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# A Dynamical Cliché

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*Haec placuit semel, haec decies repetita placebit*
*Quintus Horatius Flaccus*

The paper contains a discussion of Euler's dynamical equations in the case of a single mechanical constraint imposed on the rigid body, together with a *definitio per enumerationem simplicem* of such a constraint and an analysis of the 10 particular cases this very definition is based upon. At that, no particular hypothesis concerning the dynamical nature of the constraint is made (as, for instance, smoothness, ideality, holonomity, etc.), in other words, no restrictions are imposed on the reactions generated by the mechanical constraints in question.

The date September 3, 1750 may be proclaimed to be the Birthday of Solid Dynamics: as G. Eneström [1, S. 44] lets know, "eine Abhandlung mit diesem Titel wurde nach C. G. J. Jacobi am 3. September 1750 der Berliner Akademie vorgelegt". The title in question is that of L. Euler's article [2] containing the first announce of "Euler's dynamical equations" of the motion of a rigid body around its center of gravity. The fundamentality of these equations cannot be overestimated. As C. Truesdell [3] narrates:

" ... the fact that fifty years passed before any improvement over [Newton's] results ... was made shows that he himself had gotten the most out of the subject that his own methods and conseptions could produce; to wrest so much from so primitive a formulation of mechanics required the genius of a Newton; none of his disciples, who might reasonably have been expected to build on his foundation could raise the structure an inch higher. Before the next real advance, a half century of abstraction, precision, and generalization of Newton's concepts was necessary. The first to go substantially beyond Newton ... was the man who found out how to set up mechanical problems once and for all as definite mathematical problems, and this man was Euler." (p. 90).

And *alibi*:

"Except for certain simple if important special problems, Newton *gives no evidence of being able to set up differential equations of motion for mechanical systems* ... In Newton Principia occur no equations of motion for systems of more than two free mass-points or more than one constrained mass-point; Newton theories of fluids are largely false; and the spinning top, the bent spring, lie altogether outside Newton's range ... what Newton really did for mechanics: far from *completing*" the formal enunciation of the mechanical principles now generally accepted" [as Mach states], he *began* it ... a large part of the literature of mechanics for sixty years following the *Principia* searches various principles with a view to finding the equations of motion for the systems Newton had studied and for other systems nowadays thought of as governed by the "Newtonian" equations" (p. 92–93).

In order to fix the ideas let us remind that one of the first, simplest, and most efficient of Euler's mechanical inventions — a true stroke of genius, as any of his dynamical performances in actual fact — consists in the bright whim to consider simultaneously two orthonormal right-hand orientated Cartesian systems of reference $Oxyz$ and $\Omega\xi\eta\zeta$, the first one playing the role of an *observatory* for all proceeding mechanical phenomena, and the second one *invariably connected* with the solid $S$ the motion of which is studied. The second of Euler's inventions, smoothing the way to his dynamical program consists in introducing the angles nowadays named after him. Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and $\bar{\xi}^0, \bar{\eta}^0, \bar{\zeta}^0$ be the unit vectors of the axes $Ox, Oy, Oz$ and $\Omega\xi, \Omega\eta, \Omega\zeta$ respectively; then by hypothesis

$$\mathbf{i}^2 = \mathbf{j}^2 = 1, \quad \mathbf{ij} = 0, \quad \mathbf{k} = \mathbf{i} \times \mathbf{j}, \tag{1}$$

$$(\bar{\xi}^0)^2 = (\bar{\eta}^0)^2 = 1, \quad \bar{\xi}^0 \cdot \bar{\eta}^0 = 0, \quad \bar{\zeta}^0 = \bar{\xi}^0 \times \bar{\eta}^0 \tag{2}$$

If

$$\mathbf{k} \times \bar{\zeta}^0 \neq 0, \tag{3}$$

then E u l e r ' s a n g l e s are defined by

$$\cos\theta = \mathbf{k}\bar{\zeta}^0 \quad (0 < \theta < \pi), \tag{4}$$

$$\cos\psi = \mathbf{i}\bar{\gamma}^0, \quad \sin\psi = \mathbf{j}\bar{\gamma}^0 \quad (0 \le \psi < 2\pi), \tag{5}$$

$$\cos\phi = \bar{\xi}^0\bar{\gamma}^0, \quad \sin\phi = -\bar{\eta}^0\bar{\gamma}^0 \quad (0 \le \phi < 2\pi) \tag{6}$$

provided

$$\bar{\gamma}^0 = \frac{\mathbf{k} \times \bar{\zeta}^0}{\sin\theta}. \tag{7}$$

The third of Euler's cardinal mechanical inventions is the instantaneous angular velocity

$$\bar{\omega} = \frac{1}{2}\,(\bar{\xi}^0 \times \dot{\bar{\xi}}^0 + \bar{\eta}^0 \times \dot{\bar{\eta}}^0 + \bar{\zeta}^0 \times \dot{\bar{\zeta}}^0) \tag{8}$$

of $\Omega\xi\eta\zeta$ with respect to $Oxyz$, dots denoting derivatives with respect to the time $t$ with regard to $Oxyz$; as it is well known, if by definition

$$\bar{\omega} = \omega_\xi \bar{\xi} + \omega_\eta \bar{\eta}^0 + \omega_\zeta \bar{\zeta}^0, \tag{9}$$

then the relations

$$\begin{cases} \omega_\xi = \dot{\psi} \sin\phi \sin\theta + \dot{\theta} \cos\phi, \\ \omega_\eta = \dot{\psi} \cos\phi \sin\theta - \dot{\theta} \sin\phi, \\ \omega_\zeta = \dot{\psi} \cos\theta + \dot{\phi} \end{cases} \tag{10}$$

are called Euler's kinematical equations. Last but not least of Euler's dynamical bright ideas consists in the introduction and systematic use of moments of inertia and moments of deviation of the solid $S$ with respect to $\Omega\xi\eta\zeta$. As Truesdell notes:

"While the problem of oscillation of a heavy rigid body about a fixed axis has been solved correctly by Huygens, and while a more satisfactory method containing the germ of several later principles had been created by James Bernoulli in 1703, in 1750 it could not be said that the general motion of a rigid body was understood at all. Even for motion about a fixed axis, the reaction of the body upon its support could not be calculated, and no method for determining the behavior of a spinning top was known.

Euler's "first principles" changed the scene overnight. As just mentioned, in the paper where these principles are published Euler obtained the general equations of motion of a rigid mody [sic] about its center of gravity.He applied the "first principles" to the elements of mass making up the body, at the same time replacing the acceleration of the element by its expression in terms of the angular velocity vector, which makes its first appearance here. Taking moments about the center of gravity then yields, after some reduction, the differential equations of motion known as "Euler's equations" for a rigid body, subject to assigned torque about its center of mass. In the process arise naturally the six components of what is now called the "tensor of inertia". The "moment of inertia" had occured, of course, in the older researches and had been named by Euler much earlier" [3, p. 117–118].

Let $P$ be any point of the solid $S$, i. e.

$$\dot{\xi} = \dot{\eta} = \dot{\zeta} = 0 \quad (\forall t) \tag{11}$$

provided by definition $\bar{\rho} = \bar{\Omega P}$ and

$$\bar{\rho} = \xi\bar{\xi}^0 + \eta\bar{\eta}^0 + \zeta\bar{\zeta}^0, \tag{12}$$

and let $V_S$ denote the standart vector space of all such points. Let by definition

$$I_{\xi\xi} = \int(\eta^2+\zeta^2)dm, \quad I_{\eta\eta} = \int(\zeta^2+\xi^2)dm, \quad I_{\zeta\zeta} = \int(\xi^2+\eta^2)dm, \tag{13}$$

$$I_{\eta\zeta} = \int \eta\zeta dm, \quad I_{\zeta\xi} = \int \zeta\xi dm, \quad I_{\xi\eta} = \int \xi\eta dm, \tag{14}$$

$dm$ denoting elementary mass of $S$ and the integral being taken over $V_S$. If $G$ denotes the mass-center of $S$ and if by definition $\bar{\rho}_G = \bar{\Omega G}$ and

$$\bar{\rho}_G = \xi_G\bar{\xi}^0 + \eta_G\bar{\eta}^0 + \zeta_G\bar{\zeta}^0, \tag{15}$$

then let by definition

$$J_{\xi\xi} = m(\eta_G^2+\zeta_G^2), \quad J_{\eta\eta} = m(\zeta_G^2+\xi_G^2), \quad J_{\zeta\zeta} = m(\xi_G^2+\eta_G^2), \tag{16}$$

$$J_{\eta\zeta} = m\eta_G\zeta_G, \quad J_{\zeta\xi} = m\zeta_G\xi_G, \quad J_{\xi\eta} = m\xi_G\eta_G, \tag{17}$$

$m$ denoting the mass of $S$. Now the moments of inertia $A, B, C$ and the moments of deviation $D, E, F$ of $S$ with respect to $\Omega\xi\eta\zeta$ are defined by means of the relations

$$A = I_{\xi\xi} - J_{\xi\xi}, \quad B = I_{\eta\eta} - J_{\eta\eta}, \quad C = I_{\zeta\zeta} - J_{\zeta\zeta}, \tag{18}$$

$$D = I_{\eta\zeta} - J_{\eta\zeta}, \quad E = I_{\zeta\xi} - J_{\zeta\xi}, \quad F = I_{\xi\eta} - J_{\xi\eta}. \tag{19}$$

Supposing for the time being that the notion of liaison or constraint imposed on a solid is familiar to the reader at least on an intuitive level, let us note that the forces aplied on a particular rigid body in a particular dynamical problem are divided into two classes usually called active and passive forces respectively; these appelations are untoward ones being psychologically misleading: the only mathematical difference between the first and the second category of forces is that the active forces are *known* vector quantities being wholly determined by the very conditions of the dynamical problem under consideration for any position of the solid at any moment of the time and for any velocities the points of the solid may have, whereas the passive forces are *unknown* vector quantities the determination of which is a part and parcel and one of the main aims of the solution of any dynamical problem.

The *existence of the active forces* is warranted by the said conditions of the dynamical problem, while the *existence of the passive forces* is warranted by a dynamical axiom established *ad hoc*:

**AxR.** *Any constraint imposed on a solid generates a force acting on the latter, the directrix of which runs through the point of contact of the solid with the constraint.*

**Df R.** The force of Ax R is called the reaction of the geometrical constraint on the rigid body.

Let by hypothesis the active forces

$$\vec{F}_\mu = (\mathbf{F}_\mu, \mathbf{M}_\mu) \qquad (\mu = 1, 2, \ldots, m) \tag{20}$$

be applied on the solid $S$ and let $S$ be submitted to $n$ constraints generating the reactions

$$\vec{R}_\nu = (\mathbf{R}_\nu, \mathbf{N}_\nu) \qquad (\nu = 1, 2, \ldots, n) \tag{21}$$

respectively, all moments $\mathbf{M}_\mu$ and $\mathbf{N}_\nu$ $(\mu = 1, 2, \ldots, m;\ \nu = 1, 2, \ldots, n)$ being taken with respect to $O$. Besides, let by definition

$$\mathbf{F} = \sum_{\mu=1}^{m} \mathbf{F}_\mu, \quad \mathbf{R} = \sum_{\nu=1}^{n} \mathbf{R}_\nu \tag{22}$$

$$\mathbf{M} = \sum_{\mu=1}^{m} \mathbf{M}_\mu, \quad \mathbf{N} = \sum_{\nu=1}^{n} \mathbf{N}_\nu \tag{23}$$

be the moments with respect to $O$ of the systems of forces (20) and (21) respectively.

All these preliminaries settled, the basic problem of solid dynamics may now be formulated as follows:

If there are given:

i) A solid $S$,

ii) Certain constraints imposed on $S$, and

iii) Certain active forces applied on $S$, then determine

i) The reactions of the constraints, and

ii) The motion of $S$, *if any.*

(Some authors haughtily neglect this basic problem of solid dynamics and pretend to proceed directly to the more general case of motion of something they designate by the enigmatic term of mechanical system. Since according to their arrogant though vague explanations three bodies presumably constitute a mechanical system, such pretensions amount to the claim that one may come to know everything about the dynamical behaviour of three bodies without knowing anything about the dynamical behaviour of a single body. Such a conduct a sage saw qualifies by the words "a begar on horseback".)

At that, the meaning of the phrase "if any" in the above formulation will be explained later.

Euler solved the basic problem of solid dynamics as stated above from the Alpha to the Omega. Before proceeding to the technicalities in this connection, let us quote a most instructive excerpt from Truesdell's *Essays* [3]:

"His special case of the principle of moment of momentum did not lead Daniel Bernoulli to the equations of motion of a rigid body. In a letter of December 4, 1745 to Euler he described the general problem as "extremely difficult, which will not be solved easily by anybody ... One might ask how to determine the axis of rotation by a summation sign, such that the centrifugal forces destroy each other". While Bernoulli dropped the problem, Euler through exploration of more and more general cases approached the equations of motion of an arbitrary rigid body. He finally achieved them in a paper written in 1750. This paper, *Discovery of a new principle of mechanics*, lays down the principle of linear momentum or the "Newtonian" equation $F = m\ddot{x}$, as the axiom which "includes all the laws of mechanics", if applied to every several element of mass $m$ in every body. The end product of the paper is the "Eulerian equations of motion" for rigid bodies [2,p. 213]. How can this be? Euler states that the forces "include both such external forces as act upon the body from without, and also the internal forces binding the parts of the body to each other, so as to prevent them from changing their relative positions. But it is to be noted that the internal forces mutually destroy each other, so that the continuation of the motion does not require any external forces, except insofar as those forces do not mutually destroy each other". Euler thereupon obtains the equations of motion by taking moments of the equations of linear momentum for elements of mass, leaving altogether out of account any mutual forces that may be present. His method, then, is now akin to that of Daniel Bernoulli, but while he sees that mutual forces have no effect, he does not make any special hypothesis about their functional dependences or their directions. In effect, he *asserts* that $\dot{H} = L$ follows from the principle of linear momentum when the mutual forces are such as to maintain rigidity. The modern theorist of mechanics, who has constantly in mind the general case of a deformable body, when $\dot{H} = L$ does not generally follow from the principle of linear momentum, sees the ridgepole that Euler walked here; as usual, he did not slip off" (p. 257–259).

This is a crucial point in the intellectual history of mankind — the climax of the efforts of the best minds of the epoch in the course of more than half a century as paradoxal as to be looked upon nowadays as tragicomical, following Truesdell's methaphorical expression [4]. Indeed, could one assess otherwise Newton's proverbial declaration *Hypotheses non fingo* in a book containing the *praedicatum*:

*Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimatur.*

If this is not a hypothesis, then what does, for God's sake, a hypothesis mean? To the same extent Euler's discovery $\dot{H} = L$, using Truesdell's notation, is a hypothesis of first water and by no means a *veritas veritatum*, that is to say a proved mathematical truth. If one does not get to the bottom of this fact, then one's mechanical *Weltanschauung* stands in danger to be as sound *velut aegri somnia.*

Let by definition

$$\mathbf{K} = \int \mathbf{v}\, dm, \quad \mathbf{L} = \int \mathbf{r} \times \mathbf{v}\, dm \tag{24}$$

be the momentum and the moment of momentum respectively of a rigid body $S$ with regard to $Oxyz$; the latter proviso means that, $P$ denoting any point of $S$ and $\mathbf{r} = \mathbf{OP}$ by definition, then the velocity $\mathbf{v} = \dot{\mathbf{r}}$ of $P$ is taken with respect to $Oxyz$. If $S$ is under the action of the active forces (20) and of the passive forces (21), then Euler's greatest discovery in solid dynamics consists in the equations

$$\dot{\mathbf{K}} - \mathbf{F} - \mathbf{R} = 0, \qquad \dot{\mathbf{L}} - \mathbf{M} - \mathbf{N} = 0 \tag{25}$$

provided (22), (23), the derivatives in (25) being taken with respect to $Oxyz$. Now if Euler did not prove (25), then what on Earth did he accomplish?

Two things.

First, he invented the dynamical laws (25).

Second, he substantiated his discoveries (25) physically by means of mathematical arguments.

Neither more, nor less.

The dynamical equations (25) are as provable and disprovable as Euclid's Fifth Postulate. They are Dynamical Axioms. They may be accepted, or they may be rejected in accordance with the laws of supply and demand: take them, or make yourself scarce.

A last remark in connection with (25) concerns the system of reference $Oxyz$. It is easily proved that if (25) hold for $Oxyz$ and $\Sigma$ is a Cartesian system of reference moving with respect to $Oxyz$, then (25) are by no means necessarily true if referred to $\Sigma$, i. e. if (24) are computed with respect to $\Sigma$ and if the differentiation in (25) is accomplished with respect to $\Sigma$: a necessary and sufficient condition that (25) might hold for $\Sigma$ too is the motion of $\Sigma$ towards $Oxyz$ to be a rectilinear uniform translation.

In such a manner there is an infinite variety of Cartesian systems of reference for which (25) *do not hold.* Are there Cartesian systems of reference for which (25) *do hold*? This question is *a matter of principle, not of proof.* It include two things: first, *something* that may hold, and second, *systems of reference*, with respect to which this something holds. Now Euler's great discovery in solid dynamics consists in the invention of this something, namely the left-hand sides of equation (25), that must be equal to zero, and in his deep persuasion that there exist Cartesian systems of reference $Oxyz$ for which these quantities vanish - a hypothesis that is by no means transparent, ordinary, and even plausibly convincing. All those systems of reference are called inertial.

After all these historical and ideological preliminaries, let us proceed to the mathematical formalization of Euler's dynamical axioms (25). To this end let by definition

$$\mathbf{r}_G = x_G\mathbf{i} + y_G\mathbf{j} + z_G\mathbf{k}, \tag{26}$$

(27) $$F = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k},$$

(28) $$R = R_x\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k},$$

and let

(29) $$\mathbf{M}_G = \mathbf{M} + \mathbf{F}\times\mathbf{r}_G, \qquad \mathbf{N}_G = \mathbf{N} + \mathbf{R}\times r_G$$

be the moments with regard to $G$ of the systems of forces (20) and (21) respectively. Besides, let by definition

(30) $$\mathbf{M}_G = M_{G\xi}\bar{\xi}^0 + M_{G\eta}\bar{\eta}^0 + M_{G\zeta}\bar{\zeta}^0,$$

(31) $$\mathbf{N}_G = N_{G\xi}\bar{\xi}^0 + N_{G\eta}\bar{\eta}^0 + N_{G\zeta}\bar{\zeta}^0.$$

Now (9), (18), (19), (26)–(31) imply that (25) may be written in the form

(32) $$m\ddot{x}_G = F_x + R_x, \quad m\ddot{y}_G = F_y + R_y, \quad m\ddot{z}_G = F_z + R_z$$

and

(33) $$\begin{cases} A\dot{\omega}_\xi - (B-C)\,\omega_\eta\omega_\zeta - D\,(\omega_\eta^2 - \omega_\zeta^2) \\ \quad - E\,(\dot{\omega}_\zeta + \omega_\xi\omega_\eta) - F\,(\dot{\omega}_\eta - \omega_\zeta\omega_\xi) = M_{G\xi} + N_{G\xi}, \\ B\dot{\omega}_\eta - (C-A)\,\omega_\zeta\omega_\xi - E\,(\omega_\zeta^2 - \omega_\xi^2) \\ \quad - F\,(\dot{\omega}_\xi + \omega_\eta\omega_\zeta) - D\,(\dot{\omega}_\zeta - \omega_\xi\omega_\eta) = M_{G\eta} + N_{G\eta}, \\ C\dot{\omega}_\zeta - (A-B)\,\omega_\xi\omega_\eta - F\,(\omega_\xi^2 - \omega_\eta^2) \\ \quad - D\,(\dot{\omega}_\eta + \omega_\zeta\omega_\xi) - E\,(\dot{\omega}_\xi - \omega_\eta\omega_\zeta) = M_{G\zeta} + N_{G\zeta} \end{cases}$$

respectively.

Euler's dynamical equations (32), (33) reduce formally any dynamical problem concerning a rigid body $S$ to *ein Schablon*, as regards the composing of the algebraic-differential equations of motion themselves. This circumstance should not lead one astray to such a degree as to think that what remains to be done in a particular dynamical problem is a trivial job: the price of such a delusion is a true mathematical collapse. On the contrary: any particular dynamical problem concerning a particular solid $S$ submitted to particular constraints and subject to the action of a particular system of active forces proposes particular mathematical difficulties and requires a mathematical approach *ad hoc* in order to surmount them. One of the most arduous tasks in the solution of the particular solid dynamical problem is the proof that such a solution *exists*.

The *existence problem* is often overlooked in solid dynamics, in particular, and in rational mechanics, in general, and this curcumstance menaces with a mathematical devaluation lots of thousand printed pages of mechanical books and journals. The meaning of the phrase "if any" used above in the formulation of the *basic problem* of solid dynamics becomes now crystal-clear: if a solution of of the particular dynamical problem *exists*.

One may figuratively say that Euler's dynamical equations (32), (33) are a kind of a cliché for solving solid dynamics problems. Now the aim of the present paper is to propose a cliché within this cliché. The meaning of this intention may be explained in the following manner. The quantities at hand in the left-hand sides of equations (33), specific for any solid $S$, are its moment of inertia $A, B, C$ and moments of deviation $D, E, F$; they are completely independent of the constraints imposed on $S$. The same holds, to a certain degree at least, as regards the components $M_{G\xi}, M_{G\eta}, M_{G\zeta}$ of the moment $\mathbf{M}_G$ with respect to $G$ of the system of active forces (20) applied on the solid $S$. Quite on the contrary, the moment $\mathbf{N}_G$ with respect to $G$ of the system (21) of reactions of the constraints and hence its components $N_{G\xi}, N_{G\eta}, N_{G\zeta}$ figurating in the right-hand sides of equations (33) depend essentially on those constraints as it will be seen immediately. Now the aim of this paper is to reduce the negatice computational sequences of this dependence to the minimal possible degree. In other words, our goal is to reduce to minimum the necessity of the *ad hoc* calculations in solid dynamics problems.

The method in question is most effective when only one constraint is imposed on the solid $S$. Let its point of contact be $A$ and let by definition $\mathbf{r}_A = \mathbf{OA}$, $\bar{\rho}_A = \overline{\Omega A}$ and

$$\mathbf{r}_A = x_A\mathbf{i} + y_A\mathbf{j} + z_A\mathbf{k}, \tag{34}$$

$$\bar{\rho}_A = \xi_A\bar{\xi}^0 + \eta_A\bar{\eta}^0 + \zeta_A\bar{\zeta}^0. \tag{35}$$

Besides, let

$$\vec{R} = (\mathbf{R}, \mathbf{N}) \tag{36}$$

be the reaction of the constraint, generated by virtue of **Ax R**, the moment **N** being taken with respect to $O$. Since the directrix of (36) runs through $A$, the condition

$$\mathbf{N} = \mathbf{r}_A \times \mathbf{R} \tag{37}$$

is satisfied. If by definition $\mathbf{r}_\Omega = \overline{O\Omega}$, then the identities $\mathbf{OG} = \overline{O\Omega} + \overline{\Omega G}$ and $\mathbf{OA} = \overline{O\Omega} + \overline{\Omega A}$ imply

$$\mathbf{r}_G = \mathbf{r}_\Omega + \bar{\rho}_G, \quad \mathbf{r}_A = \mathbf{r}_\Omega + \bar{\rho}_A, \tag{38}$$

whence

$$r_A - r_G = \bar{\rho}_A - \bar{\rho}_G. \tag{39}$$

On the other hand, (37) and (29) imply

$$\mathbf{N}_G = (\mathbf{r}_A - \mathbf{r}_G) \times \mathbf{R} \tag{40}$$

and (40), (39) imply

$$\mathbf{N}_G = (\bar{\rho}_A - \bar{\rho}_G) \times \mathbf{R}. \tag{41}$$

The standart approach to the solution of the system of equations (32), (33) consists in elimination of $R_x, R_y, R_z$ by means of the following conventional process. Let by definition

$$\mathbf{R} = R_\xi \bar{\xi}^0 + R_\eta \bar{\eta}^0 + R_\zeta \bar{\zeta}^0 . \tag{42}$$

Then (42), (28) and the definitional relations

$$\begin{cases} \mathbf{i} = a_{11}\bar{\xi}^0 + a_{12}\bar{\eta}^0 + a_{13}\bar{\zeta}^0 , \\ \mathbf{j} = a_{21}\bar{\xi}^0 + a_{22}\bar{\eta}^0 + a_{23}\bar{\zeta}^0 , \\ \mathbf{k} = a_{31}\bar{\xi}^0 + a_{32}\bar{\eta}^0 + a_{33}\bar{\zeta}^0 \end{cases} \tag{43}$$

imply

$$\begin{cases} R_\xi = a_{11}R_x + a_{21}R_y + a_{31}R_z , \\ R_\eta = a_{12}R_x + a_{22}R_y + a_{32}R_z , \\ R_\zeta = a_{13}R_x + a_{23}R_y + a_{33}R_z , \end{cases} \tag{44}$$

and (32), (44) imply

$$\begin{cases} R_\xi = m\,(a_{11}\ddot{x}_G + a_{21}\ddot{y}_G + a_{31}\ddot{z}_G) - F_\xi , \\ R_\eta = m\,(a_{12}\ddot{x}_G + a_{22}\ddot{y}_G + a_{32}\ddot{z}_G) - F_\eta , \\ R_\zeta = m\,(a_{13}\ddot{x}_G + a_{23}\ddot{y}_G + a_{33}\ddot{z}_G) - F_\zeta , \end{cases} \tag{45}$$

where

$$(46)\qquad \begin{cases} F_\xi &= a_{11}F_x \;+\; a_{21}F_y \;+\; a_{31}F_z\,, \\ F_\eta &= a_{12}F_x \;+\; a_{22}F_y \;+\; a_{32}F_z\,, \\ F_\zeta &= a_{13}F_x \;+\; a_{23}F_y \;+\; a_{33}F_z\,, \end{cases}$$

whence

$$(47)\qquad \mathbf{F} \;=\; F_\xi \bar{\xi}^0 \;+\; F_\eta \bar{\eta}^0 \;+\; F_\zeta \bar{\zeta}^0$$

in accordance with (43).

On the other hand, the relations (41), (35), (15), (42), (31) imply

$$(48)\qquad \begin{cases} N_{G\xi} &= (\eta_A - \eta_G)\,R_\zeta \;-\; (\zeta_A - \zeta_G)\,R_\eta\,, \\ N_{G\eta} &= (\zeta_A - \zeta_G)\,R_\xi \;-\; (\xi_A - \xi_G)\,R_\zeta\,, \\ N_{G\zeta} &= (\xi_A - \xi_G)\,R_\eta \;-\; (\eta_A - \eta_G)\,R_\xi \end{cases}$$

and (48), (45) imply

$$(49)\qquad \begin{cases} N_{G\xi} &= (\eta_A - \eta_G)(m\,(a_{13}\ddot{x}_G + a_{23}\ddot{y}_G + a_{33}\ddot{z}_G) - F_\zeta) \\ &\quad -(\zeta_A - \zeta_G)(m\,(a_{12}\ddot{x}_G + a_{22}\ddot{y}_G + a_{32}\ddot{z}_G) - F_\eta)\,, \\ N_{G\eta} &= (\zeta_A - \zeta_G)(m\,(a_{11}\ddot{x}_G + a_{21}\ddot{y}_G + a_{31}\ddot{z}_G) - F_\xi) \\ &\quad -(\xi_A - \xi_G)(m\,(a_{13}\ddot{x}_G + a_{23}\ddot{y}_G + a_{33}\ddot{z}_G) - F_\zeta)\,, \\ N_{G\zeta} &= (\xi_A - \xi_G)(m\,(a_{12}\ddot{x}_G + a_{22}\ddot{y}_G + a_{32}\ddot{z}_G) - F_\eta) \\ &\quad -(\eta_A - \eta_G)(m\,(a_{11}\ddot{x}_G + a_{21}\ddot{y}_G + a_{31}\ddot{z}_G) - F_\xi) \end{cases}$$

provided (46). Besides, (38), (26), (15), (43) imply

$$(50)\qquad \begin{cases} x_G &= x_\Omega \;+\; a_{11}\xi_G \;+\; a_{12}\eta_G \;+\; a_{13}\zeta_G\,, \\ y_G &= y_\Omega \;+\; a_{21}\xi_G \;+\; a_{22}\eta_G \;+\; a_{23}\zeta_G\,, \\ z_G &= z_\Omega \;+\; a_{31}\xi_G \;+\; a_{32}\eta_G \;+\; a_{33}\zeta_G \end{cases}$$

providis by definition

$$(51)\qquad \mathbf{r}_\Omega \;=\; x_\Omega\mathbf{i} \;+\; y_\Omega\mathbf{j} \;+\; z_\Omega\mathbf{k}.$$

Since $G$ is a point of the solid $S$, the definition (15) implies

$$(52)\qquad \dot{\xi}_G \;=\; \dot{\eta}_G \;=\; \dot{\zeta}_G \;=\; 0 \qquad (\forall t)$$

and (52), (50) imply

$$(53)\qquad \begin{cases} \ddot{x}_G &= \ddot{x}_\Omega \;+\; \ddot{a}_{11}\xi_G \;+\; \ddot{a}_{12}\eta_G \;+\; \ddot{a}_{13}\zeta_G\,, \\ \ddot{y}_G &= \ddot{y}_\Omega \;+\; \ddot{a}_{21}\xi_G \;+\; \ddot{a}_{22}\eta_G \;+\; \ddot{a}_{23}\zeta_G\,, \\ \ddot{z}_G &= \ddot{z}_\Omega \;+\; \ddot{a}_{31}\xi_G \;+\; \ddot{a}_{32}\eta_G \;+\; \ddot{a}_{33}\zeta_G\,. \end{cases}$$

Now, as it is well known, the cosine-directors $a_{\mu\nu}$ $(\mu, \nu = 1,2,3)$ of the systems $Oxyz$ and $\Omega\xi\eta\zeta$ are completely determined functions

$$a_{\mu\nu} = a_{\mu\nu}(\psi, \phi, \theta) \quad (\mu, \nu = 1,2,3) \tag{54}$$

of Euler's angles $\psi, \phi, \theta$, namely

$$\begin{cases} a_{11} = \cos\psi \cos\phi - \sin\psi \sin\phi \cos\theta, \\ a_{12} = -\cos\psi \sin\phi - \sin\psi \cos\phi \sin\theta, \\ a_{13} = \sin\psi \sin\theta, \\ a_{21} = \sin\psi \cos\phi + \cos\psi \sin\phi \cos\theta, \\ a_{22} = -\sin\psi \sin\phi + \cos\psi \cos\phi \cos\theta, \\ a_{23} = -\cos\psi \sin\theta, \\ a_{31} = \sin\phi \sin\theta, \\ a_{32} = \cos\phi \sin\theta, \\ a_{33} = \cos\theta. \end{cases} \tag{55}$$

In such a manner, $\ddot{a}_{\mu\nu}$, $(\mu, \nu = 1,2,3)$ are completely determined functions

$$\ddot{a}_{\mu\nu} = \ddot{a}_{\mu\nu}(\psi, \phi, \theta; \dot{\psi}, \dot{\phi}, \dot{\theta}; \ddot{\psi}, \ddot{\phi}, \ddot{\theta}), \quad (\mu, \nu = 1,2,3) \tag{56}$$

of the variables

$$\psi, \phi, \theta; \dot{\psi}, \dot{\phi}, \dot{\theta}; \ddot{\psi}, \ddot{\phi}, \ddot{\theta} \tag{57}$$

obtainable from the relations (55).

As it is well known, the quantities

$$x_\Omega,\ y_\Omega,\ z_\Omega,\ \psi,\ \phi,\ \theta \tag{58}$$

are called the canonical parameters of the solid $S$ (with respect to the systems of reference $Oxyz$ and $\Omega\xi\eta\zeta$). By hypothesis, $S$ is submitted to a single constraint with a point of contact $A$. Now a constraint imposed on a rigid body diminishes its degree of freedom 6 by one, two, or at most three units, that is to say one, two, or three of the canonic parameters (58) of $S$ are functions of the remaining five, four, or three respectively. This implies that, on principle at least, if a single constraint is imposed on a solid $S$, then Euler's angles $\psi$, $\phi$, $\theta$ may always be chosen in the capacity of independent parameters of $S$.

Summing up, we may now state that we are faced with the following mathematical situation. By virtue of (49) the unknown components $R_x$, $R_y$, $R_z$ of the reaction $\mathbf{R}$ of the single constraint are completely expelled from the equations (33). On the other hand, the left-hand sides of these equations are, in view of Euler's kinematical equations (10), wholly determined functions of quantities (57). As regards the right-hand sides of (33), the following curcumstance must be taken into account.

First, a canonical for solid dynamics hypothesis states that the active forces (20) applied on the solid $S$ may depend on the position of $S$ in space, on the velocities of the points of $S$, and possibly on the time $t$. In other words, $\mathbf{F}_\mu$ and $\mathbf{M}_\mu$ ($\mu = 1, 2, \dots m$), and thence $\mathbf{F}$ and $\mathbf{M}$ by virtue of (22), (23), are completely determined functions of (58), of

$$(59) \qquad \dot{x}_\Omega,\ \dot{y}_\Omega,\ \dot{z}_\Omega,\ \dot{\psi},\ \dot{\phi},\ \dot{\theta}$$

and possibly of $t$. In view of (26), (50), (55), (52) and (29) the same holds for $\mathbf{M}_G$ and consequently for $M_{G\xi}, M_{G\eta}, M_{G\zeta}$. At last, the relations (49), (53), (56) imply that the only place, whence $\ddot{x}_\Omega, \ddot{y}_\Omega$ or $\ddot{z}_\Omega$ may take their appearance in the right-hand sides of equations (33) are the quantities (53).

The above considerations display that the elimination of $R_x, R_y, R_z$ from (33) is a rather knotty task. We shall now seek another way to disentangle this skein. To this end let us first note that a vectorial approach to the determination of $\mathbf{N}_G$ is possible instead of the scalar one exposed above. In order to acomplish it the vectorial record

$$(60) \qquad m\ddot{\mathbf{r}}_G = \mathbf{F} + \mathbf{R}$$

of (32) must be used. Then (60) and (41) imply

$$(61) \qquad \mathbf{N}_G = (m\ddot{\mathbf{r}}_G - \mathbf{F}) \times (\bar{\rho}_G - \bar{\rho}_A).$$

Let us remind the connection between the derivatives $\frac{d}{dt}$ and $\frac{\delta}{\delta t}$ of a vector function $\mathbf{a}$ with respect to the systems of reference $Oxyz$ and $\Omega\xi\eta\zeta$ respectively:

$$(62) \qquad \frac{d\mathbf{a}}{dt} = \bar{\omega} \times \mathbf{a} + \frac{\delta \mathbf{a}}{\delta t}\,.$$

At that, if

$$\mathbf{a} \;=\; a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \;=\; a_\xi\bar{\xi}^0 + a_\eta\bar{\eta}^0 + a_\zeta\bar{\zeta}^0 \,, \tag{63}$$

then by definition

$$\dot{\mathbf{a}} \;=\; \frac{d\mathbf{a}}{dt} \;=\; \dot{a}_x\mathbf{i} + \dot{a}_y\mathbf{j} + \dot{a}_z\mathbf{k}\,, \quad \frac{\delta\mathbf{a}}{\delta t} \;=\; \dot{a}_\xi\bar{\xi}^0 + \dot{a}_\eta\bar{\eta}^0 + \dot{a}_\zeta\bar{\zeta}^0 \,. \tag{64}$$

Now (62) implies

$$\frac{d^2\mathbf{a}}{dt^2} \;=\; \dot{\bar{\omega}} \times \mathbf{a} \;+\; \bar{\omega} \times (\bar{\omega} \times \mathbf{a}) \;+\; 2\bar{\omega} \times \frac{\delta\mathbf{a}}{\delta t} \;+\; \frac{\delta^2\mathbf{a}}{\delta t^2}\,, \tag{65}$$

as well as

$$\frac{d\bar{\omega}}{dt} \;=\; \frac{\delta\bar{\omega}}{\delta t}\,, \tag{66}$$

and (66), (9), (64) imply

$$\dot{\bar{\omega}} \;=\; \dot{\omega}_\xi\bar{\xi}^0 + \dot{\omega}_\eta\bar{\eta}^0 + \dot{\omega}_\zeta\bar{\zeta}^0 \,. \tag{67}$$

Applied to (39), the identity (65) implies

$$\ddot{r}_G \;=\; \ddot{r}_A + \dot{\bar{\omega}} \times (\bar{\rho}_G - \bar{\rho}_A) + \bar{\omega} \times (\bar{\omega} \times (\bar{\rho}_G - \bar{\rho}_A)) - 2\bar{\omega} \times \frac{\delta\bar{\rho}_A}{\delta t} - \frac{\delta^2\bar{\rho}_A}{\delta t^2}\,, \tag{68}$$

since (15), (52), (64) imply

$$\frac{\delta\bar{\rho}_G}{\delta t} \;=\; 0 \qquad (\forall t), \tag{69}$$

and (69) implies

(70)

$$\begin{aligned}\ddot{\mathbf{r}}_G \times (\bar{\rho}_G - \bar{\rho}_A) \;=\; & \ddot{\mathbf{r}}_A \times (\bar{\rho}_G - \bar{\rho}_A) \;+\; (\dot{\bar{\omega}}(\bar{\rho}_G - \bar{\rho}_A))(\bar{\rho}_G - \bar{\rho}_A) \\ & - \; (\bar{\rho}_G - \bar{\rho}_A)^2\dot{\bar{\omega}} \;+\; (\bar{\omega}(\bar{\rho}_G - \bar{\rho}_A)\bar{\omega} \times (\bar{\rho}_G - \bar{\rho}_A) \\ & - \; 2(\bar{\omega}(\bar{\rho}_G - \bar{\rho}_A))\frac{\delta\bar{\rho}_A}{\delta t} \;+\; 2(\frac{\delta\bar{\rho}_A}{\delta t}(\bar{\rho}_G - \bar{\rho}_A))\bar{\omega} \;+\; (\bar{\rho}_G - \bar{\rho}_A) \times \frac{\delta^2\bar{\rho}_A}{\delta t^2}\,.\end{aligned}$$

If by definition

$$\mathbf{s}_1 \;=\; (m\ddot{\mathbf{r}}_A \;-\; \mathbf{F}) \times (\bar{\rho}_G \;-\; \bar{\rho}_A), \tag{71}$$

$$(72)\qquad \mathbf{s}_2 = m(\dot{\bar\omega}(\bar\rho_G - \bar\rho_A))(\bar\rho_G - \bar\rho_A), \quad \mathbf{s}_3 = -m(\bar\rho_G - \bar\rho_A)^2\dot{\bar\omega},$$

$$(73)\quad \mathbf{s}_4 = m(\bar\omega(\bar\rho_G - \bar\rho_A))\bar\omega \times (\bar\rho_G - \bar\rho_A), \quad \mathbf{s}_5 = -2m(\bar\omega(\bar\rho_G - \bar\rho_A))\frac{\delta\bar\rho_A}{\delta t},$$

$$(74)\qquad \mathbf{s}_6 = 2m(\frac{\delta\bar\rho_A}{\delta t}(\bar\rho_G - \bar\rho_A))\bar\omega, \quad \mathbf{s}_7 = m(\bar\rho_G - \bar\rho_A)\times\frac{\delta^2\bar\rho_A}{\delta t^2}$$

then (61), (70)–(74) imply

$$(75)\qquad \mathbf{N}_G = \sum_{\nu=1}^{7}\mathbf{s}_\nu\,.$$

In order to compute $\mathbf{N}_G$ let by definition

$$(76)\qquad \mathbf{s}_\nu = s_{\nu\xi}\bar\xi^0 + s_{\nu\eta}\bar\eta^0 + s_{\nu\zeta}\bar\zeta^0 \quad (\nu = 1,\dots,7).$$

The definitions (34), (27) and (43) imply

$$(77)\qquad \begin{aligned} m\ddot{\mathbf{r}}_A - \mathbf{F} = {}& (a_{11}(m\ddot x_A - F_x) + (a_{21}(m\ddot y_A - F_y) + (a_{31}(m\ddot z_A - F_z))\,\bar\xi^0 \\ &+ (a_{12}(m\ddot x_A - F_x) + (a_{22}(m\ddot y_A - F_y) + (a_{32}(m\ddot z_A - F_z))\,\bar\eta^0 \\ &+ (a_{13}(m\ddot x_A - F_x) + (a_{23}(m\ddot y_A - F_y) + (a_{33}(m\ddot z_A - F_z))\,\bar\zeta^0, \end{aligned}$$

and (71), (76), (77), (15), (35) imply

$$(78)\qquad \begin{aligned} s_{1\xi} = {}& (a_{12}(m\ddot x_A - F_x) + (a_{22}(m\ddot y_A - F_y) + (a_{32}(m\ddot z_A - F_z))(\zeta_G - \zeta_A) \\ &- (a_{13}(m\ddot x_A - F_x) + (a_{23}(m\ddot y_A - F_y) + (a_{33}(m\ddot z_A - F_z))(\eta_G - \eta_A). \end{aligned}$$

On the other hand, (73), (76), (9), (15), (35) imply

$$(79)\quad s_{2\xi} = -m((\xi_G - \xi_A)^2 + (\eta_G - \eta_A)^2 + (\zeta_G - \zeta_A)^2)\dot\omega_\xi\,,$$

$$(80)\quad s_{3\xi} = -m(\dot\omega_\xi(\xi_G - \xi_A) + \dot\omega_\eta(\eta_G - \eta_A) + \dot\omega_\zeta(\zeta_G - \zeta_A))(\xi_G - \xi_A)\,.$$

Similarly, (73), (74), (9), (15), (35) imply

$$(81)\qquad \begin{aligned} s_{4\xi} = {}& m(\omega_\xi(\xi_G - \xi_A) + \omega_\eta(\eta_G - \eta_A) + \omega_\zeta(\zeta_G - \zeta_A))\times \\ &\times(\omega_\eta(\zeta_G - \zeta_A) - \omega_\zeta(\eta_G - \eta_A)). \end{aligned}$$

Besides, (73), (74), (76), (9), (15), (35), (64) imply

$$(82)\qquad s_{5\xi} = -2m(\omega_\xi(\xi_G - \xi_A) + \omega_\eta(\eta_G - \eta_A) + \omega_\zeta(\zeta_G - \zeta_A))\dot\xi_A\,,$$

$$(83)\qquad s_{6\xi} = 2m(\dot\xi_A(\xi_G - \xi_A) + \dot\eta_A(\eta_G - \eta_A) + \dot\zeta_A(\zeta_G - \zeta_A))\omega_\xi\,,$$

$$(84)\qquad s_{7\xi} = m((\eta_G - \eta_A)\ddot\zeta_A - (\zeta_G - \zeta_A)\ddot\eta_A)\,.$$

Now (75), (31), (78)–(84) imply

$$
\begin{aligned}
N_{G\xi} &= (a_{12}(m\ddot{x}_A - F_x) + a_{22}(m\ddot{y}_A - F_y) + a_{32}(m\ddot{z}_A - F_z))(\zeta_G - \zeta_A) \\
&\quad - (a_{13}(m\ddot{x}_A - F_x) + a_{23}(m\ddot{y}_A - F_y) + a_{33}(m\ddot{z}_A - F_z))(\eta_G - \eta_A) \\
&\quad - m((\eta_G - \eta_A)^2 + (\zeta_G - \zeta_A)^2)\dot{\omega}_\xi \\
&\quad + m(\xi_G - \xi_A)(\dot{\omega}_\eta(\eta_G - \eta_A) + \dot{\omega}_\zeta(\zeta_G - \zeta_A)) \\
&\quad + m(\omega_\xi(\xi_G - \xi_A) + \omega_\eta(\eta_G - \eta_A) + \omega_\zeta(\zeta_G - \zeta_A)) \times \\
&\quad \times (\omega_\eta(\zeta_G - \zeta_A) - \omega_\zeta(\eta_G - \eta_A)) \\
&\quad - 2m(\omega_\eta(\eta_G - \eta_A) + \omega_\zeta(\zeta_G - \zeta_A))\dot{\xi}_A \\
&\quad + 2m(\dot{\eta}_A(\eta_G - \eta_A) + \dot{\zeta}_A(\zeta_G - \zeta_A))\omega_\xi \\
&\quad + m((\eta_G - \eta_A)\ddot{\zeta}_A - (\zeta_G - \zeta_A)\ddot{\eta}_A).
\end{aligned}
\tag{85}
$$

Similar expresions, with self-evident *mutatis mutandis*, are obtained for $N_{G\eta}$ and $N_{G\zeta}$.

Considerable simplifications in (85) set in if $G$ is chosen in the capacity of origin of the system of reference $\Omega\xi\eta\zeta$ invariably connected with the solid $S$ - a choice always possible and not infrequently favourite in dynamical praxis. In such a case obviously

$$
\xi_G = \eta_G = \zeta_G = 0 \quad (\forall t)
\tag{86}
$$

and (85) takes the form

$$
\begin{aligned}
N_{G\xi} &= (a_{13}(m\ddot{x}_A - F_x) + a_{23}(m\ddot{y}_A - F_y) + a_{33}(m\ddot{z}_A - F_z))\eta_A \\
&\quad - (a_{12}(m\ddot{x}_A - F_x) + a_{22}(m\ddot{y}_A - F_y) + a_{32}(m\ddot{z}_A - F_z))\zeta_A \\
&\quad - m(\eta_A^2 + \zeta_A^2)\dot{\omega}_\xi + m\xi_A(\eta_A\dot{\omega}_\eta + \zeta_A\dot{\omega}_\zeta) \\
&\quad + m(\omega_\xi\xi_A + \omega_\eta\eta_A + \omega_\zeta\zeta_A)(\omega_\eta\zeta_A - \omega_\zeta\eta_A) \\
&\quad + 2m(\omega_\eta\eta_A + \omega_\zeta\zeta_A)\dot{\xi}_A - 2m(\eta_A\dot{\eta}_A + \zeta_A\dot{\zeta}_A)\omega_\xi \\
&\quad + m(\zeta_A\ddot{\eta}_A - \eta_A\ddot{\zeta}_A).
\end{aligned}
\tag{87}
$$

In the case when the point of contact $A$ of the solid $S$ with the constraint is fixed in $S$, i. e. when

$$
\dot{\xi}_A = \dot{\eta}_A = \dot{\zeta}_A = 0 \quad (\forall t),
\tag{88}
$$

the quantities $\xi_A, \eta_A, \zeta_A$ are known constants of the dynamical problem under consideration, and (87) takes the form

$$
\begin{aligned}
N_{G\xi} &= \eta_A(a_{13}(m\ddot{x}_A - F_x) + a_{23}(m\ddot{y}_A - F_y) + a_{33}(m\ddot{z}_A - F_z)) \\
&\quad - \zeta_A(a_{12}(m\ddot{x}_A - F_x) + a_{22}(m\ddot{y}_A - F_y) + a_{32}(m\ddot{z}_A - F_z)) \\
&\quad - m(\eta_A^2 + \zeta_A^2)\dot{\omega}_\xi + m\xi_A(\eta_A\dot{\omega}_\eta + \zeta_A\dot{\omega}_\zeta) \\
&\quad + m(\omega_\xi\xi_A + \omega_\eta\eta_A + \omega_\zeta\zeta_A)(\omega_\eta\zeta_A - \omega_\zeta\eta_A)\,.
\end{aligned} \tag{89}
$$

A slight simplification of the expression (87) may be attained in the following manner. As it is well known, Euler's equations (33) represent, purely and simply, projections of the axes $\Omega\xi, \Omega\eta, \Omega\zeta$ respectively of the equation

$$\dot{\mathbf{L}}_\Gamma = \mathbf{M}_G + \mathbf{N}_G\,, \tag{90}$$

where by definition

$$\mathbf{L}_\Gamma = \int \bar{\rho} \times (\omega \times \bar{\rho})\, dm - m\bar{\rho}_G \times (\bar{\omega} \times \bar{\rho}_G)\,. \tag{91}$$

Now (29), (41) and (39) imply

$$\mathbf{M}_G + \mathbf{N}_G = \mathbf{M} + \mathbf{F} \times \mathbf{r}_A + (+\mathbf{F} + \mathbf{R}) \times (\bar{\rho}_G - \bar{\rho}_A)\,. \tag{92}$$

On the other hand

$$\mathbf{M}_A = \mathbf{M} + \mathbf{F} \times \mathbf{r}_A \tag{93}$$

represents the moment with respect to $A$ of the system of active forces (20) applied on the solid $S$, and (90), (92), (93) imply

$$\dot{\mathbf{L}}_\Gamma = \mathbf{M}_A + \mathbf{N}_A\,, \tag{94}$$

where by definition

$$\mathbf{N}_A = (\mathbf{F} + \mathbf{R}) \times (\bar{\rho}_G - \bar{\rho}_A)\,. \tag{95}$$

Let by definition

$$\mathbf{M}_A = M_{A\xi}\bar{\xi}^0 + M_{A\eta}\bar{\eta}^0 + M_{A\zeta}\bar{\zeta}^0\,, \tag{96}$$

$$\mathbf{N}_A = N_{A\xi}\bar{\xi}^0 + N_{A\eta}\bar{\eta}^0 + N_{A\zeta}\bar{\zeta}^0\,. \tag{97}$$

Since these transformations do not affect the left-hand sides $\dot{L}_{\Gamma\xi}, \dot{L}_{\Gamma\eta}, \dot{L}_{\Gamma\zeta}$ of (33) provided by definition

$$\dot{\mathbf{L}}_\Gamma = \dot{L}_{\Gamma\xi}\bar{\xi}^0 + \dot{L}_{\Gamma\eta}\bar{\eta}^0 + \dot{L}_{\Gamma\zeta}\bar{\zeta}^0 , \tag{98}$$

the relations (94) and (96)–(98) imply that Euler's equations (33) may be written in the form

$$\dot{L}_{\Gamma\xi} = M_{A\xi} + N_{A\xi}, \quad \dot{L}_{\Gamma\eta} = M_{A\eta} + N_{A\eta}, \quad \dot{L}_{\Gamma\zeta} = M_{A\zeta} + N_{A\zeta} , \tag{99}$$

reminding once more that the left-hand sides of (99) represent a symbolic — let us say, stenographic — record of the left-hand sides of (33). This settled, we are now in the position to write down, say, $N_{A\xi}$. Indeed, in the case (86), for instance, the equations (61), (60), (95) and (87) imply

$$\begin{aligned} \frac{1}{m} N_{A\xi} &= (a_{13}\ddot{x}_A + a_{23}\ddot{y}_A + a_{33}\ddot{z}_A)\eta_A - (a_{12}\ddot{x}_A + a_{22}\ddot{y}_A + a_{32}\ddot{z}_A)\zeta_A \\ &\quad - (\eta_A^2 + \zeta_A^2)\dot{\omega}_\xi + \xi_A(\eta_A\dot{\omega}_\eta + \zeta_A\dot{\omega}_\zeta) + (\zeta_A\ddot{\eta}_A - \eta_A\ddot{\zeta}_A) \\ &\quad + (\omega_\xi\xi_A + \omega_\eta\eta_A + \omega_\zeta\zeta_A)(\omega_\eta\zeta_A - \omega_\zeta\eta_A) \\ &\quad + 2(\omega_\eta\eta_A + \omega_\zeta\zeta_A)\dot{\xi}_A - 2(\eta_A\dot{\eta}_A + \zeta_A\dot{\zeta}_A)\omega_\xi . \end{aligned} \tag{100}$$

*Mutatis mutandis*, similar expressions are obtained for $N_{A\eta}$ and $N_{A\zeta}$.

In the case (88) the relation (100) takes the form

$$\begin{aligned} \frac{1}{m} N_{A\xi} &= (a_{13}\ddot{x}_A + a_{23}\ddot{y}_A + a_{33}\ddot{z}_A)\eta_A - (a_{12}\ddot{x}_A + a_{22}\ddot{y}_A + a_{32}\ddot{z}_A)\zeta_A \\ &\quad - (\eta_A^2 + \zeta_A^2)\dot{\omega}_\xi + \xi_A(\eta_A\dot{\omega}_\eta + \zeta_A\dot{\omega}_\zeta) \\ &\quad + (\omega_\xi\xi_A + \omega_\eta\eta_A + \omega_\zeta\zeta_A)(\omega_\eta\zeta_A - \omega_\zeta\eta_A) . \end{aligned} \tag{101}$$

Let us, *exampli gratia*, write down the first equation (99) in the most general case, i.e. when the origin $\Omega$ of the system $\Omega\xi\eta\zeta$, invariably connected with the solid $S$, does not necessarily coincide with its mass-center $G$. Since the left-hand side of the first equation is identical with the left-hand side of the first equation (33), we obtain

$$
\begin{aligned}
&A\dot{\omega}_\xi - (B-C)\omega_\eta\omega_\zeta - D(\omega_\eta^2 - \omega_\zeta^2) \\
&- E(\dot{\omega}_\zeta + \omega_\xi\omega_\eta) - F(\dot{\omega}_\eta - \omega_\zeta\omega_\xi) \\
&= M_{A\xi} + m(a_{12}x_A + a_{22}y_A + a_{32}z_A)(\zeta_G - \zeta_A) \\
&- m(a_{13}x_A + a_{23}y_A + a_{33}z_A)(\eta_G - \eta_A) \\
&- m((\eta_G - \eta_A)^2 + (\zeta_G - \zeta_A)^2)\,\dot{\omega}_\xi + m((\eta_G - \eta_A)\ddot{\zeta}_A - (\zeta_G - \zeta_A)\ddot{\eta}_A) \\
&+ m(\xi_G - \xi_A)(\dot{\omega}_\eta(\eta_G - \eta_A) + \dot{\omega}_\zeta(\zeta_G - \zeta_A)) \\
&+ m(\omega_\xi(\xi_G - \xi_A) + \omega_\eta(\eta_G - \eta_A) + \omega_\zeta(\zeta_G - \zeta_A)) \times \\
&\times\ (\omega_\eta(\zeta_G - \zeta_A) - \omega_\zeta(\eta_G - \eta_A)) \\
&- 2m(\omega_\eta(\eta_G - \eta_A) + \omega_\zeta(\zeta_G - \zeta_A))\,\dot{\xi}_A \\
&+ 2m(\dot{\eta}_A(\eta_G - \eta_A) + \dot{\zeta}_A(\zeta_G - \zeta_A))\omega_\xi\,,
\end{aligned} \tag{102}
$$

the other two equations (99) being obtainable from (102) by a cyclic lexicographic change of letters.

Until now we have exposed the general traces of a strategic, if we may say so, scheme of obtaining "pure" differential equations of motion of a solid $S$, submited to a single constraint with a point of contact $A$, by an elimination of unknown reactions of the constraint from Euler' equations (33) through the mediation of (32), under most general conditions at that of the dynamical problem concerned. Let us now come down to earth by complementing those total considerations with some tactical particularities.

First and foremost we must fix our attention on the notion of a *mechanical constraint* imposed on a solid.

The time-honoured statical and dynamical praxis has shaped 10 fundamental cases when a rigid body $S$ is submitted to such external impacts that the conformation of their availability in statical and dynamical phenomena has ultimately led to the idea of k i n e t i c a l  c o n s t r a i n t s or l i a i s o n s imposed on $S$. The adjective *kinetical* is used here quite deliberately in contrast to the so called p u r e l y  g e o m e t r i c a l constraints or liaisons one may imagine imposed on a *geometrical* solid. In order to emphasize the radical distinguishing features between *geometrical* and *kinetical* liaisons, let us consider a most instructive special case.

The motion of a rigid body with a fixed point has always been a pet subject of mechanicians from the very beginning of dynamics, especially its particular case called *spinning top.* Now the very formulation of a solid-with-a-fixed-point problem provides the occasion for curious, non-standart and perspicatious mechanical meditations concerning the conceptual aspects of the notion liaison imposed on a rigid body.

Let $A$ be the fixed point of the solid $S$ and let us suppose that $A$ is *inertial* according to the very conditions of the dynamical problem under consideration. The meaning of this supposition is that $A$ may be connected *invariably* with an *inertial* system of reference $Oxyz$, i.e. that

$$\dot{x}_A = \dot{y}_A = \dot{z}_A = 0 \quad (\forall t) \tag{103}$$

provided (34). The point $A$ being by hypothesis fixed in $S$, too, the definition (35) implies the relations (88). Now (103) and (88) suggest that, for the sake of simplicity, one may choose $O = \Omega = A$. This proviso adopted, it is immediately seen that $S$ has 3 degrees of freedom and that any of its positions is determined by means of Euler's angles $\psi, \phi, \theta$ between $Axyz$ and $A\xi\eta\zeta$.

On the other hand, it is crystal-clear that, $P$ being any point of $S$ different from $A$, $P$ is compeled to remain, during any motion of $S$, on a sphere with center $A$ and radius $AP$. In such a manner, one concludes that the initial constraint

$$r_A = \bar{\rho}_A = 0 \tag{104}$$

imposed on $S$ entails an innumerable multitude of secondary constraints imposed on $S$: as a matter of fact, these by-product constraints are thrust on all and sundry points of $S$ other that $A$.

Vice versa, let $P_\nu(\nu = 1, 2, 3)$ be points of $S$ and let $A$ be such a point that

$$(\bar{\rho}_\nu - \bar{\rho}_A)^2 = a^2 \quad (\nu = 1, 2, 3; \forall t), \tag{105}$$

$$(\bar{\rho}_1 - \bar{\rho}_A) \times (\bar{\rho}_2 - \bar{\rho}_A) \cdot (\bar{\rho}_3 - \bar{\rho}_A) \neq 0 \quad (\forall t) \tag{106}$$

provided $\bar{\rho}_\nu = \overline{\Omega P_\nu}$ and $\bar{\rho}_A = \overline{\Omega A}$, $a$ being a constant with respect to the time $t$. Since by condition

$$\frac{\delta \bar{\rho}_\nu}{\delta t} = 0 \quad (\nu = 1, 2, 3; \forall t), \tag{107}$$

the relations (105) imply

$$(108) \qquad (\bar{\rho}_\nu - \bar{\rho}_A) \frac{\delta \bar{\rho}_A}{\delta t} = 0 \qquad (\nu = 1,2,3; \forall t),$$

and (108), (106) imply

$$(109) \qquad \frac{\delta \bar{\rho}_A}{\delta t} = 0 \qquad (\forall t),$$

i. e. $A$ is a point of $S$.

In such a manner, inversely, the immobility of $A$ is implied by the following constraints imposed on $S$: three points $P_\nu$ ($\nu = 1,2,3$) of the solid $S$ are compeled to remain on a sphere with a center $A$ and radius $a$, being non-complanar with $A$.

Consequently, both problems — namely the conditions (88), (103) on the one hand, and the conditions (105)–(107) on the other hand — are geometrically identical. Are they identical mechanically, too?

Mechanically those are quite different mathematical problems.

According to Ax R, the motion of the solid $S$ in the case (88), (103) is governed by equations (60) and

$$(110) \qquad \dot{\mathbf{L}}_\Gamma = \mathbf{M}_G + \mathbf{R} \times \bar{\rho}_G$$

provided (104) by virtue of (90) and (41), the reaction in $A$ being (36). On the contrary, in the case (105)–(107), again according to Ax R, there are 3 reactions of the constraints

$$(111) \qquad \vec{R}_\nu = (\mathbf{R}_\nu, \mathbf{N}_\nu) \qquad (\nu = 1,2,3)$$

the moments $\mathbf{N}_\nu$ ($\nu = 1,2,3$) being taken with respect to $O = \Omega = A$. In other words,

$$(112) \qquad \mathbf{N}_\nu = \bar{\rho}_\nu \times \mathbf{R}_\nu \qquad (\nu = 1,2,3)$$

whence, according to (29) and (22), (23),

$$(113) \qquad \mathbf{N}_G = \mathbf{N} + \mathbf{R} \times \bar{\rho}_G = \sum_{\nu=1}^{3} \bar{\rho}_\nu \times \mathbf{R}_\nu + \sum_{\nu=1}^{3} \mathbf{R}_\nu \times \bar{\rho}_G ,$$

i. e.

$$(114) \qquad \mathbf{N}_G = \sum_{\nu=1}^{3} (\bar{\rho}_\nu - \bar{\rho}_G) \times \mathbf{R}_\nu .$$

Hence, the motion of the solid $S$ in the case (105)–(107) is governed by the equations

$$(115) \qquad m\ddot{\bar{\rho}}_G = \mathbf{F} + \sum_{\nu=1}^{3} \mathbf{R}_\nu ,$$

$$(116) \qquad \dot{\mathbf{L}}_\Gamma = \mathbf{M}_G + \sum_{\nu=1}^{3} (\bar{\rho}_\nu - \bar{\rho}_G) \times \mathbf{R}_\nu .$$

Now what!

*Cujusvis hominis est errare, nullius, nisi insipientis, in errore perseverare,* observed once Cicero. The first lesson the sapient one must derive from this simple contrast is that *there is a profound distinction between a geometrical and a mechanical constraint.* A geometrical constraint is a necessary, but by no means sufficient, condition for the availability of a mechanical constraint. One may state figuratively that a mechanical constraint is a geometrical constraint plus Ax R. This not quite precise a formulation stands in need of some particularization.

First of all, let us note that the conclusion in the case (104) that any point of $S$ other that $A$ is compelled to remain on a sphere with center $A$ is due to the specific characteristics of the rigid body concept rather than to any outside interference. Indeed, a solid $S$ may be described as such a mechanical entity, for which

$$(117) \qquad \frac{d}{dt}(\mathbf{r}_1 - \mathbf{r}_2)^2 = 0 \qquad (\forall t)$$

holds for any two points $P_1$ and $P_2$ of $S$ provided $r_\nu = OP_\nu (\nu = 1,2)$. Now it is enough to substitute $P$ and $A$ for $P_1$ and $P_2$ respectively in order to derive

$$(118) \qquad (\mathbf{r} - \mathbf{r}_A)^2 = a^2 \qquad (\forall t)$$

from (117), $a^2$ being a constant of integration.

Second, it is not pointless to brood a while over the physical motivations of the mechanical liaison concept. Any constraint imposed on a *physical* rigid body is feasible by means of another *physical* rigid body, and the *physical* result, as well as cause in the same time, of their mutual contact are two forces, equal in magnitude and opposite in direction according to Newton's *Lex III*, acting on either of the bodies. In such a manner the *reaction* of the constraint is an inalienable attribute of the mechanical liaison concept — as necessary as its geometrical description.

All those considerations once grasped, they lead to the following two important inferences.

First, it is most desirable to give prominence to the essential difference between geometrical and mechanical constraints imposed on a solid by an explicit indication of the fact in any particular instance; this may be achieved by the indispensable use of the adjectives *geometrical* or *mechanical* in any concrete case. In the sequel we shall adopt the term mechanical liaison.

Second, a mechanical liaison is outlined against all possible concomitant geometrical constraints (leading to the same restrictions of the solid $S$ in space, consequently to the same degree of freedom $l$ of $S$ and to the same mutually independent parameters $q_\lambda$ $(\lambda = 1, \dots l)$ of $S$) by the mere proclaiming it as such one. The very promulgation of a particular geometrical constraint for a mechanical liaison from among a infinite variety of other equivalent geometrical constraints (as in the above example of a solid with a fixed point) applying for the dynamical rank, or state, or title, immediately calls forth arouse Ax R, generating a reaction through the corresponding point of contact, while all other failed candidates purely and simply are falling into disuse. *Cum grano salis* the situation is to a certain degree similar to that of a the-bell-of-the-ball-competition.

Reverting to the theme touched some pages above, namely about the 10 traditional cases of mechanical constraints established by the statical and dynamical praxis, let us formulate them pedantically from beginning to end with a view to their application to equations (102) and suchlike.

**Mechanical constraint No 1.** A point $A$ fixed in the solid is constrained to remain on a surface given in space (possibly changeable in the course of time).

**Mechanical constraint No 2.** A surface fixed in the solid is constrained to pass through a point $A$ given in space (possibly changeable in the course of time).

**Mechanical constraint No 3.** A point $A$ fixed in the solid is constrained to remain on a curve line given in space (possibly changeable in the course of time).

**Mechanical constraint No 4.** A curve line fixed in the solid is constrained to pass through a point $A$ given in space (possibly changeable in the course of time).

**Mechanical constraint No 5.** A point $A$ fixed in the solid is constrained to coincide with a given point in space (possibly changeable in the course of time).

**Mechanical constraint No 6.** A surface fixed in the solid is constrained to touch a surface given in space (possibly changeable in the course of time), the common point $A$ of the surfaces being in the general case variable on both of them.

**Mechanical constraint No 7.** A surface fixed in the solid is constrained to touch a curve line given in space (possibly changeable in the course of time), the common point $A$ of the surface and the curve being in the general case variable on both of them.

**Mechanical constraint No 8.** A curve line fixed in the solid is constrained to touch a surface given in space (possibly changeable in the course of time), the common point $A$ of the curve and the surface being in the general case variable on both of them.

**Mechanical constraint No 9.** A curve line fixed in the solid is constrained to intersect a curve line given in space (possibly changeable in the course of time), the common point $A$ of the curve lines being in the general case variable on both of them.

**Mechanical constraint No 10.** A curve line fixed in the solid is constrained to touch a curve line given in space (possibly changeable in the course of time), the common point $A$ of the curve lines being in the general case variable on both of them.

In any of those 10 cases the geometrical entity external to the solid $S$ (the surfaces in Nos 1, 6 and 8; the points in Nos 2, 4 and 5; the curve lines in Nos 3, 7, 9 and 10) is called a *mechanical constraint* imposed on $S$, and the common point $A$ of $S$ and the corresponding mechanical constraint is called their point of contact.

Let us formulate mathematically those descriptive definitions. To this end let first $P$ be any point of space and, provided $\mathbf{r} = \mathbf{OP}$, $\bar{\rho} = \overline{\Omega P}$, let by definition

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{119}$$

and (12) hold.

Second, let $F(x,y,z,t), F_\nu(x,y,z,t), f(\xi,\eta,\zeta), f_\nu(\xi,\eta,\zeta)$ $(\nu = 1,2)$ be given twofold differentiable functions of the indicated arguments with

$$\operatorname{grad} F \neq 0\,, \tag{120}$$

$$\operatorname{grad} F_1 \times \operatorname{grad} F_2 \neq 0\,, \tag{121}$$

$$\operatorname{grad} f \neq 0\,, \tag{122}$$

$$\operatorname{grad} f_1 \times \operatorname{grad} f_2 \neq 0\,, \tag{123}$$

provided by definition

$$\operatorname{grad} F = \frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y},\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k} \tag{124}$$

$$\operatorname{grad} f = \frac{\partial f}{\partial \xi}\bar{\xi}^0 + \frac{\partial f}{\partial \eta}\bar{\eta}^0 + \frac{\partial f}{\partial \zeta}\bar{\zeta}^0\,, \tag{125}$$

etc.

Third, let $\mathbf{r}_A = \mathbf{OA}$, $\bar{\rho}_A = \overline{\Omega A}$, $r_\Omega = \overline{O\Omega}$ and let by definition (34), (35) and

$$\mathbf{r}_\Omega = x_\Omega\,\mathbf{i} + y_\Omega\,\mathbf{j} + z_\Omega\,\mathbf{k} \tag{126}$$

hold. Now the identity $\mathbf{r}_A = \mathbf{r}_\Omega + \bar{\rho}_A$ and the relations (43) imply

$$\begin{cases} x_A = x_\Omega + a_{11}\xi_A + a_{12}\eta_A + a_{13}\zeta_A\,, \\ y_A = y_\Omega + a_{21}\xi_A + a_{22}\eta_A + a_{23}\zeta_A\,, \\ z_A = z_\Omega + a_{31}\xi_A + a_{32}\eta_A + a_{33}\zeta_A\,. \end{cases} \tag{127}$$

Under the above provisoes the mechanical constraints Nos 1–10 can be expressed in the following manner.

**Constraint No 1.** The equation of the surface being

$$F(x, y, z, t) = 0, \tag{128}$$

the only restriction imposed on the solid $S$ by the mechanical constraint is

$$F(x_A, y_A, z_A, t) = 0, \tag{129}$$

the constants

$$\xi_A, \eta_A, \zeta_A \tag{130}$$

in (127) being determined by the conditions of the dynamical problem under consideration.

**Constraint No 2.** The equation of the surface being

$$f(\xi, \eta, \zeta) = 0, \tag{131}$$

the only restriction imposed on the solid $S$ by the mechanical constraint is

$$f(\xi_A, \eta_A, \zeta_A) = 0, \tag{132}$$

the functions

$$x_A(t), y_A(t), z_A(t) \tag{133}$$

being determined by the conditions of the dynamical problem under consideration.

**Constraint No 3.** The equations of the curve line being

$$F_\nu(x, y, z, t) = 0 \quad (\nu = 1, 2), \tag{134}$$

the only restrictions imposed on the solid $S$ by this mechanical constraint are

$$F_\nu(x_A, y_A, z_A, t) = 0 \quad (\nu = 1, 2), \tag{135}$$

the constants (130) being determined by the conditions of the dynamical problem under consideration.

**Constraint No 4.** The equations of the curve line being

$$f_\nu(\xi, \eta, \zeta) = 0 \quad (\nu = 1, 2), \tag{136}$$

the only restrictions imposed on the solid $S$ by this mechanical constraint are

$$f_\nu(\xi_A, \eta_A, \zeta_A, t) = 0 \qquad (\nu = 1, 2), \tag{137}$$

the functions (133) being determined by the conditions of the dynamical problem under consideration.

**Constraint No 5.** The only restrictions imposed on the solid $S$ by this mechanical constraint are (127), the constants (130) and the functions (133) being determined by the conditions of the dynamical problem under consideration.

**Constraint No 6.** The equations of the surfaces being (131) and (128) respectively, the only restrictions imposed on the solid $S$ by this mechanical constraint are (132), (129) and

$$\operatorname{grad}_A F \times \operatorname{grad}_A f = 0\,, \tag{138}$$

the index $A$ indicating that the corresponding quantities must be computed for the point $A$.

**Constraint No 7.** The equations of the surface and the curve line being (131) and (134) respectively, the only restrictions imposed on the solid $S$ by this mechanical constraint are (132), (135) and

$$\operatorname{grad}_A F_1 \times \operatorname{grad}_A F_2 \cdot \operatorname{grad}_A f = 0\,. \tag{139}$$

**Constraint No 8.** The equations of the curve line and the surface being (136) and (128) respectively, the only restrictions imposed on the solid $S$ by this mechanical constraint are (137), (129) and

$$\operatorname{grad}_A F \cdot \operatorname{grad}_A f_1 \times \operatorname{grad}_A f_2 = 0\,. \tag{140}$$

**Constraint No 9.** The equations of the curve lines being (136) and (134) respectively, the only restrictions imposed on the solid $S$ by this mechanical constraint are (137), (135), since the condition of intersection

$$(\operatorname{grad}_A F_1 \times \operatorname{grad}_A F_2) \times (\operatorname{grad}_A f_1 \times \operatorname{grad}_A f_2) \neq 0 \tag{141}$$

is a prohibition rather than an obligation.

**Constraint No 10.** The equations of the curve lines being (136) and (134) respectively, the only restrictions imposed on the solid $S$ by this mechanical constraint are (137), (135) and

$$(\operatorname{grad}_A F_1 \times \operatorname{grad}_A F_2) \times (\operatorname{grad}_A f_1 \times \operatorname{grad}_A f_2) = 0\,. \tag{142}$$

The above analysis is enough and to spare with a view to the mathematical situation one is faced with when any particular of the mechanical constraints Nos 1–10 is separately at hand in a particular solid dynamical problem. Nevertheless we shall propose a second mathematical formalization of the circumstances attendant these constraints which, being equivalent to the former one, has several technical advantages, in some cases at least. It is closely connected with the parametric representation of the surfaces and curve lines involved in those constraints.

To this end let $\mathbf{r}(u,v), \mathbf{r}(s)$ and $\bar{\rho}(\alpha,\beta), \bar{\rho}(\sigma)$ be given twofold differentiable functions of the indicated arguments with

(143) $$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \neq 0,$$

(144) $$\frac{d\mathbf{r}}{ds} \neq 0,$$

and

(145) $$\frac{\partial \bar{\rho}}{\partial \alpha} \times \frac{\partial \bar{\rho}}{\partial \beta} \neq 0,$$

(146) $$\frac{d\bar{\rho}}{d\sigma} \neq 0.$$

Under those provisoes the mechanical constraints Nos 1–10 can be expressed in the following manner.

**Constraint No 1.** The equation of the surface being

(147) $$\mathbf{r} = \mathbf{r}(u,v,t)$$

where $\mathbf{r}$ in the left-hand side represents the radius-vector (119) of any of its points $P$, the only restriction imposed on the solid $S$ by this mechanical constraint is

(148) $$\mathbf{r}_A = \mathbf{r}(u_A, v_A, t),$$

$u_A$ and $v_A$ representing newly introduced parameters of $S$. Since the right-hand side of (147) represents a given function of $u, v, t$, the relation

(149) $$\mathbf{r}(u,v,t) = x(u,v,t)\,\mathbf{i} + y(u,v,t)\,\mathbf{j} + z(u,v,t)\,\mathbf{k}$$

implied by (119), (147) suggests that the components $x(u,v,t)$, $y(u,v,t)$, $z(u,v,t)$ are given functions of $u, v, t$, too. Now the relations (127), (148), (34) imply

$$(150) \qquad \begin{cases} x_\Omega &= x(u_A, v_A, t) \; - \; (a_{11}\xi_A \, + \, a_{12}\eta_A \, + \, a_{13}\zeta_A)\,, \\ y_\Omega &= y(u_A, v_A, t) \; - \; (a_{21}\xi_A \, + \, a_{22}\eta_A \, + \, a_{23}\zeta_A)\,, \\ z_\Omega &= z(u_A, v_A, t) \; - \; (a_{31}\xi_A \, + \, a_{32}\eta_A \, + \, a_{33}\zeta_A)\,, \end{cases}$$

where the constants (130) are determined by the condition of the dynamical problem under consideration. In such a manner, in view of (55), the canonic parameters

$$(151) \qquad x_\Omega, y_\Omega, z_\Omega$$

of the solid $S$ are found to be completely determined functions of the time $t$ and of the quantities

$$(152) \qquad \psi, \phi, \theta, u_A, v_A\,,$$

which in turn are mutually wholly independent.

Summing up, we conclude that in the case of mechanical constraint No 1 the solid $S$ has 5 degrees of freedom and that the quantities (152) may be chosen in the capacity of its independent parameters.

**Constraint No 2.** The equation of the surface being

$$(153) \qquad \bar{\rho} \; = \; \bar{\rho}(\alpha, \beta)$$

where $\bar{\rho}$ in the left-hand side represents the radius-vector (12) of any of its points $P$, the only restriction imposed on the solid $S$ by this mechanical constraint is

$$(154) \qquad \bar{\rho}_A \; = \; \bar{\rho}(\alpha_A, \beta_A),$$

$\alpha_A$ and $\beta_A$ representing newly introduced parameters of $S$. Since the right-hand side of (153) represents a given function of $\alpha$ and $\beta$, the relation

$$(155) \qquad \bar{\rho}(\alpha, \beta) \; = \; \xi(\alpha, \beta)\,\bar{\xi}^0 \, + \, \eta(\alpha, \beta)\,\bar{\eta}^0 \; + \; \zeta(\alpha, \beta)\,\bar{\zeta}^0,$$

implied by (12), (153) suggests that the components $\xi(\alpha, \beta)$, $\eta(\alpha, \beta)$, $\zeta(\alpha, \beta)$ are given functions of $\alpha$ and $\beta$, too. On the other hand,

$$(156) \qquad \mathbf{r}_A \; = \; \mathbf{r}_A(t),$$

and consequently (133) by virtue of (34), are completely determined functions of the time $t$ by the conditions of the dynamical problem under consideration. Now the relations (127), (154), (156) imply

(157)
$$\begin{cases} lx_\Omega = x_A(t) - (a_{11}\xi_A(\alpha_A,\beta_A) + a_{12}\eta_A(\alpha_A,\beta_A) + a_{13}\zeta_A(\alpha_A,\beta_A)), \\ y_\Omega = y_A(t) - (a_{21}\xi_A(\alpha_A,\beta_A) + a_{22}\eta_A(\alpha_A,\beta_A) + a_{23}\zeta_A(\alpha_A,\beta_A)), \\ z_\Omega = z_A(t) - (a_{31}\xi_A(\alpha_A,\beta_A) + a_{32}\eta_A(\alpha_A,\beta_A) + a_{33}\zeta_A(\alpha_A,\beta_A)). \end{cases}$$

In such a manner, in view of (55), the canonic parameters (151) of the solid $S$ are found to be completely determined functions of the time and of the quantities

(158)
$$\psi, \phi, \theta, \alpha_A, \beta_A,$$

which in turn are mutually wholly independent.

Summing up, we conclude that in the case of mechanical constraint No 2 the solid $S$ has 5 degrees of freedom and that the quantities (158) may be choosen in the capacity of its independent parameters.

**Constraint No 3**. The equation of the curve line being

(159)
$$\mathbf{r} = \mathbf{r}(s,t)$$

where $r$ in the left-hand side represents the radius-vector (119) of any of its points $P$, the only restriction imposed on the solid $S$ by this mechanical constraint is

(160)
$$\mathbf{r}_A = \mathbf{r}(s_A,t),$$

$s_A$ representing a newly introduced parameter of $S$. Since the right-hand side of (159) represents a given function of $s,t$, the relation

(161)
$$\mathbf{r}(s,t) = x(s,t)\mathbf{i} + y(s,t)\mathbf{j} + z(s,t)\mathbf{k}$$

implied by (119), (159) suggests that the components $x(s,t)$, $y(s,t)$, $z(s,t)$ are given functions of $s$ and $t$, too. Now the relations (127), (160), (34) imply

(162)
$$\begin{cases} x_\Omega = x(s_A,t) - (a_{11}\xi_A + a_{12}\eta_A + a_{13}\zeta_A), \\ y_\Omega = y(s_A,t) - (a_{21}\xi_A + a_{22}\eta_A + a_{23}\zeta_A), \\ z_\Omega = z(s_A,t) - (a_{31}\xi_A + a_{32}\eta_A + a_{33}\zeta_A), \end{cases}$$

where the constants (130) are determined by the condition of the dynamical problem under consideration. In such a manner, in view of (55), the canonic parameters (151) of the solid $S$ are found to be completely determined functions of the time $t$ and of the quantities

$$\psi, \phi, \theta, s_A, \tag{163}$$

which in turn are mutually wholly independent.

Summing up, we conclude that in the case of mechanical constraint No 3 the solid $S$ has 4 degrees of freedom and that the quantities (163) may be choosen in the capacity of its independent parameters.

**Constraint No 4.** The equation of the curve line being

$$\bar{\rho} = \bar{\rho}(\sigma), \tag{164}$$

where $\bar{\rho}$ in the left-hand side represents the radius-vector (12) of any of its points $P$, the only restriction imposed on the solid $S$ by this mechanical constraint is

$$\bar{\rho}_A = \bar{\rho}(\sigma_A), \tag{165}$$

$\sigma_A$ representing a newly introduced parameter of $S$. Since the right-hand side of (164) represents a given function of $\sigma$, the relation

$$\bar{\rho}(\sigma) = \xi(\sigma)\bar{\xi}^0 + \eta(\sigma)\bar{\eta}^0 + \zeta(\sigma)\bar{\zeta}^0, \tag{166}$$

implied by (12), (164) suggests that the components $\xi(\sigma)$, $\eta(\sigma)$, $\zeta(\sigma)$ are given functions of $\sigma$, too. On the other hand, (156) and consequently (133) by virtue of (34) are completely determined functions of the time $t$ by the very conditions of the dynamical problem under consideration. Now the relations (127), (165), (156) imply

$$\begin{cases} x_\Omega = x_A(t) - (a_{11}\xi_A(\sigma_A) + a_{12}\eta_A(\sigma_A) + a_{13}\zeta_A(\sigma_A)), \\ y_\Omega = y_A(t) - (a_{21}\xi_A(\sigma_A) + a_{22}\eta_A(\sigma_A) + a_{23}\zeta_A(\sigma_A)), \\ z_\Omega = z_A(t) - (a_{31}\xi_A(\sigma_A) + a_{32}\eta_A(\sigma_A) + a_{33}\zeta_A(\sigma_A)). \end{cases} \tag{167}$$

In such a manner, in view of (55), the canonic parameters (151) of the solid $S$ are found to be completely determined functions of the time and of the quantities

$$\psi, \phi, \theta, \sigma_A, \tag{168}$$

which in turn are mutually wholly independent.

Summing up, we conclude that in the case of mechanical constraint No 4 the solid $S$ has 4 degrees of freedom and that the quantities (168) may be choosen in the capacity of its independent parameters.

**Constraint No 5.** The only restriction imposed on the solid $S$ by this mechanical constraint is

$$\mathbf{r}_A(t) = \mathbf{r}_\Omega + \bar{\rho}_A , \tag{169}$$

where the left-hand side and consequently (133) by virtue of (34), are completely determined functions of the time $t$ by the conditions of the dynamical problem under consideration, whereas (130) are wholly determined constants by the same conditions. Now (169) and (127) imply

$$\begin{cases} x_\Omega = x_A(t) - (a_{11}\xi_A + a_{12}\eta_A + a_{13}\zeta_A), \\ y_\Omega = y_A(t) - (a_{21}\xi_A + a_{22}\eta_A + a_{23}\zeta_A), \\ z_\Omega = z_A(t) - (a_{31}\xi_A + a_{32}\eta_A + a_{33}\zeta_A). \end{cases} \tag{170}$$

In such a manner, in view of (55), the canonic parameters (151) of the solid $S$ are found to be completely determined functions of the time $t$ and of the parameters

$$\psi, \phi, \theta, \tag{171}$$

of $S$, which in turn are mutually wholly independent.

Summing up, we conclude that in the case of mechanical constraint No 5 the solid $S$ has 3 degrees of freedom and that the quantities (171) may be choosen in the capacity of its independent parameters.

**Constraint No 6.** The equations of the surface fixed in the solid $S$ and that given in space being (153) and (147) respectively, the only restrictions imposed on $S$ are

$$\mathbf{r}(u_A, v_A, t) = \mathbf{r}_\Omega + \bar{\rho}(\alpha_A, \beta_A) \tag{172}$$

and

$$\left(\frac{\partial \mathbf{r}}{\partial u_A} \times \frac{\partial \mathbf{r}}{\partial v_A}\right) \times \left(\frac{\partial \bar{\rho}}{\partial \alpha_A} \times \frac{\partial \bar{\rho}}{\partial \beta_A}\right) = 0, \tag{173}$$

the symbols $u_A$, $v_A$, and $\alpha_A$, $\beta_A$ having the same meaning as in (148) and (154) respectively, equation (173) expressing the tangentiality of the surfaces (153) and (147) at $A$. Now (172), (12), (119), (126) imply

(174)
$$\begin{cases} lx_\Omega = x(u_A, v_A, t) - (a_{11}\xi_A(\alpha_A, \beta_A) + a_{12}\eta_A(\alpha_A, \beta_A) + a_{13}\zeta_A(\alpha_A, \beta_A)) \\ y_\Omega = y(u_A, v_A, t) - (a_{21}\xi_A(\alpha_A, \beta_A) + a_{22}\eta_A(\alpha_A, \beta_A) + a_{23}\zeta_A(\alpha_A, \beta_A)) \\ z_\Omega = z(u_A, v_A, t) - (a_{31}\xi_A(\alpha_A, \beta_A) + a_{32}\eta_A(\alpha_A, \beta_A) + a_{33}\zeta_A(\alpha_A, \beta_A)) \end{cases}$$

In such a manner, in view of (55), the canonic parameters (151) of the solid $S$ are found to be completely determined functions of the time $t$ and of the quantities

$$\psi, \phi, \theta, u_A, v_A, \alpha_A, \beta_A . \tag{175}$$

On the other hand, these quantities are not quite independent mutually by virtue of equation (173) which is equivalent to two independent scalar relations.

Indeed, (173) is a vector equation of the kind

$$\mathbf{a} \times \mathbf{b} = 0 \qquad (\mathbf{a} \neq 0,\ \mathbf{b} \neq 0) \tag{176}$$

by virtue of (143) and (145). On the other hand, the left-hand side of (176) is not quite independent by virtue of the identities

$$\mathbf{a} \times \mathbf{b} \cdot \mathbf{a} = 0, \quad \mathbf{a} \times \mathbf{b} \cdot \mathbf{b} = 0. \tag{177}$$

The relations (177) are, however, not independent in view of (176), In such a manner, only one of the identities (177), say the first one, is authoritative; owing to this, the vector equation (176) is equivalent with 2, rather than with 3, scalar equations.

This peculiarity of equation (173) once grasped, let us consider this condition somewhat closer. First of all, let us note that it may be written in the form

$$\left(\frac{\partial \mathbf{r}}{\partial u_A} \cdot \frac{\partial \bar{\rho}}{\partial \alpha_A} \times \frac{\partial \bar{\rho}}{\partial \beta_A}\right) \frac{\partial \mathbf{r}}{\partial v_A} = \left(\frac{\partial \mathbf{r}}{\partial v_A} \cdot \frac{\partial \bar{\rho}}{\partial \alpha_A} \times \frac{\partial \bar{\rho}}{\partial \beta_A}\right) \frac{\partial \mathbf{r}}{\partial u_A} . \tag{178}$$

On the other hand, the relations

$$\begin{aligned} \frac{\partial \bar{\rho}}{\partial \alpha_A} &= \frac{\partial \xi}{\partial \alpha_A} \bar{\xi}^0 + \frac{\partial \eta}{\partial \alpha_A} \bar{\eta}^0 + \frac{\partial \zeta}{\partial \alpha_A} \bar{\zeta}^0, \\ \frac{\partial \bar{\rho}}{\partial \beta_A} &= \frac{\partial \xi}{\partial \beta_A} \bar{\xi}^0 + \frac{\partial \eta}{\partial \beta_A} \bar{\eta}^0 + \frac{\partial \zeta}{\partial \beta_A} \bar{\zeta}^0 \end{aligned} \tag{179}$$

imply

$$\frac{\partial \bar{\rho}}{\partial \alpha_A} \times \frac{\partial \bar{\rho}}{\partial \beta_A} = \begin{vmatrix} ccc\bar{\xi}^0 & \bar{\eta}^0 & \bar{\zeta}^0 \\ \frac{\partial \xi}{\partial \alpha_A} & \frac{\partial \eta}{\partial \alpha_A} & \frac{\partial \zeta}{\partial \alpha_A} \\ \frac{\partial \xi}{\partial \beta_A} & \frac{\partial \eta}{\partial \beta_A} & \frac{\partial \zeta}{\partial \beta_A} \end{vmatrix} . \tag{180}$$

At last, the first of the relations

$$\frac{\partial \mathbf{r}}{\partial u_A} = \frac{\partial x}{\partial u_A}\mathbf{i} + \frac{\partial y}{\partial u_A}\mathbf{j} + \frac{\partial z}{\partial u_A}\mathbf{k}, \quad \frac{\partial r}{\partial v_A} = \frac{\partial x}{\partial v_A}\mathbf{i} + \frac{\partial y}{\partial v_A}\mathbf{j} + \frac{\partial z}{\partial v_A}\mathbf{k}, \tag{181}$$

along with (43), implies

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial u_A} = {} & (a_{11}\frac{\partial x}{\partial u_A} + a_{21}\frac{\partial y}{\partial u_A} + a_{31}\frac{\partial z}{\partial u_A})\bar{\xi}^0 \\ & + (a_{12}\frac{\partial x}{\partial u_A} + a_{22}\frac{\partial y}{\partial u_A} + a_{32}\frac{\partial z}{\partial u_A})\bar{\eta}^0 \\ & + (a_{13}\frac{\partial x}{\partial u_A} + a_{23}\frac{\partial y}{\partial u_A} + a_{33}\frac{\partial z}{\partial u_A})\bar{\zeta}^0 \end{aligned} \tag{182}$$

and (178), (180), (182) (with a similar expression for $\frac{\partial \bar{r}}{\partial v_A}$ ) imply

(183)

$$\begin{aligned} & \{(a_{11}\frac{\partial x}{\partial u_A} + a_{21}\frac{\partial y}{\partial u_A} + a_{31}\frac{\partial z}{\partial u_A})(\frac{\partial \eta}{\partial \alpha_A}\frac{\partial \zeta}{\partial \beta_A} - \frac{\partial \zeta}{\partial \alpha_A}\frac{\partial \eta}{\partial \beta_A}) \\ & + (a_{12}\frac{\partial x}{\partial u_A} + a_{22}\frac{\partial y}{\partial u_A} + a_{32}\frac{\partial z}{\partial u_A})(\frac{\partial \zeta}{\partial \alpha_A}\frac{\partial \xi}{\partial \beta_A} - \frac{\partial \xi}{\partial \alpha_A}\frac{\partial \zeta}{\partial \beta_A}) \\ & + (a_{13}\frac{\partial x}{\partial u_A} + a_{23}\frac{\partial y}{\partial u_A} + a_{33}\frac{\partial z}{\partial u_A})(\frac{\partial \xi}{\partial \alpha_A}\frac{\partial \eta}{\partial \beta_A} - \frac{\partial \eta}{\partial \alpha_A}\frac{\partial \xi}{\partial \beta_A})\}\frac{\partial \bar{\mathbf{r}}}{\partial v_A} \\ = & \{(a_{11}\frac{\partial x}{\partial v_A} + a_{21}\frac{\partial y}{\partial v_A} + a_{31}\frac{\partial z}{\partial v_A})(\frac{\partial \eta}{\partial \alpha_A}\frac{\partial \zeta}{\partial \beta_A} - \frac{\partial \zeta}{\partial \alpha_A}\frac{\partial \eta}{\partial \beta_A}) \\ & + (a_{12}\frac{\partial x}{\partial v_A} + a_{22}\frac{\partial y}{\partial v_A} + a_{32}\frac{\partial z}{\partial v_A})(\frac{\partial \zeta}{\partial \alpha_A}\frac{\partial \xi}{\partial \beta_A} - \frac{\partial \xi}{\partial \alpha_A}\frac{\partial \zeta}{\partial \beta_A}) \\ & + (a_{13}\frac{\partial x}{\partial v_A} + a_{23}\frac{\partial y}{\partial v_A} + a_{33}\frac{\partial z}{\partial v_A})(\frac{\partial \xi}{\partial \alpha_A}\frac{\partial \eta}{\partial \beta_A} - \frac{\partial \eta}{\partial \alpha_A}\frac{\partial \xi}{\partial \beta_A})\}\frac{\partial \mathbf{r}}{\partial u_A} . \end{aligned}$$

In the light of all aforesaid it is clear that the vector equation (183) is equivalent to 2 scalar equations between the quantities (175) and the time $t$. In such a manner, only 5 of (175), say

$$\psi, \phi, \theta, u_A, \alpha_A \tag{184}$$

are mutually independent, while the other two, say $v_A$ and $\beta_A$, are completely determined functions of (184) and $t$.

Summing up, we conclude that in the case of mechanical constraint No 6 the solid $S$ has 5 degrees of freedom and that the quantities (184) may be choosen in the capacity of its independent parameters.

**Constraint No 7.** The equations of the surface and of the curve line being (153) and (159) respectively, the only restrictions imposed on $S$ are

$$\mathbf{r}(s_A,t) = \mathbf{r}_\Omega + \bar{\rho}(\alpha_A,\beta_A) \tag{185}$$

and

$$\frac{\partial \mathbf{r}}{\partial s_A} \cdot \frac{\partial \bar{\rho}}{\partial \alpha_A} \times \frac{\partial \bar{\rho}}{\partial \beta_A} = 0, \tag{186}$$

the equation (186) expressing the tangentiality of the surface (153) with the curve line (159) at $A$. Now (185), (12), (119), (129) imply

$$\begin{cases} x_\Omega = x(s_A,t) - (a_{11}\xi(\alpha_A,\beta_A) + a_{12}\eta(\alpha_A,\beta_A) + a_{13}\zeta(\alpha_A,\beta_A)), \\ y_\Omega = y(s_A,t) - (a_{21}\xi(\alpha_A,\beta_A) + a_{22}\eta(\alpha_A,\beta_A) + a_{23}\zeta(\alpha_A,\beta_A)), \\ z_\Omega = z(s_A,t) - (a_{31}\xi(\alpha_A,\beta_A) + a_{32}\eta(\alpha_A,\beta_A) + a_{33}\zeta(\alpha_A,\beta_A)). \end{cases} \tag{187}$$

In such a manner, in view of (55), the canonic parameters (151) of the solid $S$ are found to be completely determined functions of the time and of the quantities

$$\psi, \phi, \theta, s_A, \alpha_A, \beta_A\,. \tag{188}$$

On the other hand, these quantities are not quite independent mutually by virtue of equation (186). Following a computational process similar to the above one, by means of which equation (183) has been derived, it is immediately seen that equation (186) is equivalent to

$$\begin{aligned} &\{(a_{11}\frac{\partial x}{\partial s_A} + a_{21}\frac{\partial y}{\partial s_A} + a_{31}\frac{\partial z}{\partial s_A})(\frac{\partial \eta}{\partial \alpha_A}\frac{\partial \zeta}{\partial \beta_A} - \frac{\partial \zeta}{\partial \alpha_A}\frac{\partial \eta}{\partial \beta_A}) \\ &+ (a_{12}\frac{\partial x}{\partial s_A} + a_{22}\frac{\partial y}{\partial s_A} + a_{32}\frac{\partial z}{\partial s_A})(\frac{\partial \zeta}{\partial \alpha_A}\frac{\partial \xi}{\partial \beta_A} - \frac{\partial \xi}{\partial \alpha_A}\frac{\partial \zeta}{\partial \beta_A}) \\ &+ (a_{13}\frac{\partial x}{\partial s_A} + a_{23}\frac{\partial y}{\partial s_A} + a_{33}\frac{\partial z}{\partial s_A})(\frac{\partial \xi}{\partial \alpha_A}\frac{\partial \eta}{\partial \beta_A} - \frac{\partial \eta}{\partial \alpha_A}\frac{\partial \xi}{\partial \beta_A})\} = 0. \end{aligned} \tag{189}$$

In the light of all aforesaid it is clear that (189) is a relation between the quantities (188) and the time $t$. In such a manner, only 5 of (188), say

$$\psi, \phi, \theta, s_A, \alpha_A \tag{190}$$

are mutually independent, while the other one, say $\beta_A$, is a completely determined function of (190) and $t$.

Summing up, we conclude that in the case of mechanical constraint No 7 the solid $S$ has 5 degrees of freedom and that the quantities (190), say, may be choosen in the capacity of its independent parameters.

**Constraint No 8.** The equations of the curve line and the surface being (164) and (147) respectively, the only restrictions imposed on $S$ are

$$\mathbf{r}(u_A, v_A, t) = \mathbf{r}_\Omega + \bar{\rho}(\sigma_A) \tag{191}$$

and

$$\frac{\partial \mathbf{r}}{\partial u_A} \times \frac{\partial \mathbf{r}}{\partial v_A} \cdot \frac{d\bar{\rho}}{d\sigma_A} = 0, \tag{192}$$

equations (192) expressing the tangentiality of the curve line (164) with the surface (147). Now (191), (12), (119), (126) imply

$$\left\{\begin{aligned} x_\Omega &= x(u_A, v_A, t) - (a_{11}\xi(\sigma_A) + a_{12}\eta(\sigma_A) + a_{13}\zeta(\sigma_A)), \\ y_\Omega &= y(u_A, v_A, t) - (a_{21}\xi(\sigma_A) + a_{22}\eta(\sigma_A) + a_{23}\zeta(\sigma_A)), \\ z_\Omega &= z(u_A, v_A, t) - (a_{31}\xi(\sigma_A) + a_{32}\eta(\sigma_A) + a_{33}\zeta(\sigma_A)). \end{aligned}\right. \tag{193}$$

In such a manner, in view of (55), the canonic parameters (151) of the solid $S$ are found to be completely determined functions of the time $t$ and of the quantities

$$\psi, \phi, \theta, u_A, v_A, \sigma_A . \tag{194}$$

On the other hand, these quantities are not quite independent mutually by virtue of equation (192). Following a computational process similar to the above one, by means of which equation (189) has been derived, it is immediately seen that equation (192) is equivalent to

$$\begin{aligned} &\left(\frac{\partial y}{\partial u_A}\frac{\partial z}{\partial v_A} - \frac{\partial z}{\partial u_A}\frac{\partial y}{\partial v_A}\right)\left(a_{11}\frac{d\xi}{d\sigma_A} + a_{12}\frac{d\eta}{d\sigma_A} + a_{13}\frac{d\zeta}{d\sigma_A}\right) \\ &+ \left(\frac{\partial z}{\partial u_A}\frac{\partial x}{\partial v_A} - \frac{\partial x}{\partial u_A}\frac{\partial z}{\partial v_A}\right)\left(a_{21}\frac{d\xi}{d\sigma_A} + a_{22}\frac{d\eta}{d\sigma_A} + a_{23}\frac{d\zeta}{d\sigma_A}\right) \\ &+ \left(\frac{\partial x}{\partial y_A}\frac{\partial y}{\partial x_A} - \frac{\partial y}{\partial u_A}\frac{\partial x}{\partial v_A}\right)\left(a_{31}\frac{d\xi}{d\sigma_A} + a_{32}\frac{d\eta}{d\sigma_A} + a_{33}\frac{d\zeta}{d\sigma_A}\right) = 0. \end{aligned} \tag{195}$$

In the light of all aforesaid it is clear that (195) is a relation between the quantities (194) and the time $t$. In such a manner, only 5 of (194), say

$$\psi, \phi, \theta, u_A, \sigma_A \tag{196}$$

are mutually independent, while the other one, say $v_A$, is a completely determined function of (196) and $t$.

Summing up, we conclude that in the case of mechanical constraint No 8 the solid $S$ has 5 degrees of freedom and that the quantities (196), say, may be choosen in the capacity of its independent parameters.

**Constraint No 9.** The equations of the curve line invariably connected with $S$ being (164) and that of the curve line given in space being (159), the only restriction imposed on $S$ is

$$\mathbf{r}(s_A, t) = \mathbf{r}_\Omega + \bar{\rho}(\sigma_A), \tag{197}$$

the prohibition

$$\frac{\partial \mathbf{r}}{\partial s_A} \times \frac{d\bar{\rho}}{d\sigma_A} \neq 0 \tag{198}$$

being a guarantee that the curves intersect at $A$. Now (197), (12), (119), (126) imply

$$\begin{cases} x_\Omega = x(s_A, t) - (a_{11}\xi(\sigma_A) + a_{12}\eta(\sigma_A) + a_{13}\zeta(\sigma_A)), \\ y_\Omega = y(s_A, t) - (a_{21}\xi(\sigma_A) + a_{22}\eta(\sigma_A) + a_{23}\zeta(\sigma_A)), \\ z_\Omega = z(s_A, t) - (a_{31}\xi(\sigma_A) + a_{32}\eta(\sigma_A) + a_{33}\zeta(\sigma_A)). \end{cases} \tag{199}$$

In such a manner, in view of (55), the canonic parameters (151) of the solid $S$ are found to be completely determined functions of the time $t$ and of the quantities

$$\psi, \phi, \theta, s_A, \sigma_A. \tag{200}$$

Since they are mutually independent, we conclude that in the case of mechanical constraint No 9 the solid $S$ has 5 degrees of freedom and that the quantities (200) may be choosen in the capacity of its independent parameters.

**Constraint No 10.** Under the notations of the previous case the restrictions of the solid $S$ are (199) and

$$\frac{\partial r}{\partial s_A} \times \frac{d\bar{\rho}}{d\sigma_A} = 0, \tag{201}$$

instead of (198), the equation (201) expressing the tangentiality of the curve lines. Similar considerations of those apropos of (176) imply that vector equation (201) is equivalent to 2 independent scalar equations. Now

(202)

$$\frac{d\bar{\rho}}{d\sigma_A} = (a_{11}\frac{d\xi}{d\sigma_A} + a_{21}\frac{d\eta}{d\sigma_A} + a_{31}\frac{d\zeta}{d\sigma_A})\mathbf{i}$$
$$+ (a_{12}\frac{d\xi}{d\sigma_A} + a_{22}\frac{d\eta}{d\sigma_A} + a_{32}\frac{d\zeta}{d\sigma_A})\mathbf{j} + (a_{13}\frac{d\xi}{d\sigma_A} + a_{23}\frac{d\eta}{d\sigma_A} + a_{33}\frac{d\zeta}{d\sigma_A})\mathbf{k}$$

and

$$\frac{\partial \mathbf{r}}{\partial s_A} = \frac{\partial x}{\partial s_A}\mathbf{i} + \frac{\partial y}{\partial s_A}\mathbf{j} + \frac{\partial z}{\partial s_A}\mathbf{k} \tag{203}$$

imply that (201) is equivalent to

$$\begin{aligned} &\frac{\partial x}{\partial s_A}\left(a_{12}\frac{d\xi}{d\sigma_A} + a_{22}\frac{d\eta}{d\sigma_A} + a_{32}\frac{d\zeta}{d\sigma_A}\right) \\ &- \frac{\partial y}{\partial s_A}\left(a_{11}\frac{d\xi}{d\sigma_A} + a_{21}\frac{d\eta}{d\sigma_A} + a_{31}\frac{d\zeta}{d\sigma_A}\right) = 0 \end{aligned} \tag{204}$$

and

$$\begin{aligned} &\frac{\partial y}{\partial s_A}\left(a_{13}\frac{d\xi}{d\sigma_A} + a_{23}\frac{d\eta}{d\sigma_A} + a_{33}\frac{d\zeta}{d\sigma_A}\right) \\ &- \frac{\partial z}{\partial s_A}\left(a_{12}\frac{d\xi}{d\sigma_A} + a_{22}\frac{d\eta}{d\sigma_A} + a_{32}\frac{d\zeta}{d\sigma_A}\right) = 0, \end{aligned} \tag{205}$$

say. In the light of all aforesaid it is clear that (204), (205) are independent relations between the quantities (200) and the time $t$. In such a manner, only 3 of them, say (171), are mutually independent, while the remaining two, say $s_A$ and $\sigma_A$, are functions of (171) and $t$.

Summing up, we conclude that in the case of mechanical constraint No 10 the solid $S$ has 3 degrees of freedom and that the quantities (171), say, may be choosen in the capacity of its independent parameters.

In the light of the above analysis it is immediately seen how in any of the particular cases Nos 1–10 the results obtained concerning the independent parameters (152), (158), (163), (168), (171), (184), (190), (196), (200) and (171) respectively may be applied to the equations (102), etc. It is essential that in any set of independent parameters just now listed Euler's angles $\psi, \phi, \theta$ may be included (*exceptis excipiendis*, of course); in other words, the left-hand sides of the equations (102) etcetera are, if one can put it like this, stable with respect to $\psi, \phi, \theta$.

The following three concluding remarks in connection with the equations (102) etcetera are a matter of principle.

First, as it has been proved in a previous article [5], the following fundamental identities of Lagrangean formalism

$$\begin{aligned} &(\dot{\mathbf{K}} - \mathbf{F} - \mathbf{R})\frac{\partial \mathbf{r}_G}{\partial q_\lambda} + (\dot{\mathbf{L}}_\Gamma - \mathbf{M}_G - \mathbf{N}_G)\frac{\partial \bar{\omega}}{\partial \dot{q}_\lambda} \\ &= \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\lambda} - \frac{\partial T}{\partial q_\lambda} - Q_\lambda - P_\lambda \quad (\lambda = 1, \ldots, l) \end{aligned} \tag{206}$$

hold, provided $q_\lambda$ $(\lambda = 1, \ldots, l)$ denote the independent parameters of the solid $S$, submitted to holonomic constraints,

$$T = \frac{1}{2} \int v^2 \, dm \tag{207}$$

denotes the kinetic energy of $S$ with respect to $Oxyz$, and by definition

$$Q_\lambda = \sum_{\mu=1}^{m} \mathbf{F}_\mu \frac{\partial \mathbf{r}_\mu}{\partial q_\lambda}, \quad P_\lambda = \sum_{\nu=1}^{n} \mathbf{R}_\nu \frac{\partial \mathbf{r}_{m+\nu}}{\partial q_\lambda}, \quad (\lambda = 1, \ldots, l) \tag{208}$$

$\mathbf{r}_\mu$ and $\mathbf{r}_{m+\nu}$ $(\mu = 1, \ldots, m;\ \nu = 1, \ldots, n)$ denoting points of $S$, incident at the moment of time $t$ with the directrices of the corresponding active forces (20) and passive forces (21) respectively. Now (206) and Euler's dynamical axioms, i. e. the first relation (25) and (90), imply

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\lambda} - \frac{\partial T}{\partial q_\lambda} - Q_\lambda - P_\lambda = 0 \quad (\lambda = 1, \ldots, l). \tag{209}$$

Equations (209) coincide with Lagrange's dynamical equations

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\lambda} - \frac{\partial T}{\partial q_\lambda} - Q_\lambda = 0 \quad (\lambda = 1, \ldots, l). \tag{210}$$

if and only if the mechanical constraints imposed on $S$ are smooth, i. e.

$$P_\lambda = 0 \quad (\lambda = 1, \ldots, l). \tag{211}$$

In the case (211) the mathematical process leading from Euler's dynamical axioms (25), (90) towards Lagrange's dynamical equations (210) via identities (206) represents, as a matter of fact, a *Schablon* for the automatical elimination of the *smooth* reactions (21) from Euler's dynamical equations (32), (33). Now if the constraints are *non-smooth* and the conditions (211) are violated, this process fails to accomplish elimination of reactions since $P_\lambda$ $(\lambda = 1, \ldots, l)$ remain at hand in (209). In such a sense, the elimination of the passive forces in (102) cannot be attained by means of (206).

Second, in all cases of mechanical constraints Nos 1–4 and 6–9 the number of the parameters of the solid $S$ exceeds the number 3 of the "pure" differential equations (102) etcetera available for their determination by 1 (Nos 3 and 4) or by 2 (Nos 1, 2 and 6–9) units. If one puts

$$u_A = u_{A\psi} \dot{\psi} + u_{A\phi} \dot{\phi} + u_{A\theta} \dot{\theta} + u \tag{212}$$

in the cases (152), (184), (196); or

(213) $$v_A = v_{A\psi}\dot{\psi} + v_{A\phi}\dot{\phi} + v_{A\theta}\dot{\theta} + v$$

in the case (152); or

(214) $$\alpha_A = \alpha_{A\psi}\dot{\psi} + \alpha_{A\phi}\dot{\phi} + \alpha_{A\theta}\dot{\theta} + \alpha$$

in the cases (158), (184), (190); or

(215) $$\beta_A = \beta_{A\psi}\dot{\psi} + \beta_{A\phi}\dot{\phi} + beta_{A\theta}\dot{\theta} + \beta$$

in the case (158); or

(216) $$s_A = s_{A\psi}\dot{\psi} + s_{A\phi}\dot{\phi} + s_{A\theta}\dot{\theta} + s$$

in the cases (163), (190), (200); or

(217) $$\sigma_A = \sigma_{A\psi}\dot{\psi} + \sigma_{A\phi}\dot{\phi} + \sigma_{A\theta}\dot{\theta} + \sigma$$

in the cases (168), (196), (200) — all coeficients and free terms in (212)–(217) being twofold differentiable functions of $\psi, \phi, \theta$ — then after replacing (212)–(217) in the equations (102) etcetera one obtains an irreproachable system of 3 differential equations of second order with respect to the three unknown functions $\psi, \phi, \theta$ of the time $t$. This is an estimate of the *degree of arbitrariness* the mechanical constraints Nos 1–10 introduce in a particular dynamical problem, without inserting the reactons of the constraints in such an assessment.

Last, but not least, let it be noted that the put-up affairs (212)– (217) enable one to disclose once more the intenability of some modern traditional "definitions" of the notion of *non-holonomic constraints* imposed on rigid bodies.

We shall confine us to quote only two of the numerous literary sources typical in this respect, both published relatively recently at that.

In the monograph [8] on non-holonomic dynamics one reads:

" **Представим себе, что мы мысленно или фактически сняли с системы ряд имеющихся у нее связей и что после этого ее геометрическое положение определяется $n$ величинами $q_1, q_2, \ldots, q_n$, называемыми обобщенными координатами. Тогда произвольному изменению этих обобщеных координат во времени соответствует некоторое движение освобожденной системы. Если теперь вновь наложить на систему снятые связи, то уже не всяким изменениям обобщенных координат $q_1, q_2, \ldots, q_n$ будут соответствОвать некоторые движения системы. Изменения обобщенных координат и их значения должны теперь подчиняться ряду условий, нарушение которых означало бы нарушение наложенных связей. Эти условия могут, в частности, выражаться системой неравенств вида**

$$f_\alpha(q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t) \geq 0 \qquad (1.3)$$

$(\alpha = 1, 2, \ldots, m)$ и тогда соответствующие им связи называются односторонными, неудерживающими или освобождающими, или уравнения вида

$$f_\alpha(q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t) = 0 \tag{1.4}$$

$(\alpha = 1, 2, \ldots, m)$ и тогда соответствующие им связи называются двусторонными, удерживающими или неосвобождающими. Удерживающие связи в свою очередь делят на геометрические и кинематические, зависящие и независящие от времени, соответственно тому, входит или не входит явно время в их уравнения. Связи называются геометрическими, если они выражаются уравнениями вида

$$f_\alpha(q_1, q_2, \ldots, q_n; t) = 0 \tag{1.5}$$

$(\alpha = 1, 2, \ldots, m)$ и кинематическими, если выражающие их уравнения имеют вид

$$f_\alpha(\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; q_1, q_2, \ldots, q_n; t) = 0 \tag{1.6}$$

$(\alpha = 1, 2, \ldots, m)$ (p. 11)."

We shall not comment the logical level of these "definitions" which call to mind an observation of Truesdell in his *Essays* [3]:

"The busy modern calls for culture in predigested quintessence pills, packaged in abridged paperbacks, explained by folksy prefaces in pellet-paragraphs of sugared baby-talk, lullabies to smugness" (p. 3).

In spite of all their flaws, they top by a head the descriptions one can find in the treatise [7] pretending to expose analytical dynamics "as it now stands" (p. VII). Indeed, the "definitions" of [6] are leastway applicable to rigid bodies, whereas Pars' descriptions are dedicated to mass-point systems, yielding tribute to the superstitious belief that "it was natural to conceive of a rigid body as an aggregate of particles. The idea of a rigid body in the classical dynamics is a collection of particles set in a rigid and imponderable frame. Similarly, we shall think of the general dynamical system as a collection of particles acted on by given forces and controlled by various kinds of constraints" [7, p. 20].

The climax of the mechanical philosophy of constrained systems in the treatise [7] is brilliantly lit up by the following formulation:

"The possible displacements of particles are not arbitrary (except in the *problem of* $\nu$ *bodies*, where we are concerned with a swarm of $\nu$ free particles) but are subject to $L$ equations of constraints

$$\text{(2.2.4)} \qquad \sum_{s=1}^{N} A_{rs}\, dx_s + A_r\, dt = 0,$$

$r = 1, 2, \ldots, L < N$ "

Now the following question quite naturally arises: the relations (212)–(217) satisfy the conditions (1.6) of [6] and, though somewhat forcedly, the equations (2.2.4) of [7] — are then (212)–(217) non-holonomic constraints imposed on the solid $S$ ?

As Juvenalis once observed, *difficile est satiram non scribere.*

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