q^x$ and $\beta = q^y$ in (9) we get

$$\frac{(q)^3_\infty(q^{x+y})_\infty}{(q^x)^2_\infty(q^y)^2_\infty} = \frac{1}{2}\sum_{-\infty}^{\infty} \frac{(2n+1)(q^{1-x})_n q^{nx}}{(q^y)_{n+1}}. \tag{11}$$

Observe that

$$\frac{(q)_\infty}{(q^{x+y})_\infty}B^2_q(x,y) = \frac{(q)_\infty\Gamma^2_q(x)\Gamma^2_q(y)}{(q^{x+y})_\infty\Gamma^2_q(x+y)}$$

$$= \frac{(1-q)^2(q)^3_\infty(q^{x+y})_\infty}{(q^x)^2_\infty(q^y)^2_\infty}. \tag{12}$$

Using (12) and (11) we get the required identity.

Corollary 1. *Putting $x+y=1$ in Theorem 1 we get*

$$(\Gamma_q(x)\Gamma_q(1-x))^2 = \frac{(1-q)^2}{2}\sum_{-\infty}^{\infty}(2n+1)\frac{q^{nx}}{(1-q^{n+1-x})}.$$

It may be noted that, employing Ramanujan's ${}_1\Psi_1$ - summation Askey [3] has shown that

$$\Gamma_q(x)\Gamma(1-x) = \frac{(-c)_\infty(-q/c)_\infty}{(-cq^x)_\infty(-q^{1-x}/c)_\infty}(1-q)\sum_{-\infty}^{\infty}\frac{q^{nx}}{1+cq^n}.$$

Proof of Theorem 2.

Putting $\alpha = q^x$ and $\beta = 0$ in (9) we get

$$\frac{(q)_\infty^3}{(q^x)_\infty^2} = \frac{1}{2}\sum_{-\infty}^{\infty}(2n+1)(q^{1-x})_n q^{nx}. \tag{13}$$

Using the definition of q-gamma function in (13) we get the required result.

Now we write (6) as q-integral. The right hand side of (6) can be written as

$$\frac{(1-q)^3}{(1-q)^{x+y}\Gamma_q(x+y)}[L(x,y)+L(y,x)] \tag{14}$$

where

$$L(x,y) = \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)\frac{(q^{1-x})_n q^{nx}}{(q^y)_{n+1}}.$$

Using the definition of q-integral, we have

$$L(x,y) = \frac{1}{2}\frac{\Gamma_q(y)(1-q)^{x+y}}{\Gamma_q(1-x)(1-q)^2\log q}\int_0^1 f(t)d_qt \tag{15}$$

where

$$f(t) = \log(t^2q)\frac{(tq^{y+1})_\infty}{(tq^{1-x})_\infty}t^{x-1}.$$

Substituting (15) in (14), (6) can be written as

$$B_q^2(x,y) = \frac{(1-q)}{\log(q)\Gamma_q(x+y)}[I(x,y)+I(y,x)] \tag{16}$$

where

$$I(x,y) = \frac{1}{2}\frac{\Gamma_q(y)}{\Gamma_q(1-x)}\int_0^1 f(t)d_qt.$$

Letting q to 1 in (16) we get

$$B^2(x,y) = J(x,y) + J(y,x), \quad 0 < x < 1, \quad 0 < y < 1, \tag{17}$$

where

$$J(x,y) = \frac{\Gamma(y)}{\Gamma(1-x)\Gamma(x+y)}\int_0^1 \frac{\log(1/t)\cdot t^{x-1}}{(1-t)^{x+y}}dt.$$

Putting $x + y = 1$ in (17) we obtain

$$B^2(x, 1-x) = \int_0^1 \frac{\log(1/t)[t^{x-1} + t^{-x}]}{(1-t)}dt, \quad 0 < x < 1. \tag{18}$$

Putting $x = \frac{1}{2}$ in (18) and using $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we get

$$\frac{\pi^2}{2} = \int_0^1 \frac{\log(1/t)}{\sqrt{t}(1-t)}dt. \tag{19}$$

Now, expanding $1/(1-t)$ in (19) and by integration by parts we have

$$\frac{\pi^2}{2} = \sum_{n=0}^{\infty}\frac{1}{(n+1/2)^2}. \tag{20}$$

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