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A Note on the Hypergeometric Identities of Cayley and Orr

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The object of the present note is to give a rather simple proof of the classical hypergeometric identities of Cayley and Orr, which entirely avoids the use both of differential equations and of relations between generalized hypergeometric series. Our alternative derivations provide a novel insight into the various other hypergeometric identities of Cayley and Orr types.

1. Introduction

In 1858, while discussing certain relations in planetary theory, A. Cayley [3] stated without proof the interesting theorem:
If

$$(1.1)\qquad (1-z)^{a+b-c}{}_2F_1(2a,2b;2c;z)=\sum_{n=0}^{\infty}\delta_n z^n\ ,$$

then

$$(1.2)\qquad {}_2F_1\left(a,b;c+\frac{1}{2};z\right){}_2F_1\left(c-a,c-b;c+\frac{1}{2};z\right)=\sum_{n=0}^{\infty}\frac{(c)_n}{\left(c+\frac{1}{2}\right)_n}\delta_n z^n\ .$$

Forty years later, while investigating the differential equation satisfied by the product of two hypergeometric series, W. M. Orr [6] gave a proof of Cayley's theorem and of several other relations of a similar nature. In particular, he showed that:

If

$$(1.3)\qquad (1-z)^{a+b-c-\frac{1}{2}}{}_2F_1(2a-1,2b;2c-1;z)=\sum_{n=0}^{\infty}\lambda_n z^n\ ,$$

then

$$(1.4)\qquad {}_2F_1(a,b;c;z){}_2F_1\left(c-a+\frac{1}{2},c-b-\frac{1}{2};c;z\right)=\sum_{n=0}^{\infty}\frac{(c-\frac{1}{2})_n}{(c)_n}\lambda_n z^n\ ;$$

Furthermore, if

$$(1.5)\qquad (1-z)^{a+b-c-\frac{1}{2}}{}_2F_1(2a,2b;2c;z)=\sum_{n=0}^{\infty}\mu_n z^n\ ,$$

then

$$(1.6)\qquad {}_2F_1(a,b;c;z){}_2F_1\left(c-a+\frac{1}{2},c-b+\frac{1}{2};c+1;z\right)=\sum_{n=0}^{\infty}\frac{(c+\frac{1}{2})_n}{(c+1)_n}\mu_n z^n\ .$$

The special case $c=a+b$ of (1.2) is a famous formula for the square of a Gauss hypergeometric series:

$$(1.7)\qquad \left[{}_2F_1\left(a,b;a+b+\frac{1}{2};z\right)\right]^2={}_3F_2\left[\begin{matrix}2a, & 2b, & a+b & ; \\ & & & z \\ & 2a+2b, & a+b+\frac{1}{2} & ;\end{matrix}\right],$$

which was published by T. C z l a u s e n [4] as long ago as 1828. The usefulness of the Clausen identity (1.7) in various fields of analysis has recently been pointed out by one of us [8].

The differential equation approach to the above identities involves a certain amount of algebraic work [2]. An alternative procedure, based on the transformation theory of generalized hypergeometric series [1, pp. 84-86], has been widely used in the literature. In this note we present a particular *modus operandi*, requiring very simple tools, which yields (1.2) and (1.4) in a somewhat thaumaturgic way. As we shall see, the key to achieve this goal is the elementary generating function (cf. [5, p.85]; see also [9, p.138 (8) with $r=s=1$]):

$$(1.8)\qquad \sum_{r=0}^{\infty}\frac{(A)_r}{r!}y^r{}_2F_1(-r,B;C;u)=(1-y)^{-A}{}_2F_1\left(A,B;C;-\frac{yu}{1-y}\right)\quad (|y|<1)$$

2. Description of the procedure

First of all, we rewrite Euler's transformation [9, p.33(21)] in the form:

$$(1-z)^{2(a+b-c)}{}_2F_1(2a,2b;2c;z) = {}_2F_1(2c-2a,2c-2b;2c;z) = \tag{2.1}$$
$$= \sum_{r=0}^{\infty} \frac{(2c-2a)_r(c-b)_r}{(c)_r r!} z^r \sum_{s=0}^{[\frac{r}{2}]} \frac{(-\frac{r}{2})_s(\frac{1-r}{2})_s(b)_s}{(c+\frac{1}{2})_s(1-c+b-r)_s s!},$$

which follows readily from Saalschutz's theorem [9,p.95].

Making use of the elementary series identity [9, p.101 (6)]:

$$\sum_{r=0}^{\infty}\sum_{s=0}^{[\frac{r}{2}]} g(r,s) = \sum_{r,s=0}^{\infty} g(r+2s,s), \tag{2.2}$$

we find from (2.1), after a straightforward manipulation, that

$$(1-z)^{2(a+b-c)}{}_2F_1(2a,2b;2c;z) = \tag{2.3}$$
$$= \sum_{s=0}^{\infty} \frac{(c-a)_s(c-a+\frac{1}{2})_s}{(c+\frac{1}{2})_s s!}(-z^2)^s \frac{(b)_s(c-b)_s}{(c)_{2s}}$$
$$\times\ {}_2F_1(2c-2a+2s, c-b+s; c+2s; z)\ .$$

Since

$${}_2F_1(2c-2a+2s,c-b+s;c+2s;z) = (1-z)^{2a+b-2c-s}{}_2F_1(2a-c,b+s;c+2s;z)$$
$$(|arg(1-z)| \le \pi-\epsilon;\ 0<\epsilon<\pi)\ ,$$

Equation (2.3) can be written as

$$(1-z)^{b}{}_2F_1(2a,2b;2c;z) = \tag{2.4}$$
$$= \sum_{s=0}^{\infty} \frac{(c-a)_s(c-a+\frac{1}{2})_s}{(c+\frac{1}{2})_s s!}\left(-\frac{z^2}{1-z}\right)^s \frac{(b)_s(c-b)_s}{(c)_{2s}}$$
$$\times\ {}_2F_1(2a-c,b+s;c+2s;z)\ .$$

By expressing the ${}_2F_1$ in the right-hand side as an Eulerian integral:

$${}_2F_1(2a-c,b+s;c+2s;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b+s)}\frac{(c)_{2s}}{(b)_s}$$
$$\times \int_0^1 t^{b+s-1}(1-t)^{c-b+s-1}(1-zt)^{c-2a}dt$$
$$(Re(c) > Re(b) > 0;\ s=0,1,2\ldots;\ |arg(1-z)| \le \pi-\epsilon;\ 0<\epsilon<\pi),$$

we arrive at

(2.5)

$$(1-z)^b {}_2F_1(2a,2b;2c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}$$

$$\times\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{c-2a}{}_2F_1\left(c-a,c-a+\frac{1}{2};c+\frac{1}{2};-\frac{z^2t(1-t)}{1-z}\right)dt$$

$$(Re(c)>Re(b)>0;\ |arg(1-z)|\leq\pi-\epsilon;\ 0<\epsilon<\pi)$$

or, equivalently,

$$(1-z)^{2a+2b-c}{}_2F_1(2a,2b;2c;z)=$$

$$=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{c-b-1}(1-t)^{b-1}(1-zt)^{2a-c} \tag{2.6}$$

$$\times\ {}_2F_1\left(a,a+\frac{1}{2};c+\frac{1}{2};-\frac{z^2t(1-t)}{1-z}\right)dt,$$

where we have made use of Euler's transformation referred to above.

Next, we apply the generating function (1.8) and observe that

$${}_2F_1\left(a,a+\frac{1}{2};c+\frac{1}{2};-\frac{z^2t(1-t)}{1-z}\right)=\left(\frac{1-z}{1-zt}\right)^a\sum_{r=0}^{\infty}\frac{(a)_r}{r!}\left(\frac{z(1-t)}{1-zt}\right)^r \tag{2.7}$$

$$\times\ {}_2F_1\left(-r,a+\frac{1}{2};c+\frac{1}{2};zt\right)$$

By substituting from (2.7) into (2.6), it follows that

$$(1-z)^{a+b-c}{}_2F_1(2a,2b;2c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{r=0}^{\infty}\frac{(a)_r}{r!}z^r$$

$$\times\int_0^1 t^{c-b-1}(1-t)^{b+r-1}(1-zt)^{a-c-r}{}_2F_1\left(-r,a+\frac{1}{2};c+\frac{1}{2};zt\right)dt=$$

$$=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{r=0}^{\infty}\frac{(a)_r}{r!}z^r\int_0^1 t^{c-b-1}(1-t)^{b+r-1}$$

$$\times\ {}_2F_1\left(c+r+\frac{1}{2},c-a;c+\frac{1}{2};zt\right)dt$$

$$(Re(c)>Re(b)>0)$$

which, after explicit by evaluating the integral by means of [8, p.716 (1.18)], yields

$$(2.8)\quad \begin{aligned}(1-z)^{a+b-c}{}_2F_1(2a,2b;2c;z) &= \sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c)_r r!}z^r\\ &\times {}_3F_2\left[\begin{matrix} c+r+\frac{1}{2}, & c-a, & c-b & ; \\ & & & & z \\ & c+\frac{1}{2}, & c+r & ; \end{matrix}\right].\end{aligned}$$

As a final step, we replace z in (2.8) by zt, multiply both sides by $t^{c-1}(1-t)^{-\frac{1}{2}}$, and then integrate with respect to t over the interval $(0,1)$. An amazing simplification occurs, because (cf. [8, p.716 (1.18)])

$$\begin{aligned}&\int_0^1 t^{c+r-1}(1-t)^{-\frac{1}{2}}{}_3F_2\left[\begin{matrix} c+r+\frac{1}{2}, & c-a, & c-b & ; \\ & & & & zt \\ & c+\frac{1}{2}, & c+r & ; \end{matrix}\right]dt =\\ &= B\left(c+r,\frac{1}{2}\right){}_2F_1\left(c-a,c-b;c+\frac{1}{2};z\right)\\ &\quad (Re(c)>0;\ r=0,1,2,..)\end{aligned}$$

and we get the integral formula:

$$(2.9)\quad \begin{aligned}&{}_2F_1\left(a,b;c+\frac{1}{2};z\right){}_2F_1\left(c-a,c-b;c+\frac{1}{2};z\right) =\\ &= \frac{\Gamma(c+\frac{1}{2})}{\Gamma(c)\Gamma(\frac{1}{2})}\int_0^1 t^{c-1}(1-t)^{-\frac{1}{2}}(1-zt)^{a+b-c}{}_2F_1(2a,2b;2c;zt)dt\\ &\quad (Re(c)>0;\ |arg(1-z)|\le \pi-\epsilon;\ 0<\epsilon<\pi)\end{aligned}$$

which, as one immediately recognizes, is equivalent to Cayley's theorem. The various parametric and variable constraints (involved in our alternative derivation) can, of course, be waived by appealing to the principle of analytic continuation.

The derivation of Orr's first identity [Equations (1.3) and (1.4)] proceeds along similar lines. If, in (2.6), we replace a by $a-\frac{1}{2}$ and c by $c-\frac{1}{2}$, we obtain

$$(2.10)\quad \begin{aligned}&(1-z)^{2a+b-c-\frac{1}{2}}{}_2F_1(2a-1,2b;2c-1;z) = \frac{\Gamma(c-\frac{1}{2})}{\Gamma(b)\Gamma(c-b-\frac{1}{2})}\\ &\times\int_0^1 t^{c-b-\frac{3}{2}}(1-t)^{b-1}(1-zt)^{2a-c-\frac{1}{2}}{}_2F_1\left(a,a-\frac{1}{2};c;-\frac{z^2t(1-t)}{1-z}\right)dt\\ &\quad \left(Re(c)-\frac{1}{2}>Re(b)>0;\ |arg(1-z)|\le\pi-\epsilon;\ 0<\epsilon<\pi\right).\end{aligned}$$

Next, we use (1.8) once again to write

$$
\begin{aligned}
{}_2F_1\left(a, a-\frac{1}{2}; c; -\frac{z^2 t(1-t)}{1-z}\right) &= \left(\frac{1-z}{1-zt}\right)^a \sum_{r=0}^{\infty} \frac{(a)_r}{r!} \left(\frac{z(1-t)}{1-zt}\right)^r \\
&\quad \times {}_2F_1\left(-r, a-\frac{1}{2}; c; zt\right)
\end{aligned} \tag{2.11}
$$

and repeat, *mutatis mutandis*, the steps leading to Equation (2.8), which now takes the form:

$$
\begin{aligned}
&(1-z)^{a+b-c-\frac{1}{2}} {}_2F_1(2a-1, 2b; 2c-1; z) = \\
&= \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(c-\frac{1}{2})_r r!} z^r {}_3F_2 \left[\begin{matrix} c+r, & c-a+\frac{1}{2}, & c-b-\frac{1}{2} & ; \\ & & & & z \\ & c, & c+r-\frac{1}{2} & ; \end{matrix}\right]
\end{aligned} \tag{2.12}
$$

Finally, we replace z by zt, multiply both sides by $t^{c-\frac{3}{2}}(1-t)^{-\frac{1}{2}}$, and then integrate with respect to t over the interval $(0,1)$. The outcome is the integral formula:

(2.13)

$$
\begin{aligned}
&{}_2F_1(a, b; c; z) {}_2F_1\left(c-a+\frac{1}{2}, c-b-\frac{1}{2}; c; z\right) = \\
&= \frac{\Gamma(c)}{\Gamma(c-\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^1 t^{c-\frac{3}{2}}(1-t)^{-\frac{1}{2}}(1-zt)^{a+b-c-\frac{1}{2}} {}_2F_1(2a-1, 2b; 2c-1; zt)\,dt \\
&\quad \left(Re(c) > \frac{1}{2};\ |arg(1-z)| \leq \pi - \epsilon;\ 0 < \epsilon < \pi\right),
\end{aligned}
$$

which is equivalent to the required result.

Orr's second identity [Equations (1.5) and (1.6)] is easily obtained from (2.13), with $a \to a+1$, so that

$$
\begin{aligned}
&B\left(c-\frac{1}{2}, \frac{1}{2}\right) {}_2F_1(a+1, b; c; z) {}_2F_1\left(c-a-\frac{1}{2}, c-b-\frac{1}{2}; c; z\right) = \\
&= \int_0^1 t^{c-\frac{3}{2}}(1-t)^{-\frac{1}{2}}(1-zt)^{a+b-c+\frac{1}{2}} {}_2F_1(2a+1, 2b; 2c-1; zt)\,dt \\
&\quad \left(Re(c) > \frac{1}{2};\ |arg(1-z)| \leq \pi - \epsilon;\ 0 < \epsilon < \pi\right),
\end{aligned} \tag{2.14}
$$

and from (2.9), with $c \to c - \frac{1}{2}$, so that

$$B\left(c-\frac{1}{2},\frac{1}{2}\right){}_2F_1(a,b;c;z){}_2F_1\left(c-a-\frac{1}{2},c-b-\frac{1}{2};c;z\right) =$$

$$(2.15) \qquad = \int_0^1 t^{c-\frac{3}{2}}(1-t)^{-\frac{1}{2}}(1-zt)^{a+b-c+\frac{1}{2}}{}_2F_1(2a,2b;2c-1;zt)dt$$

$$\left(Re(c) > \frac{1}{2};\ |arg(1-z)| \le \pi - \epsilon;\ 0 < \epsilon < \pi\right).$$

By recalling the well-known relation [5, p.103 (35)]

$$(c-a-1){}_2F_1(a,b;c;z) + a{}_2F_1(a+1,b;c;z) = (c-1){}_2F_1(a,b;c-1;z)$$

between contiguous hypergeometric functions, equations (2.14) and (2.15) together imply

$$B\left(c-\frac{1}{2},\frac{1}{2}\right){}_2F_1(a,b;c-1;z){}_2F_1\left(c-a-\frac{1}{2},c-b-\frac{1}{2};c;z\right) =$$

$$= \int_0^1 t^{c-\frac{3}{2}}(1-t)^{-\frac{1}{2}}(1-zt)^{a+b-c+\frac{1}{2}}{}_2F_1(2a,2b;2c-2;zt)dt$$

$$\left(Re(c) > \frac{1}{2};\ |arg(1-z)| \le \pi - \epsilon;\ 0 < \epsilon < \pi\right)$$

and this (with $c \to c+1$) completes the proof.

3. Concluding remarks

Many other theorems of a type similar to Cayley's and Orr's theorems exist in the literature [7], and it would be interesting to prove them in the spirit of the present note (i.e., by avoiding differential equations or relations between hypergeometric ${}_3F_2$ and ${}_4F_3$ functions). We hope to come back to this subject in a future communication.

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$$y = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1) \cdot \beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + etc.$$

ein Quadrat von der Form

$$z = 1 + \frac{\alpha' \cdot \beta' \cdot \delta'}{1 \cdot \gamma' \cdot \epsilon'} x + \frac{\alpha'(\alpha'+1) \cdot \beta'(\beta'+1) \cdot \delta'(\delta'+1)}{1 \cdot 2 \cdot \gamma'(\gamma'+1) \cdot \epsilon'(\epsilon'+1)} x^2 + etc.$$

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