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On the Convergence of Rational Functions of Best Uniform Approximation with Unbounded Number of the Poles

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Let f be a function, continuous and real valued on the segment Δ, $\Delta \subset (-\infty, \infty)$ and $\{R_n\}$ be the sequence of the rational functions of best uniform approximation to f on Δ of order (n,n). In the present work, the convergence of $\{R_n\}$ in the complex plane is considered for the special cases when f is "an almost entire function", or the poles of $\{R_n\}$ accumulate in the terms of weak convergence of measures to a compact set of zero capacity. As a consequence, sufficient conditions for the holomorphic and the meromorphic continuability of f are given.

1. Introduction

Let Δ be a real segment, $\Delta = [\alpha, \beta]$, $\infty \notin \Delta$, and let f, $f \not\equiv 0$, be a function, continuous and real valued on Δ ($f \in C_R(\Delta)$). In the following, we shall always assume that f is not rational. For each positive integer n we denote by $\Re_n$ the set of the real rational functions of order (n,n), i.e. $R = \{p/q : p, q \in p, q \not\equiv 0\}$. Let R_n, $R_n = R_n(f)$ be the function of best uniform approximation to f on Δ in $\Re_n$; that is $\|f - R_n\|_\Delta = \inf_{r \in \Re_n} \|f - r\|_\Delta$, where $\|...\|_\Delta$ is the uniform (sup) norm on Δ.

It is well known (see [1]), that for each n, $n \in N$, the function R_n always exists and is uniquely determined by the alternation theorem of Chebyshev. The sequence $\{R_n\}$ converges to the function f uniformly (in the sup-norm) on the segment Δ.

Let K be a compact set in the complex plane C. We say, that K is r e g u l a r, if the unbounded component K^c of $\bar{C} - K$ is regular with respect to the Dirichlet problem. Denote by $G_K(z, \infty)$ the Green's function with pole at infinity and by Cap(K) the logarithmic capacity of K. It is known (see [2]) that the compact set K is regular iff Cap(K) is positive. It is given by

$$\mathrm{Cap}(K) = \lim \exp\left(\log |z| - G(z, \infty)\right), \text{ as } z \to \infty$$

In accordance to [2], the compact set e, $e \subset C$, is of zero (outer) capacity (Cap$(e) = 0$), if there is a set of regular compact sets $\{K_n\}$, $K_n \supset K_{n+1} \supset e$, such that Cap$(K_n) \to 0$ as $n \to \infty$. (Compare [3]).

Further, if K is a compact set in C, we define the m_1-m e a s u r e of K as follows (see [3]): $m_1(K) = \inf \sum |U_\nu|$, where the infimum is taken over all the coverings $\{U_\nu\}$ of K by disks; $|U_\nu|$ is the radius of the disk U_ν.

Now let Ω be an open set in C and the functions φ_n, $n \in N$, be continuous in $\bar{C}$ on Ω. We say (see [4]) that the sequence $\{\varphi_n\}$ converges to the function φ in capacity on compact subsets in Ω, if for any positive number ϵ and each compact subset K of Ω, there holds that Cap$(\{z\, z \in K,\, |(\varphi_n - \varphi)(z)| \geq \epsilon\}) \to 0$ as $n \to \infty$. The convergence in m_1-measure is defined in analogous way. From Cartan's inequality Cap$(\cdot) \geq C.m_1(\cdot)$ (see [3]), where C is a positive constant, it derives that the convergence in capacity implies a convergence in m_1-measure (on compact subsets) in Ω.

Further, we say that $\{\varphi_n\}$ converges m_1-a l m o s t u n i f o r m l y on the compact set K, if for each positive ϵ there is a compact subset K_ϵ of K such that $_1m(K - K_\epsilon) < \epsilon$ and $\{\varphi_n\}$ converges uniformly on K_ϵ.

All the detailed explanations concerning the convergence in capacity and in m_1-measure may be found in [4]. We restrict ourselves in reminding of the fact that for the special instance when the functions φ_n are holomorphic in Ω ($\varphi \in A(\Omega)$), then the convergence in m_1-measure yields uniform convergence in Ω (on the compact subsets); thus the limit function φ is holomorphic in Ω, as well.

2. New results

The first result in the present paper provides information about the behavior of $\{R_n\}$ in the complex plane C (on compact sets) in the case when the function f is analytic "almost everywhere" (except a set of zero capacity) in C.

Theorem 1. *Let Δ be a real segment, $\infty \notin \Delta$, and $f_R \in C(\Delta)$, $f \# 0$. Assume further, that $f \in A(C - e)$, where the compact set e is of zero capacity and $e \cap \Delta = \emptyset$.*

Then the sequence $\{R_n\}$ converges to f, as $n \to \infty$, in capacity on compact sets in C.

This theorem is an analog of the theorem of Pommerenke concerning the convergence in capacity of the diagonal sequence $\{\pi_n(f)\}$ of the Padé table corresponding to the function f (see [5]). A special case has been considered in [6].

Further, let μ be a positive measure with a compact support suppμ. We denote by $U^\mu(z)$ the corresponding logarithmic potential:

$$U^\mu(z) = -\int \log|z-t|d\mu(t)$$

As it is known (see [3]), $U^\mu(z)$ is a harmonic function in $C-\text{supp}\mu$, subharmonic in $\bar{C} - \text{supp}\ \mu$ and superharmonic in C; $U^\mu(\infty) = -\infty$ and $U^\mu(z) > -\infty$ at each $z \in C$.

Let $\{\mu_n\}$, $n \in \Theta$, $\Theta \subseteq N$ be a sequence of measures. We say that $\{\mu_n\}$ converges weakly, as $n \in \Theta$, to the measure μ, if for any continuous function ψ with compact support there holds

$$\int \psi d\mu_n \to \int \psi d\mu, \text{ as } n \in \Theta.$$

Let $R_n = P_n/Q_n$, where both polynomials P_n and Q_n do not have common divisor; set

$$P_n(z) = a_n \prod_{k=1}^{m_n}(z - \xi_{n,k}), \quad m_n \leq n,$$

and

$$Q_n(z) = b_n \prod_{k=1}^{l_n}(z - \zeta_{n,k}), \quad l_n \leq n;$$

Let δ_z be Dirac measure at the complex point z. Then the unique measures ν_n and μ_n associated with the polynomials P_n and Q_n, respectively can be defined a follows:

$$\nu_n = (1/n).\sum_{k=1}^{m_n} \delta_{\xi_{n,k}}.\ \text{and}\ \mu_n = (1/n).\sum_{k=1}^{l_n} \delta_{\zeta_{n,k}}.$$

In the present paper we prove

Theorem 2. *Let $f \in C_R(\Delta)$ and suppose there is a compact set e, $e \subset C$, $e \cap \Delta = \emptyset$, such that for each compact set K in $\bar{C} - e$ there holds $\mu_n(K) = o(1/\log n)$, as $n \to \infty$; furthermore, let one of two following condition holds:*

a) $\mathrm{Cap}(e) = 0$ *and the sequence $\{\mu_n\}$ converges, as $n \to \infty$, weakly to a measure μ,* $\mathrm{supp}\mu = e$ *or*

b) for each infinite sequence Λ, $\Lambda \subset N$, there is a subsequence Λ_1, such that $\{\mu_n\}$ converges, as $n \in \Lambda$, weakly to a measure $\mu(\Lambda_1)$, $\mathrm{supp}\mu(\Lambda_1) \subseteq e$ *and there is a point ζ_0, $\zeta_0 \in e$ at which*

$$U^{\mu_n}(\zeta_0) \to \infty, \ \ as \ n \to \infty.$$

Then the sequence $\{R_n\}$ of rational functions defined above converges, as $n \to \infty$, m_1-almost uniformly, on compact sets in $C - e$.

Combining this theorem and the fact that $\{R_n\}$ converges on Δ uniformly to f, we obtain sufficient conditions for the continuability of the given function f into $C - e$, namely:

Corollary 1. *In the conditions of Theorem 1, let K be a compact set in $C - e$ and ϵ an arbitrary number. Then there is a compact subset K_ϵ of K, such that $m_1(K - K_\epsilon) < \epsilon$ and $f \in A(K_\epsilon)$.*

Corollary 2. *In the conditions of Theorem 2 on f, e and μ, assume that $R_n \in A(K)$ for any compact set K in $\bar{C} - e$ and all n starting at a number $n(K)$, $n(K) \in N$.*

Then $f \in A(\bar{C} - e)$.

Corollary 3. *In the conditions of Theorem 2 on f, e and μ, assume that there is a compact set K in $\bar{C} - e$ and an integer $m(K)$ such that*

$$\limsup n.\mu_n(K) \leq m(K), \ \ as \ n \to \infty$$

Then the function f admits continuation as a meromorphic function in K with not more that $m(K)$ poles.

For the special case when the set e consists of a finite number of points Corollary 2 has been proved by K. Lungu (see [7]).

Applying the same considerations as in the Proof of Theorem 2 to the zeros of R_n, one can get information concerning the asymptotical behavior of the sequence $\{R_n\}$, in the terms of the zeros of $\{R_n\}$, as well. We restrict ourselves in formulation an analog to Theorem 3, namely

Theorem 3. *Let $f \in C_R(\Delta)$ and let e be a compact set, $e \subset C$, $e \cup \Delta = \emptyset$, such that for each compact set K in $\bar{C} - e$ there holds $\nu_n(K) = o(1/\log n)$, as $n \to \infty$; furthermore, let one of two following condition holds:*

a) $\text{Cap}(e) = 0$ *and the sequence* $\{\nu_n\}$ *converges, as* $n \to \infty$, *weakly to a measure* ν, *supported on e or*

b) for each infinite sequence Z, $Z \subset N$, there is a subsequence Z_1, such that $\{\nu_n\}$ converges, as $n \in Z_1$, weakly to a measure $\nu(Z_1)$, supported in e and there is a point η, $\nu_0 \in e$ for which

$$U^{\nu_n}(\zeta_0) \to \infty, \text{ as } n \to \infty.$$

Then the sequence $\{R_n\}$ converges m_1-almost uniformly, as $n \to \infty$, on compact sets in $C - e$.

From Theorem 3, it derives

Corollary 4. *In the conditions of Theorem 2 on f, e and ν, assume that for any compact set K in $\bar{C} - (e \cup D)$ there holds*

$$\liminf \left(\min_{z \in K} |R_n(z)| \right) > 0, \text{ as } n \to \infty,$$

Then the function f admits a meromorphic continuation in $C - \Delta$.

Preliminaries:

Lemma 1 (lemma of Hermite-Lagrange). *Let Ω be a domain in C and assume the function $g \in A(\Omega) \cap C(\Omega)$. Suppose the rational function r, $r \in \Re$, $r = p/q$, interpolates g at the points a_i, $a_i \in \Omega$, $i = 1, ..., n+m+1$. Set $\Omega(z) = \prod_{i=1}^{n+m+1}(z - a_i)$.*

Then for each z, $z \in \Omega$ and $q(z) \neq 0$, the following identity holds:

$$(g - r)(z) = (1/2\pi i)) \int_{\partial\Omega} (W(z)q(t)T(t)g(t)dt) \,/\, (W(t)q(z)T(z)(t - z))$$

where T is an arbitrary polynomial of degree not grater than m.

The proof of the lemma of Hermite-Lagrange may be found in [7].

Lemma 2 (see [2]). *Let K be a regular compact set in C and $T_n \downarrow$ be the corresponding Chebyshev polynomials of degree n with zeros on K. Then*

$$\lim \|T_n(z)\|_K^{1/n} = \text{Cap}(K), \text{ as } n \to \infty$$

Lemma 3 (see [2]). *Let Q_n be a monic polynomial of degree exactly n and r a positive number. Then*

$$\mathrm{Cap}\{z, |Q_n(z)| \leq r^n\} = r.$$

Lemma 4 (see [8]). *Let K be a compact set K, $K \subset C$ and $\{\mu_n\}$, $n \in \Lambda$, be a sequence of measures with* $\mathrm{supp}\mu_n \subseteq K$. *Then $\{\mu_n\}$ contains a subsequence $\{\mu_n\}$, $n \in \Lambda_1$, $\Lambda_1 \subset \Lambda$, that converges weakly, as $n \in \Lambda$, to a measure μ,* $\mathrm{supp}\mu \subseteq K$. *The sequence of the corresponding logarithmic potentials $U^\mu n$ converges to U^μ, as $n \in \Lambda_1$, uniformly on compact sets in $C - K$; for any $z \in C$ there holds*

$$\liminf(U^{\mu_n})(z) \geq U^\mu(z), \text{ as } n \in \Lambda_1.$$

Lemma 5. (Bernstein-Walsh lemma, see [7]). *Let K be a regular compact set in C and P_n a polynomial of degree n. Then for each z, $z \in K^c$ the inequality*

$$|P_n(z)| \leq \|P_n\|_K . \exp(nG(z, \infty)) \textit{holds}.$$

P r o o f s of the results:

We say that the integer n, $n \in N$, is normal if the number of the poles of R_n in the extended complex plane $\bar{C}$ is equal to n, i.e. if $\max(m_n, l_n) = n$. Since the function f is not rational, the sequence Λ of the normal integers is infinite. Let $n(k) \in \Lambda$. By the alternation theorem of Chebyshev, the difference $(f - R_{n(k)})$ attains alternatively the value of $\pm\|f - R_{n(k)}\|_\Delta$ at $N(k)+1 == n(k)+n(k+1)+1$ (different) points on Δ (see [1, pp 66]). It is easy to show that for any $n \in \Lambda$, $n(k) < n < n(k+1)$, one has $R_{n(k)}(z) \equiv R_n(z)$. Chebyshev's theorem implies also

(1) $$(R_{n(k+1)} - R_{n(k)})(z) = A_{n(k)} w_{n(k)}(z)/(Q_{n(k)} Q_{n(k+1)})(z)$$

where $w_{n(k)}(z)$, is a monic polynomial of degree exactly $N(k)$, with simple zeros belonging to Δ and $A_{n(k)}$ is given by the formula

(2) $$A_{n(k)} = (P_{n(k+1)} Q_{n(k)} - P_{n(k)} Q_{n(k+1)})(\eta)/w_{n(k)}(\eta)$$

for any complex number η, $\eta \notin \Delta$.

To simplify notations, we shall assume everywhere in our further considerations that $\Lambda = N$. With respect to the arguments discussed above we preserve the generality of the considerations.

Proof of Theorem 1:

In the conditions of the theorem, we may assume that the function f is holomorphic at infinity (outside of a disk of sufficiently big radius) and $f(\infty) = 0$.

Let K be a compact set in C, $K \cap e = \emptyset$ and r be a positive number such that the disk $D_r = \{z, |z| < r\}$ contains the sets K, e and Δ.

For each n, $n \in N$, we decompose the polynomial $Q_n(z) = Q_n^0(z).q_n(z)$, where

(3) $$q_n(z) = \prod |\zeta_{n,i}| > 2r(z/\zeta_{n,i} - 1)$$

and Q_n^0 is monic.

For each set F, $F \supseteq e$, $F \cap (K \cup \Delta) = \emptyset$, we introduce the parameter $C(F)$ as follows:

$$C(F) := (\sup_{t \in \delta, \eta \in F} |t - \eta|)^2 / \{(\text{dist}(F, \Delta)^2 . \text{dist}(F, K))\}.$$

Obviously, for any sequence $\{F_n\}$ of compact sets with $F_n \cap K = \emptyset$, $F_n \supset F_{n+1} \supset e$, there holds $C(F_n) \to C(e)$ continuously as $n \to \infty$. Select now a compact set V, $V \supseteq e$, $V \cap (K \cup \Delta) = \emptyset$ such, that $C_1 = 2C(e) \geq C(V)$.

From here and on we shall denote by C_i, $i = 1, 2, \ldots$ constants, independent on n.

Let δ be an arbitrary positive number δ and E be a regular compact set such that $e \subset E \subset V$ and $0 < \text{Cap}E < \delta/(9C_1)$. For any n, $n \in N$, let $T_n(z)$, $\deg T_n = n$, be the corresponding Chebyshev polynomial with zeros on E. By the alternation theorem of Chebyshev, for each $n \in N$ there are $2n + 1$ points $x_{n,i}$, lying on Δ such that $f(x_{n,i}) = R_n(x_{n,i})$. Setting $W_n(z) = \prod_{i=1}^{2n+1}(z - x_{n,i})$ and applying Lemma 1, with $\Omega = D_r$ we obtain for $z \in K$ (recall that $f(\infty) = 0$.)

Setting

$$X_n(E) := \|W_n\|_E \|T_n\|_E \|q_n\|_E \|Q_n^0\|_E \,|_{\, t \in \partial E} \min\{|W_n(t)|\, |T_n(t)|\, |Q_n^0(t)|\, |q_n(t)|\}$$

we get

$$\begin{aligned} |(f - R_n)(z)| &\leq C_2\, X_n(E) \\ &\leq C_2 . C^n(E).(3r)^n.3^n.\|T_n\|_E / |Q_n^0(z)| \end{aligned}$$

In view of (3) and of Lemma 2, taking into account the choice of the positive constant C_1, we now get

(4) $$|(f - R_n)(z)| \leq C_2 . \delta / |Q_n^0(z)|$$

Note that the last inequality is valid for each z, $z \in K$ with $Q_n^0(z) \neq$ and n large enough.

Let now ϵ be an arbitrary positive number. From the last inequality w get

$$\{z, z \in K, |(f - R_n)|(z) \geq \epsilon\} \subseteq \{z, z \in K, |Q_n^0(z)| \leq C_2.\delta^n/\epsilon\}$$

Hence, by Lemma 3 and by (4)

$$\limsup \mathrm{Cap}\{z, z \in K, |(f - R)|(z) \geq \epsilon\} < \delta, \text{ as } n \to \infty.$$

This proves Theorem 1.

Proof of Theorem 2:

Let K be a compact set in C, $K \cap e = \emptyset$. Fix a compact set F, $K \subset$ l and $F \cap e = \emptyset$.

For every ρ, $\rho > 1$ we introduce the notation Φ_ρ

$$\Phi = \{z, G_\Delta(z, \infty) = \log \rho\}.$$

It the case being considered Φ_ρ is an ellipse with focuses at the points α and β.

Select now a positive number ρ such that

$$D_r = \{z, |z| < r\} \supset \{e \cup F \cup \{\Phi_\rho F, \Phi_\rho \cap F \neq \emptyset\}\}.$$

We normalize the polynomials $Q_n(z) = \prod_{|\zeta_{n,k}| \leq r}(z - \zeta_{n,k}) \cdot \prod_{|\zeta_{n,k}| > r}(z/\zeta_{n,k} - 1)$. Set $q_n^0(z) = \prod_{|\zeta_{n,k}| > r}(z/\zeta_{n,k} - 1)$. By the conditions of the theorem,

$$(5) \qquad |q_n^0(z)|^{1/n} \to 1, \text{ as } n \to \infty$$

uniformly on compact subsets of D_r.

Decompose $Q_n = q_n.q_n^0.Q_n^0$, where all the zeros of q_n are situated on F Set $k_n = \deg q_n$. In the conditions of the theorem, $k_n = o(n/\log n)$. Therefore,

$$(6) \qquad \|q_n\|_M^{1/n} \to 1, \text{ as } n \to \infty$$

for each compact set M in C.

Denote by μ_n^0 the unique measure associated with Q_n^0 and by $U^{\mu_n^0}$ th corresponding logarithmic potential. Obviously, μ_n^0 is the restriction of μ_n o $\bar{D} - F$. It follows from the conditions of the theorem that if one of the sequence $\{\mu_n\}$ or $\{\mu_n^0\}$ converges weakly for some X, $X \subseteq N$, then so does the other an the limit measures are equal.

Set now

$$\varphi_n(z) = (1/n)\log|((R_{n+1} - R_n(z))q_n(z)q_{n+1}(z)|$$

and consider the function

$$\varphi(z) = \limsup \varphi_n(z), \text{ as } n \to \infty.$$

According to (1), the functions φ_n are harmonic in $K-\Delta$, equicontinuous and uniformly bounded above there (on compact subsets). Hence, by Arzela's theorem, $\varphi(z)$ is either

1) a continuous subharmonic function in $K - \Delta$ or

2) $\varphi(z) \equiv -\infty$.

We shall show that Proposition 2 holds. For this purpose it suffices to prove that $\varphi(a) = -\infty$ for an arbitrary point a, $a \in K - \Delta$.

Fix an arbitrary point a, $a \in \partial K$. From (1) and (2), we get

$$\varphi_n(a) \le (1/n)\log|(P_{n+1}Q_n - P_nQ_{n+1})(\zeta)|+$$

$$+(1/n)\log|w_{2n+1}(a)/w_{2n+1}(\zeta)| - (1/n)|\log Q_n^0(a) + \log Q_{n+1}^0(a)|+$$

(7) $$+(-1/n)|\log q_n^0(a) + \log q_{n+1}^0(a)|.$$

where $\zeta \notin \Delta$ is an arbitrary point.

Fix the compact set E such that $e \subset E \subset D_r$ and $E \cap (\Delta \cup F) = \emptyset$. Everywhere in what follows we shall assume that $\zeta \in E$. Obviously

(8) $$\limsup(1/n)\log\|w_{2n+1}(a)/w_{2n+1}(\zeta)\|_{\zeta\in E} \le C_3, \text{ as } n \to \infty$$

and

(9) $$\limsup(-1/n)\{\{\log|Q_n^0(a)| + \log|Q_{n+1}^0(a)|\} \le C_4, \text{ as } n \to \infty.$$

Note that the constants $C_3 = C_3(a,\zeta)$ and $C_4 = C_4(a)$ do not depend on n and are finite numbers.

$$\text{Set} B_n(\zeta) = (1/n)\log|(P_{n+1}Q_n - P_nQ_{n+1})(\zeta)|$$

By (7)

(10) $$\varphi(a) \le B_n(\zeta) + C_5$$

for a suitable constant C_5, $C_5 = C_5(a, E)$.

Let us estimate $B_n(\zeta)$ for n large enough. In view of (2), we get

$$B_n(\zeta) \le (1/n)\log(|P_{n+1}(\zeta)||Q_n(\zeta)| + |Q_{n+1}(\zeta)||P_n)(\zeta)|)$$

From here, using an argument of [9], we have

$$B_n(\zeta) \leq \tag{11}$$

$$\leq (1/n)\log\left(\|P_{n+1}\|_E|Q_n^0(\zeta)|\|q_n^0\|_E\|q_n\|_E + \|P_n\|_E|Q_{n+1}^0(\zeta)|\|q_{n+1}^0\|_E\|q_{n+1}\|_E\right)$$

Applying now Bernstein-Walsh' lemma to $\|P_n\|_E$, we obtain

$$\|P_n\|_E \leq \|P_n\|_\Delta . \exp(nG_\Delta(\zeta,\infty)) \leq \|R_n\|_\Delta\|Q_n\|_\Delta\|\exp(nG_\Delta(\zeta,\infty))\|_E$$

$$\leq C_6\|Q_n\|_\Delta . C_7^n = C_8^n\|Q_n\|_\Delta, \tag{12}$$

where $C_7 = \|\exp G_\Delta(\zeta,\infty)\|_E$. Here we used the fact that $\{R_n\}$ converges uniformly to f, as $n \to \infty$. Similarly,

$$\|P_{n+1}\|_E \leq C_8^{n+1}\|Q_{n+1}\|_\Delta. \tag{13}$$

Further, we have

$$\max(\|Q_{n+1}^0\|_\Delta, \|Q_n^0\|_\Delta) \leq C_9^n \tag{14}$$

Inequalities (12), (13) and (14) are valid for each $n \in N$ starting at some index n_1, $n_1 = n_1(E,a)$.

Combining now (11), (12), (13), (5) and (6), we obtain

$$B_n(\zeta) \leq C_{10} + (1/n)\log 2 +$$

$$+\max\left\{(1/n)\log|Q_{n+1}^0(\zeta)|, (1/n)\log|Q_n^0(\zeta)|\right\}$$

where $C_{10} = C_{10}(a, E)$ is a suitable finite constant.

Applying again an argument of [9], we may write

$$\begin{aligned} B_n(\zeta) &\leq C_{11} + \max\{-U^{\mu_n^0}(\zeta), -U^{\mu_{n+1}^0}(\zeta)\} = \\ &= C_{11} - \min\{U^{\mu_n^0}(\zeta), U^{\mu_{n+1}^0}(\zeta)\} \end{aligned} \tag{15}$$

Note that the last inequality is valid for all $n \geq n_2$ and for any ζ, $\zeta \in E$. Let now $\Lambda = \Lambda(a)$ be an infinite sequence, for which

$$\lim \varphi(a) = \varphi(a), \text{ as } n \in \Lambda.$$

Then, in accordance to (10)

$$\varphi(a) \leq \limsup B_n(\zeta) + C_5, \text{ as } n \in \Lambda^0. \tag{16}$$

for any sequence Λ^0, $\Lambda^0 \subseteq \Lambda$, and any point ζ, $\zeta \in E$.

By Lemma 4 there exists an infinite subsequence Λ_1 of Λ such that $\{\mu_n^0\}$ and $\{\mu_{n+1}^0\}$ converge, as $n \in \Lambda_1$, weakly to the measures μ' and μ'', respectively. In view of the conditions of the theorem, suppμ' and suppμ'' belong both to e.

The further application of Lemma 4 to inequality (15) yields

$$\lim_{n\in\Lambda_1} \sup B_n(\zeta) \leq C_{12} - \min\{U^{\mu'}(\zeta), U^{\mu''}(\zeta)\}.$$

Combining (16) and the last inequality, we get

$$\varphi(a) \leq C_{12} - \min\{U^{\mu'}(\zeta), U^{\mu''}(\zeta)\}. \tag{17}$$

Note that C_{12} is finite constant and $\zeta \in E$ is an arbitrary point.

Now, it easily follows that

$$\varphi(a) = -\infty. \tag{18}$$

and thereby,

$$\varphi \equiv -\infty \text{ on } \partial K. \tag{19}$$

Indeed, in the case a) of the theorem we have $\mu' = \mu'' = \mu$. Since Cap$(e) = 0$, there is at least one point $\zeta_0 \in e$ at which $U^\mu(\zeta_0) = \infty$ and (18) is proved by setting in (17) $\zeta = \zeta_0$. In the case b) we select a sequence ζ tending to the point ζ_0 and pass to the limit in (17); equality (18) now results from the lower semicontinuity of the logarithmic potential at the point ζ_0. Set

$$q_n(z) = \prod_{i=1}^{k_n}(z - \zeta_{n,i})$$

Select a positive number ϵ such that $\epsilon < m_1(K)$ and introdu- ce the set $\Omega(\epsilon)$

$$\Omega(\epsilon) = \cup_n \cup_{i=1}^n \{z, z \in K, |z - \zeta_{n,i}| < \epsilon/(k_n n^2)\}$$

Obviously, $m_1(\Omega(\epsilon)) < \epsilon$ and $m_1(\Omega(\epsilon))$ decreases with ϵ. Also, for each $z \notin \Omega(\epsilon)$ there holds

$$|q_n(z)| \geq (k_n n^2/\epsilon)^k n \tag{20}$$

Let q be a fixed positive number, $q < 1$. It is not hard to see that for all n large enough the inequality

$$\|\varphi\|_{\partial K} \leq \log q$$

holds. Therefore, by the maximum principle for holomorphic functions,

$$\|R_{n+1} - R_n\|_{K-\Omega(\epsilon} \leq C_{13}.q^n / \{|q(z).q_{n+1}(z)|\}$$

for every z, $z \in K - \Omega(\epsilon)$ and for an appropriate positive constant C_{13}.

Applying (20) to the last inequality and taking into account that both k, k_{n+1} satisfy the condition k, $k_{n+1} = o(n/\log n)$, we obtain

$$\|R_{n+1} - R_n\|_{K-\Omega(\epsilon)} \leq q_1^n$$

for a some number q_1, $q < q_1 < 1$.

Therefore, the sequence $\{R_n\}$ converges uniformly on $K - \Omega(\epsilon)$. Recall that $m_1(\Omega(\epsilon)) < \epsilon$.

This proves the theorem. ■

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