Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

New Series Vol. 9, 1995, Fasc. 2-3

Existence and Continuous Dependence of the Solutions of Ordinary Differential Equations with Maximum

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1. Introduction

In the paper differential equations with maximum of the form

$$x(t) = F(t, x(t)), \quad \max_{s\in[g(t),t]} x(s)),$$

are considered, where $x(t) \in R^n, F$ and g are continuous functions. Such equations arise in the problem of automatic regulation of various real system [7], [11]. It is characteristic for them that the deviation of the argument is not previously given but depends on the unknown function x. The necessity of study of this type of equation was pointed out by A.D. Myshkis [10]. Various questions related to ordinary differential equations with maximum are considered in separate papers. For instance, theorems of existence of periodic solutions and some of their properties are studied in [12]; the averaging method is justified for differential equations with maximum in [9].

In the papers [1], [6], [8] (see also [3]) it is assumed that the initial functions are elements of a concrete functional space and the right-hand side of the equation satisfies the Lipschitz condition which ensures uniqueness of the solution. For instance in Magemedov [8] and Angelov and Bainov [1] the phase space is respectively $C([-h, 0], R^1)$ and $C((-\infty, 0, R^n)$ consisting of continuous and bounded functions.

Here an abstract phase space E is used which admits a single approach to the cases of bounded and unbounded delay of the argument. A theorem of existence is proved, which does not require a Lipshitz condition on the right-hand side of the equation considered. Sufficient conditions for continuous dependence of the solutions on the initial function are found.

2. Existence of a solution

Let $\Phi = \{\varphi\,(-\infty, 0] \to R^n\}$, $E \subset \Phi$, and let E be a linear space provided with some seminorm $\|\cdot\|$. For any function x defined in the interval $(-\infty, t] \in R$ set $x_t(s) = x(t+s)$ for $s \leq 0$. For $\tau \geq 0$ set

$$E_\tau = \{\varphi \in \Phi | \varphi \in C[-\tau, 0],\ \varphi_{-\tau} \in E\}.$$

Introduce the following conditions:

H1. If $\tau \geq 0$, $\varphi \in E_\tau$, $t \in [-\tau, 0]$, then $\varphi_t \in E$.

H2. If $\tau \geq 0$, $\varphi \in E_\tau$, then $\varphi(0)| \leq M_0\|\varphi\|$ and

$$\|\varphi\| \leq K(\tau) \max_{s\in[-\tau,0]} |\varphi(s)| + M(\tau)\|\varphi_{-\tau}\|,$$

where M_0 is a constant, K and M are continuous positive functions, and $|\cdot|$ is the Euclidean norm in R^n.

H3. If $\varphi \in E$ then φ is continuous for $s \in I_0$, where

(1) $$I_0 = \{s \leq 0\ |\{\exists c > 0\}\{\forall \varphi \in E)\}\{|\varphi(s)| < c\|\varphi\|\}.$$

Definition 1. A linear space E which satisfies conditions H1–H3 is said to be an admissible space.

Lemma 1. *Let E be a linear space satisfying conditions H1 and H2. Let $\lambda \in I_0$.*

Then there exists a number $K_0 > 0$ such that $|\varphi(s)| \leq K_0\|\varphi\|$ for $\varphi \in E$, $s \in [\lambda, 0]$.

The proof is given in [5] (Lemma 2.4).

Remark 1. From Lemma 1 it follows that if E is an admissible space, then the set I_0 is an interval, $0 \in I_0$, and there exists a positive continuous and monotone decreasing function M^* such that

$$|\varphi(s)| \leq M^*(s)\|\varphi\| \text{ for } s \in I_0, \varphi \in E.$$

A typical example of an admissible space is the linear space $\bar{C}_h^\gamma = \{\varphi \in \Phi \cap C(-h, 0 \mid (\exists d \in R)(\forall s \in (-h, 0](|\varphi(s)|e^{\gamma s} < d)\}$ provided with a seminorm defined by the equality

$$\|\varphi\| = \sup_{s \in (-h, 0]} |\varphi(s)|e^{\gamma s},$$

where $0 \leq \bar{\gamma} < +\infty$, $0 \leq h \leq +\infty$. In this case $I_0 = (-h, 0]$, $M^*(s) = e^{-\gamma s}$, $K(\tau) = 1$ for $s \leq 0 \leq \tau$, $M(\tau) = e^{-\gamma\tau}$ for $0 \leq \tau < h$, and $M(\tau) = 0$ for $\tau \geq h+1$. If E is an admissible space, where $\|\varphi\| = |\varphi(0)|$ for $\varphi \in E$, then the quotient space $E/\|\cdot\|$ is isomorphic to R^n and instead of E we shall write R^n.

Remark 2. Similar spaces have been introduced by Hale [4] Hale, Kato [5], Schumacher [14] under some additional constraints. In the papers quoted an essential role is played by the additional assumption that for $\varphi \in E_\tau$, $t \in [-\tau, 0]$ the mapping $t \to \varphi_t \in E$ is continuous in the topology of E. The last requirement excludes from consideration the spaces $\bar{C}_h^\gamma$ for $0 \leq \gamma < +\infty$, $0 < h \leq +\infty$. In order to illustrate this fact we shall note that a typical example of the space considered in Hale [4] Hale and Kato [5], Schumacher [14] is the subspace

$$C_h^\gamma = \{\varphi \in \bar{C}_h^\gamma \mid \varphi(s)e^{\gamma s} \text{ has a limit as } s \to -h, s > -h\}$$

for $0 \leq \gamma < +\infty$, $0 \leq h \leq +\infty$, and $\bar{C}_h^\gamma \setminus C_h^\gamma \neq \emptyset$ for $h > 0$. Let $0 \leq \gamma < +\infty$. If $\varphi(s) = e^{-\gamma s} \sin s$ for $s \leq 0$, then $\varphi \in \bar{C}_\infty^\gamma \setminus C_\infty^\gamma$. If $h \in (0, +\infty)$, $\psi(s) = 0$ for $s \leq -h$, and $\psi(s) = e^{-\gamma s} \sin \frac{1}{h+s}$ for $s \in (-h, 0]$, then $\psi \in \bar{C}_\infty^\gamma \setminus C_\infty^\gamma$.

For any vector-valued function $x = (x_1, x_2, ..., x_n)$ continuous in the interval $[a, b]$ introduce the notation

$$\max_{s \in [a,b]} x(s) = (\max_{s \in [a,b]} x_1(s), ..., \max_{s \in [a,b]} x_n(s)).$$

Consider the differential equation

$$\dot{x}(t) = F(t, x(t), \max_{s \in [g(t), t]} x(s)), \tag{2}$$

where $\dot{x}$ is a right-hand derivative of the unknown n- dimensional function x, and g and F are given functions defined respectively in the open sets $I \subset R^1$ and $G \subset I \times R^{2n}$. Introduce the conditions:

H4. $g \in C(I)$, $g(t) - t \in I_0$ for $t \in I \subset R$, where I is an open set.

H5. $F \in C(G)$, where G is an open set, $G \subset I \times R^{2n}$.

Lemma 2 [1]. *Let h_1, $h_2 \in C(I)$ and $h_1(t) \leq h_2(t)$ for $t \in I$. Let $t_1 \in I$, $[h_1(t_1), h_2(t_1)] \subset U$, $x \in C(U)$, where $U \subset R^1$ is an open set.*

Then the function $\max\limits_{s\in[h_1(t),h_2(t)]} x(s)$ is continuous for $t = t_1$.

Lemma 3. *Let E be an admissible space and let condition H4 hold. Let j be a functional defined for $(t,\psi) \in I \times E$ by the equality*

$$j(t,\psi) = (t, \psi(0) \max_{s\in[g(t)-t,0]} \psi(s)). \tag{3}$$

Then $j \in C(I \times E)$ and $j(t,.)$ is locally Lipschitz continuous.

Proof. For φ, $\bar{\varphi} \in E$, c, $d \in I_0$, $c \leq d$, $k = 1, 2, ..., n$ the following inequalities are valid

$$\max_{s\in[c,d]} \varphi_k(s) - \max_{s\in[c,d]} \bar{\varphi}_k(s) \leq \max_{s\in[c,d]} (\varphi_k - \bar{\varphi}_k)(s)$$

$$\leq \max_{s\in[c,d]} M^*(s)\|\varphi - \bar{\varphi}\| = M^*(c)\|\varphi - \bar{\varphi}\| \tag{4}$$

where M^* is the function of Remark 1. Here $j(t,.)$ is locally Lipschitz continuous. It remains to show that the last n components of j are continuous functionals. But this follows from conditions H3, H4, from relations (4) and Lemma 2 in view of the equality

$$\max_{s\in[g(t)-t,0]} \varphi_k(s) - \max_{s\in[g(\bar{t})-\bar{t},0]} \bar{\varphi}_k(s)$$
$$= \max_{s\in[g(t)-t,0]} \varphi_k(s) - \max_{s\in[g(t)-t,0]} \bar{\varphi}_k(s)$$
$$+ \max_{s\in[g(t)-t,0]} \bar{\varphi}_k(s) - \max_{s\in[g(\bar{t})-\bar{t},0]} \bar{\varphi}_k(s)$$

Let E be an admissible space and let conditions H4 and H5 hold. From Lemma 3 it follows that the set $\Omega = j^{-1}(G) \subset I \times E$ is open. For $(t,\psi) \in \Omega$ set

$$f(t,\psi) = F(t, \psi(t), \max_{s\in[g(t)-t,0]} \psi(s)). \tag{5}$$

Thus equality (2) takes the form

$$\dot{x}(t) = f(t, x_t). \tag{6}$$

Remark 3. Under the conditions H1-H5 the functional f in the right-hand side of equality (6) is continuous by Lemma 3.

For $(\sigma, \varphi) \in \Omega$ the initial value problem (2), $x_\sigma = \varphi$ is equivalent to the initial value problem (6), $x_\sigma = \varphi$. For the initial value problem (6), $x_\sigma = \varphi$ we shall prove an analogue of Peano's theorem of existence of a solution.

Remark 4. We shall illustrate the significance of condition H4 or more precisely of the inclusion $g(t) - t \in I_0$ for $t \in I$ which enters it. Let E be an admissible space consisting of continuous functions. Let the functions g and F be continuous respectively in I and $G \subset I \times R^n$. Suppose that $(\tau, 0, ..., 0 \in G, g(\tau)\tau$, but $g(\tau) - \tau \notin I_0$. Then the functional f defined by equality (5) is discontinuous at the point $(\tau, 0) \in I \times E$. Moreover, if $G \subset I \times \Gamma$, where the set $\Gamma \subset R^{2n}$ is bounded, then in any neighborhood V of the point $0 \in E$ there exists $\varphi \in V$ such that the right-hand side of equality (2) is not defined for $t = \tau$, $x_\tau = \varphi$. This fact follows from the relation $g(t) - \tau \notin I_0$ and from the homogeneity of the seminorm $\|\cdot\|$ in E.

Definition 2. The function $x\ (-\infty, a) \to R^n$ is said to be a solution of the initial value problem (2), $x_\sigma = \varphi$, $(\sigma, \varphi) \in I \times E$ if $\sigma < a$, $x \in C([\sigma, a))$, $x_\sigma = \varphi$ and for $t \in [\sigma, a)$ satisfies the relations $j(t, x_t) \in G$ and (2).

Lemma 4. *Let E be an admissible space and condition H4 hold. Let $\sigma < a$, $[\sigma, a) \subset I$, and let the function $x\ (-\infty, a) \to R^n$ be continuous in the interval $[\sigma, a)$ and $x_\sigma = \varphi \in E$.*

Then the vector $j(t, x_t)$ is continuous function of $t \in [\sigma, a)$

Proof. Since $x_\sigma \in E$, $x \in C([\sigma, a))$ and E is an admissible space, then $x_t \in E$ for $t \in [\sigma, a)$. From Lemma 1 it follows that two cases are possible: $I_0 = (-\tau, 0], \tau > 0$ or $I_0 = [-\tau, 0]$, $\tau \in (0, +\infty)$. If $I_0 = (-\tau, 0]$, then $x \in C((\sigma - \tau, a))$ and the assertion of the Lemma follows from condition H4 and from Lemma 2. In the case $I_0 = [-\tau, 0]$, $\tau \in (0, +\infty)$ we set $\bar{x}(t) = x(t)$ for $t \in [\sigma - \tau, a)$ and $\bar{x}(t) = x(\sigma - \tau)$ for $t < \sigma - \tau$. Then $\bar{x} \in C(-\infty, a)$ and from condition H4 and from Lemma 2 it follows that the vector $j(t, \bar{x}_t)$ is a continuous function of $t \in [\sigma, a)$ and $j(t, \bar{x}_t) = j(t, x_t)$ for $t \in [\sigma, a)$.

Lemma 5. *Let E be an admissible space and conditions H4 and H5 hold. Let the function $x\ (-\infty, a) \to R^n$ be a solution of the initial value problem (2), $x_\sigma = \varphi$, $(\sigma, \varphi) \in I \times E$.*

Then x has a continuous derivative for $t \in (\sigma, a)$.

Proof. From relation (2) and from Lemma 4 it follows that for $t \in [\sigma, a)$ the function x has a right-hand derivative $\dot{x}(t)$ which is continuous. From Rouche, Habets and Laloy ([13], Appendix 1, Corollary 2.4) it follows that for $t \in (\sigma, a)$ the function x has a continuous derivative $d/dtx(t)$.

Let $\alpha > 0$. Consider the spaces

$$C_0[0,\alpha] = \{y \in C([0,\alpha], R^n) \mid y(0) = 0\},$$

$$\bar{C}_0[0,\alpha] = \{y \in C((-\infty,\alpha], R^n) \mid y(s) = 0, \text{ for } \quad s \leq 0\}$$

provided with the standard norm

$$\|y\|_0 = \max_{s\in[0,\alpha]} |y(s)|.$$

It is obvious that these are Banach spaces and the embedding $i C_0[0,\alpha] \to \bar{C}_0[0,\alpha]$ is an isomorphism and homeomorphism. For $\beta > 0$ set

$$A(\alpha,\beta) = \{y \in C_0[0,\alpha] \,|\|y\|_0 \leq \beta\},$$

$$\bar{A}(\alpha,\beta) = i(A(\alpha,\beta)).$$

Moreover, with each $\varphi \in E$ we associate the function $\hat{\varphi}\ R^1 \to R^n$ defined by the equalities

(7) $$\hat{\varphi}_0 = \varphi,\ \hat{\varphi}(t) = \varphi(0) \text{ for } t \geq 0.$$

Lemma 6. *Let E be an admissible space and conditions H4, H5 hold. Let $(\sigma,\varphi) \in \Omega = j^{-1}(G)$ and $a > \sigma$:*

Then the function $x\ (-\infty, a)$ is a solution of the initial value problem (6), $x_\sigma = \varphi$ if and only if the function y defined for $t < a - \sigma$ by the equality

(8) $$y(t) = x(\sigma + t) - \hat{\varphi}(t)$$

satisfies the equalities

(9) $$y_0 = 0,$$

$$y(t) = \int_0^t F(j(\sigma + s, \hat{\varphi}_s + y_s))ds$$

for $t \in [0, a - \sigma)$.

The proof follows from the relations $\hat{\varphi}_s + y_s = x_{\sigma+s}, j(\sigma + s, x_{\sigma+s}) \in G$ for $s \in [0, a - \sigma)$, and from Lemmas 4 and 5.

Lemma 7. *Let E be an admissible space and conditions H4, H hold. Let $U \subset \Omega = j^{-1}(G)$, and let α, β, d be positive numbers such that the set*

(10) $$V = \{(\sigma + t, \hat{\varphi}_t + y_t) \mid (\sigma,\varphi) \in U, t \in [0,\alpha], y \in \bar{A}(\alpha,\beta)\}$$

satisfies the relations

$$V \subset \Omega, |F(j(t,\psi))| \le d \text{ for } (t,\psi) \in V \tag{11}$$

Then the formulae

$$TF(\sigma,\varphi,y) = 0 \text{ for } t \in (-\infty, 0], \tag{12}$$

$$TF(\sigma,\varphi,y)(t) = \int_0^t F(j(\sigma+s, \hat{\varphi}_s + y_s))ds, t \in [0,\alpha], \tag{13}$$

define an operation $TFU \times \bar{A}(\alpha,\beta) \to \bar{C}_0[0,\alpha]$ *which is continuous and, moreover,* $TF(U \times \bar{A}(\alpha,\beta)) \subset \bar{S}$, *where the set*

$$\bar{S} = \{y \in C_0[0,\alpha] \mid |y(t) - y(\tau)| \le d|t-\tau|, \quad |y(t)| \le d\alpha, \ t,\alpha \in [0,\alpha]\} \tag{14}$$

is compact in $\bar{C}_0[0,\alpha]$.

Proof. From relations (10), (11) it follows that equalities (12), (13) define an operator $TFU \times \bar{A}(\alpha,\beta) \to \bar{S}$ where $\bar{S}$ is a subset of $\bar{C}_0[0,\alpha]$ define by equality (14). From (14) it follows that $i^{-1}(\bar{S})$ is compact in $C_0[0,\alpha]$ by the Ascoli-Arzella theorem. Since the embedding $i\ C_0[0,\alpha] \to C_0[0,\alpha]$ is continuous, then $\bar{S}$ is a compact in $\bar{C}_0[0,\alpha]$. Let $\{(\sigma^m,\varphi^m,y^m)\}_{m=1}^{\infty} \subset U \times \bar{A}(\alpha,\beta)$ and $(\sigma^m,\varphi^m,y^m) \to (\sigma^0,\varphi^0,y^0) \in U \times \bar{A}(\alpha,\beta)$ as $m \to +\infty$. Since $\{TF(\sigma^m,\varphi^m,y^m)\}_{m=1}^{\infty} \subset \bar{S}$ and $\bar{S}$ is a compact, then there exists an accumulation point $\gamma \in \bar{S}$ and a subsequence $\{(\sigma^{m_\nu},\varphi^{m_\nu},y^{m_\nu},)\}_{\nu=1}^{\infty}$ such that for $t \in [0,\alpha]$

$$\gamma(t) = \lim_{\nu\to+\infty} \int_0^t F(j(\sigma^{m_\nu}+s, \hat{\varphi}_s^{m_\nu} + y_s^{m_\nu}))ds. \tag{15}$$

From conditions H2 and from relations (7) it follows that for $s \in [0,\alpha]$, $m \in N$ the following inequalities are valid

$$\begin{aligned} &\|\hat{\varphi}_s^m - \hat{\varphi}_s^0 + y_s^m - y_s^m\| \\ &\le K(s)|\varphi^m(0) - \varphi^0(0)| + M(s)\|\varphi^m - \varphi^0\| + K(s)\|y^m - y^0\|_0 \\ &\le (K(s)M_0 + M(s))\|\varphi^m - \varphi^0\| + K(s)\|y^m - y^0\|_0 \end{aligned}$$

Then from relations (10), (11), from conditions H4, H5 and from Lemma 3 it follows that

$$\lim_{\nu\to+\infty} F(j(\sigma^{m_{nu}}+s, \hat{\varphi}^{m_\nu} + y_s^{m_\nu})) = F(j(\sigma^0+s, \hat{\varphi}_s^0 + y_s^o)) \tag{16}$$

for $s \in [0, \alpha]$. From relations (10), (11), (15), (16) and from Lebesgue's theorem it follows that

$$\gamma(t) = \int_0^t F(j(\sigma^0 + s, \hat{\varphi}_s^0 + y_s^0))ds = TF(\sigma, \varphi^0, y^0)(t)$$

for $t \in [0, \alpha]$. Since the accumulation point γ is unique for the sequence $TF(\sigma^m, \varphi^m, \eta^m)$, then the operator TF is continuous.

Theorem 1. *Let E be an admissible space and conditions H4, H5 hold. Let $(\sigma, \varphi) \in I \times E$ and $j(\sigma, \varphi) \in G$.*

Then the initial value problem (2), $x_\sigma = \varphi$ has a solution.

Proof. Since the set G is open, $j(\sigma, \varphi \in G$ and the function F is continuous, then there exists a neighborhood G_0 of the point $j(\sigma, \varphi)$ and a positive constant d such that $|F(t, x, y)| \leq d$ for $(t, x, y) \in G_0$. From equalities (7) and from Lemma 4 it follows that the vector $j(\sigma + t, \hat{\varphi}_t)$ is a continuous function of t for $t > 0$, $\sigma + t \in I$. By Lemma 3 the function $j(s, \psi)$ is locally Lipschitz continuous with respect to ψ. Hence there exist numbers $\alpha > 0$, $\beta > 0$ such that $\alpha d \leq \beta$ and the set V defined by the equalities $U = \{(\sigma, \varphi)\}$, (10) satisfies relations (11). From the inequality $\alpha d \leq \beta$ it follows that the set $\bar{S}$ defined by equality (14) is a subset of $\bar{A}(za, \beta)$. From Lemma 7 it follows that $\bar{S}$ is a compact subset of the Banach space $\bar{C}_0[0, \alpha]$ and the operator $TF(\sigma, \varphi, .)\, \bar{S} \to \bar{S}$ defined by the equalities (12), (13) is compact. Since the set $\bar{S}$ is a compact and convex subset of $\bar{C}_0[0, \alpha]$, then by Schauder's fixed point theorem it follows that there exists $y \in \bar{S}$ such that $TF(\sigma, \varphi, y) = y$. By Lemma 6 the function x defined for $t < \sigma + \alpha$ by equality (8) is a solution of the initial value problem (2), $x_\alpha = \varphi$.

Definition 3. The function $x\ (-\infty, a) \to R^n$ is said to be a continuation of the function $\tilde{x}\ (-\infty, \tilde{a}) \to R^n$ if $a > \tilde{a}$ and $x(t) = \tilde{x}(t)$ for $t < \tilde{a}$.

Definition 4. The function $x\ (-\infty, a) \to R^n$ is said to be a noncontinuableolution of the initial value problem (2), $x_\sigma = \varphi$, $(\sigma, \varphi) \in I \times E$, if x is a solution of the initial value problem (2), $x_\sigma = \varphi$ and no solution of the initial value problem (1.3), $x_\sigma = \varphi$ is a continuation of x.

The existence of a noncontinuable solution of the initial value problem (2), $x_\sigma = \varphi$, $(\sigma, \varphi) \in I \times E$ is guaranteed by Theorem 1 and Zorn lemma. If x is a noncontinuable solution of the initial value problem (2), $x_\sigma = \varphi$, $(\sigma, \varphi) \in I \times E$, then by $b(x)$ we shall denote the right-hand endpoint of its definition domain.

Corollary 1. *Let E be an admissible space and let conditions H4, H5 hold. Let x be a noncontinuable solution of the initial value problem (2), $x_\sigma = \varphi$, $(\sigma, \varphi) \in I \times E$ such that $b = b(x) < +\infty$ and has a limit $x(b)$ as $t \to b$, $t < b$.*

Then $x_b \in E$ and $(b, x_b) \notin j^{-1}(G)$.

The proof follows from Theorem 1.

3. Continuous dependence of the solution on the initial function

Lemma 8. *Let E be an admissible space. Let conditions H4, H5 hold, where $I = (t_0, +\infty)$, $G = I \times R^{2n}$, and the function F is bounded. Let x be a noncontinuable solution of the initial value problem (2), $x_\sigma = \varphi$, $(\sigma, \varphi) \in I \times E$.*

Then $b(x) = +\infty$.

Proof. Suppose that this is not true, i.e. that $b(x) < +\infty$. For $t < b(x) - \sigma$ set

$$y(t) = x(\sigma + t) - \hat{\varphi}(t),$$

where $\hat{\varphi}(t) = \varphi(t)$ for $t \leq 0$ and $\hat{\varphi}(t) = \varphi(0)$ for $t \geq 0$. By Lemma 6 equalities (9) are valid. Since the function F is bounded, then the function y is Lipschitz continuous. Hence the function x has a limit as $t \to b(x)$, $t < b(x)$, and by means of Corollary 1 we get to a contradiction with the equality $G = \{t_0, +\infty) \times R^{2n}$.

Introduce the following condition:

H6. There exist numbers $\tau \in I$, $p \geq 0$, $\alpha > 0$ such that $[\tau - p, \tau] \subset I$, and if x is a noncontinuable solution of the initial value problem (2), $x_\sigma = 0$, $\sigma \in [\tau - p, \tau]$, then $b(x) > \sigma + \alpha$ and $x(t) = 0$ for $t \leq \sigma + \alpha$.

Theorem 2. *Let E be an admissible space. Let conditions H4, H5 hold, where $I = (t_0, +\infty)$, $G = I \times R^{2n}$, and the function F is bounded. Let, moreover, condition H6 hold.*

Then for $\epsilon > 0$ there exists a number $\delta = \delta(\tau, p, \alpha, \epsilon) \in (0, \epsilon)$ such that if x is a noncontinuable solution of the initial value problem (2), $x_\sigma = \varphi$, $\sigma \in [\tau - r, \tau]$, $\varphi \in E$, $\|\varphi\| < \delta$, then $b(x) > \sigma + \alpha$ and $\|x_t\| < \epsilon$ for $t \in [\sigma, \sigma + \alpha]$.

Proof. Let $q = \max_{s \in [0,\alpha]} (K(s) + M(s))$, where K and M are the functions of condition H2. It suffices to show that there exists a positive function δ defined for $\epsilon > 0$ so that $\delta(\epsilon) < \epsilon$ and if x is a noncontinuable solution of the initial value problem (2), $x_\sigma = \varphi \in E$, $\sigma \in [\tau - p, \tau]$, $\|\varphi\| < \delta(\epsilon)$, then $b(x) > \sigma + \alpha$ and $q|x(t)| < \epsilon$ for $t \in [\sigma, \sigma + \alpha]$.

Suppose that the assertion of the theorem is not true. Then from Lemma 8 it follows that there exists a number $\epsilon_0 > 0$ such that for any $n \in N$ there exist $\sigma^n \in [\tau - p, \tau]$, $\varphi^n \in E$, $n\|\varphi^n\| < \epsilon_0$, $n|\varphi^n(0)| < 1$, a solution $\bar{x}^n$ of the initial value problem (2), $x_{\sigma^n} = \varphi^n$ and $t^n \in [\sigma^n, \sigma^n + \alpha]$ such that $|\bar{x}^n(t^n)| = \epsilon_0$. By condition there exists a number $d > 0$ such that $|\bar{F}(t, x, y)| \leq d$ for $t \in I$, $x, y \in R^n$. From Lemma 7 it follows that formulae (12), (13) define a continuous operator $TF\, I \times E \times \bar{C}[0, \alpha] \to \bar{S}$, where $\bar{S}$ is a compact subset of $\bar{C}[0, \alpha]$ defined by equality (14).

For $n \in N$, $n > 1/\epsilon_0$, $t \leq \alpha$ set

$$y^n(t) = \bar{x}^n(\sigma^n + t) - \hat{\varphi}^n(t),$$

where $\hat{\varphi}^n(t) = \varphi^n(t)$ for $t \leq 0$, $\hat{\varphi}^n(t) = \varphi^n(0)$ for $t \geq 0$. From Lemma 6 it follows that

$$TF(\sigma^n, \varphi^n, y^n) = y^n. \tag{17}$$

Moreover, $y^n \in \bar{S}$, $\tau - p \leq \sigma^n \leq \tau$, $\sigma^n \leq t^n \leq \tau + \alpha$, $n\|\varphi^n\| < \epsilon_0$ and the sets $\bar{S}$, $[\tau - p, \tau + \alpha]$ are compact. Therefore, if we pass to a subsequence and preserve the same notation, we obtain without loss of generality that $(\sigma^n, \varphi^n, y^n) \to (\bar{\sigma}, 0, \bar{y})$, $t^n \to \bar{t}$ as $n \to \infty$, where $\bar{\sigma} \in [\tau - p, \tau]$, $\bar{t} \in [\bar{\sigma}, \tau + \alpha]$, $\bar{y} \in \bar{S}$. Since the operator TF is continuous, then passing to the limit in (17), we obtain that $TF(\bar{\sigma}, 0, \bar{y}) = \bar{y}$. From Lemma 6 and from condition H6 it follows that $\bar{y} = 0 \in \bar{C}([0, \alpha])$. On the other hand, the relations $t^n - \sigma^n \in [0, \alpha]$ and $|\bar{y}^n(t^n - \sigma^n)| = |\bar{x}(t^n) - \varphi^n(0)| \geq \epsilon_0 - |\varphi^n(0)|$ are valid. Hence $\|\bar{y}^n\| \geq \epsilon_0 - n^{-1}$. Since $\bar{y}^n \to \bar{y} = 0$ as $n \to +\infty$, we obtain that $\epsilon_0 \leq 0$ which contradicts the inequality $\epsilon_0 > 0$.

Theorem 3. *Let E be an admissible space. Let conditions H4, H5, H6 hold, where $I = (t_0, +\infty)$ and $(t, 0, 0) \in G \subset I \times R^{2n}$ for $t \in [\tau - p, \tau + \alpha]$.*

Then for $\epsilon > 0$ there exists a number $\delta = \delta(\tau, p, \alpha, \epsilon) \in (0, \epsilon)$ such that if x is a noncontinuable solution of the initial value problem (2), $x_\sigma = \varphi$, $\sigma \in [\tau - p, \tau]$, then $b(x) > \sigma + \alpha$ and $\|x_t\| < \epsilon$ for $t \in [\sigma, \sigma + \alpha]$.

Proof. Since $[\tau - p, \tau + \alpha] \times \{(0, 0)\}$ is a compact subset of $G \subset I \times R^{2n}$ and G is open, then there exists a number $\gamma > 0$ such that the open set

$$G_0 = \{(t, x, y) \in R^{2n+1} \mid t \in (\tau - p - \gamma, \tau + \alpha + \gamma),\ |x| < \gamma,\ |y| < \gamma\}$$

and its closure $\bar{G}_0$ lie in G. By Urysohn's lemma there exists a continuous and bounded function $\bar{F}$ defined in R^{2n+1} so that $F = \bar{F}$ in $\bar{G}_0$. The set $j^{-1}(G_0)$ is open by Lemma 3 and contains the compact $[\tau - p, \tau + \alpha] \times \{(0, 0)\} \subset I \times R^{2n}$. Then there exists a number $\epsilon_0 > 0$ such that the set

$$V_0 = \{(\sigma + s, \psi) \mid \sigma \in (\tau - p, \tau + \alpha),\ \psi \in E,\ |s| \leq \epsilon_0,\ \|\psi\| < \epsilon_0\}$$

lies in $j^{-1}(G_0)$, i.e. $j(V_0) \subset G_0$. For $\epsilon \in (0, \epsilon_0)$ the assertion of the theorem follows from Theorem 2 since $F = \bar{F}$ in $j(V_0)$

In the general case the continuous dependence of the solutions of the initial value problem (2), $x_\sigma = \varphi$, on the initial function φ, $\|\varphi\| \approx 0$ for fixed $\sigma \in I$ is not uniform on σ even if the null solution of (2), $x_\sigma = 0$ is unique for $\sigma \in I$ and the right-hand side of the equation is bounded.

Example 1. For $t \in I = (1, +\infty)$, $x \geq 0$ set

$$F(t,x) = \begin{cases} tx, & 0 \leq x \leq t^{-1} \\ 1, & x > t^{-1} \end{cases}$$

and $F(t, -x) = -F(t, x)$. Consider the ordinary differential equation

$$\dot{x}(t) = F(t, x(t)) \tag{18}$$

For given $q > 0$, $\epsilon \in (0, q)$ and $\delta > 0$ arbitrarily small there exists $\sigma > \delta^{-1}$ such that the solution $\bar{x}$ of the initial value problem (18), $x(\sigma) = \delta$ leaves the domain $|x| < \epsilon$.

Remark 5. Theorem 2 and Theorem 3 give sufficient conditions under which the continuous dependence of the solutions of the initial value problem (2), $x_\sigma = \varphi$ on the initial function φ, $\|\varphi\| \approx 0$ is uniform on σ running over a fixed compact. In particular, for $p = 0$ and fixed $\alpha > 0$ the function $\bar{\delta}(\sigma, \epsilon) = \delta(\sigma, 0, \alpha, p, \epsilon)$ can be chosen nonincreasingly and continuous on σ.

Introduce the following conditions:

H7. If x is a noncontinuable solution of the initial value problem (2), $x_\sigma = 0$, $\sigma \in I$, then $x(t) = 0$ for $t \in R$.

H8. There exists $\epsilon_0 > 0$ such that for $|x| + |y| < \epsilon_0$, $t \in R$ the functions $g(t) - t$ and $F(t, x, y)$ are periodic with respect to t with rationally dependent periods.

H9. The function $M^*(g(t) - t)$ is bounded for $t \in I$.

H10. There exist numbers $\epsilon_0 > 0$, $c > 0$ such that $|F(t, x, y)| \leq c(|x| + |y|)$ for $t \in I$, $|x| + |y| \leq \epsilon_0$.

Remark 6. A function r is said to be periodic with period $w = 0$ if $r(t) = const$ for $t \in R$.

Corollary 2. *Let E be an admissible space. Let conditions H4, H5, H7, H8 hold, where $I = R$, $G \supset R \times \{0\}$.*

Then for any $q > 0$, $\epsilon > 0$ there exists a number $\delta = \delta(q, \epsilon) > 0$ such that if x is a noncontinuable solution of the initial value problem (2), $x_\sigma = \varphi$, $(\sigma, \varphi) \in R^1 \times E$, $\|\varphi\| < \delta$, then $b(x) > \sigma + q$ and $\|x_t\| < \epsilon$ for $t \in [\sigma, \sigma + q]$.

P r o o f. By condition H7 there exists a number $w > 0$ such that $g(t-w) = g(t) - w$, $F(t-w, x, y) = F(t, x, y)$ for $t \in R^1$, $|x| + |y| < \epsilon_0$. Without loss of generality we suppose that

$$G = \{(t, x, y) \mid (t, x, y) \in R \times R^n \times R^n,\ |x| + |y| < \epsilon\}$$

Let $\sigma < a < +\infty$, $\bar{x}\ (-\infty, a) \to R^n$, $x^*\ (-\infty, a + w) \to R^n$ and $x^*(t) = \bar{x}(t-w)$ for $t < a + w$. Then $x^*_{\sigma+w} = \bar{x}_\sigma$ and for $t \in (\sigma + w, a + w)$ the following equalities are valid

$$F(t, x^*(t), \max_{s\in[g(t),t]} x^*(s)) = F(t-w, \bar{x}(t-w), \max_{s\in[g(t-w),t-w]} \bar{x}(s)).$$

Hence the function $\bar{x}$ is a solution of the initial value problem (2), $x_\sigma = \varphi$ if and only if the function x^* is a solution of the initial value problem (2), $x_{\sigma+w} = \varphi$. From Theorem 3 (for $p = w$) there follows the assertion of Corollary 2.

Corollary 2 gives sufficient conditions under which the number $\bar{\delta}(\sigma, \epsilon)$ in Remark 5 does not depend on $\sigma \in I$. Another sufficient condition for this is given in the following theorem.

Theorem 4. *Let E be an admissible space. Let conditions H4, H5, H7, H9, H10 hold, where $I = (t_0, +\infty)$ and $G = \{(t, x, y) \in I \times R^{2n} \mid |x| + |y| < 2\epsilon_0\}$.*

Then for any $q > 0$, $\epsilon > 0$ there exists a number $\delta = \delta(q, \epsilon) > 0$ such that if x is a noncontinuable solution of the initial value problem (2), $x_\sigma = \varphi$, $(\sigma, \varphi) \in I \times E$, $\|\varphi\| < \delta$, then $b(x) > \sigma + q$ and $\|x_t\| < \epsilon$ for $t \in [\sigma, \sigma + q]$.

P r o o f. Let $q > 0$. Taking into account that E is an admissible space and conditions H4, H9 are met, it follows that if $x\ (-\infty, a) \to R^n$, $\sigma < a$, $x_\sigma \in E$, $x \in C[\sigma, a)$, then for $t \in [\sigma, a) \cap [\sigma, \sigma + q]$ the relations $x_t \in E$ and

(19) $$\|x_t\| \le K_1 \max_{s\in[\sigma,t]} |x(s)| + M_1 \|x_\sigma\|,$$

(20) $$|\max_{s\in[g(t),t]}| \le \max_{s\in[g(t)-t,0]} |x_t(s) + | \le d\|x_t\|,$$

are valid, where $K_1 = K(\tau)$, $M_1 = \max_{\tau\in[0,q]} M(\tau)$, and $d = \max\left\{\frac{1}{2}, \sup^*_{t\in I}(g(t)-t)\right\}$. For any $\epsilon \in (0, \epsilon_0)$ there exists a number $\delta = \delta(q, \epsilon)$ which satisfies the inequalities

(21) $$0 < 2d\delta < \epsilon,\ 4M_1 d\delta < \sigma,\ 4dK_1(M_0 + 2cdqM_1)e^{2cdqK_1}\delta < \epsilon.$$

We shall prove that $\delta(q, \epsilon)$ satisfies the requirements of Theorem 4.

Let x be a noncontinuable solution of the initial value problem (2), $x_\sigma = \varphi$, $(\sigma, \varphi) \in I \times E$, $\|\varphi\| < \delta = \delta(q, \epsilon)$. Set

$$W = \left\{ t \in [\sigma, b(x)) \;\middle|\; |x(t)| + |\max_{s\in[g(t),t]} x(s)| > \epsilon_0 \right\}, \tag{22}$$

$$r = b(x) \text{ for } W = \emptyset, r = \inf W \text{ for } W \neq \emptyset.$$

From the inequalities (20) and $2d\delta < \epsilon < \epsilon_0$ it follows that $\sigma \notin W$. Hence $\sigma < r \leq b(x) \leq +\infty$. From condition H10 it follows that

$$|F(t, x(t), \max_{s\in[g(t),t]} x(s)| \leq 2c| \max_{s\in[g(t),t]} x(s)| \tag{23}$$

for $t \in [\sigma, b(x))$. Integrating (2) from σ to t, we obtain

$$|x(t)| \leq |x(\sigma)| + \int_\sigma^t |F(\lambda, x(\lambda), \max_{s\in[g(\lambda),\lambda]} x(s)| d\lambda \tag{24}$$

for $t \in [\sigma, b(x))$. From relations (19), (20), (24), (25), $x_\sigma = \varphi \in E$ it follows that for $t \in [\sigma, \sigma + q] \cap [\sigma, r)$ we have

$$|x(t)| \leq M_0\|\varphi\| + 2cd[M_1\|\varphi\|(t - \sigma) + \int_\sigma^t K_1 \max_{s\in[\sigma,\lambda]} |x(s)| d\lambda].$$

Hence

$$\max_{s\in[\sigma,t]} |x(s)| \leq (M_0 + 2cdqM_1)\|\varphi\| + 2cdK_1 \int_\sigma^t \max_{s\in[\sigma,\lambda]} |x(s)| d\lambda.$$

for $t \in [\sigma, \sigma + q] \cap [\sigma, r)$. From Lemma 2 and Gronwall's inequality it follows that

$$| \max_{s\in[\sigma,t]} x(s)| \leq (M_0 + 2cdqM_1\|\varphi\| e^{cdqK_1} \tag{25}$$

for $t \in [\sigma, \sigma + q] \cap [\sigma, r)$. From the inequalities $\|\varphi\| < \delta$, (26), (19)-(21) it follows that

$$|x(t)| + | \max_{s\in[g(t),t]} x(s)| \leq 2d\|x(t)\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon < \epsilon_0 \tag{26}$$

for $t \in [\sigma, \sigma + q] \cap [\sigma, r)$. Since $2d \geq 1$, then it suffices to prove that $r > \sigma + q$. Suppose that this is not true, i.e. $r \leq \sigma + q$. From relations (27), (2) and from condition H10 it follows that $x(t)$ has a limit $x(r)$ as $t \to r$, $t < r$. Then $r \in I$,

$x_r \in E$, $j(r, x_r) \in G$ and from Corollary 1 it follows that $r \neq b(x)$. Hence $\sigma < r < b(x)$ and by relations (22), (23) we obtain that $W \neq \emptyset$ and

$$| \max_{s \in [g(r), r]} x(s)| + |x(r)| = \epsilon_0$$

which contradicts inequality (27).

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Received 05.04.1993