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T_3 and T_4 - Objects in the Category of Bornological Spaces

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In this paper, two new generalizations of the usual topological T_2 - axiom and four various generalizations of the completely normal topological space to topological categories are given. Furthermore, an explicit characterization of each of the separation properties as well as the relationships among the various forms of T_2, T_3, T_4, and T_5 (completely normal) is given in the topological categories of prebornological and bornological spaces. Finally, some well known results that are related to the separation properties in the category of topological spaces are generalized and interpreted in these categories.

1. Introduction

Several generalizations of the usual separation properties of topology to an arbitrary topological category over sets are given in [1], [2], and [11]. There are, for example, four notions of "T_0" denoted by T_0', $\overline{T}_0$ (in [1]), T_0 and separated (in [11]), each equivalent to the classical notion of T_0 for topological spaces, eight notions of "T_2" denoted by T_2', $\bar{T}_2$, KT_2, LT_2, MT_2, NT_2, ST_2, and ΔT_2 in [1] and [4], each equivalent to the classical notion of T_2 for topological spaces, and four notions of each of "T_3" and "T_4" denoted by T_i', ST_i', $\overline{T}_i$, and $S\overline{T}_i$ in [1] for $i = 3, 4$, each equivalent to the classical notion of T_i for topological spaces, $i = 3, 4$, respectively.

In this paper, we introduce two new generalizations of Hausdorff topological spaces and four various generalizations of the completely normal, T_5, topological spaces (i.e. every subspace of normal, T_4, topological space is normal [8] p.206) to topological categories. Moreover, we establish the followings for the topological categories of prebornological spaces and bornological spaces:

1. To give an explicit characterization of each of the generalized separation properties as well as to examine the relationships among these various forms.

2. To investigate the relationships between the notion of closedness and strongly closedness.

3. To determine the invariance properties (i.e. closed under formation of products, subspaces, and quotient spaces) of each of these separation properties.

4. To generalize and interpret some well-known results in the category of topological spaces to the categories of prebornological and bornological spaces.

The utility of the category of bornological spaces for functional analysis has become apparent trough [6] and [10].

General results involving relationships among these generalized separation properties as well as interrelationships among their various forms are being investigated in [4].

Let B and E be any categories. The functor $U : E \to B$ is said to be topological if it is concrete (i.e. faithful and amnestic (i.e. if $U(f) = id$ and f is an isomorphism, then $f = id$)), has small (i.e sets) fibers, and for which every U-source has an initial lift or, equivalently, for which each U-sink has a final lift [5] p.125 or [7] p.279. The topological functor $U : E \to$ Sets, the category of sets, is said to be normalized if there is only one structure on a point and the empty set.

Recall that an epi final E-morphism is called a quotient map. A mono initial E-morphism is called an embedding, and an object Y in E is called a subspace of an object X in E if Y, as a set, is contained in X and the inclusion map is an embedding.

A Prebornological space is a pair $(A, \mathcal{F})$ where $\mathcal{F}$ is a family of subsets of A that is closed under finite union and contains all finite nonempty subsets of A (see [3] or [2] p. 384). Furthermore, if $\mathcal{F} \neq \Phi$ and $\mathcal{F}$ is hereditary closed, then $(A, \mathcal{F})$ is called a b o r n o l o g i c a l s p a c e (see [3], [6] or [2]). A morphism $(A, \mathcal{F}) \to (B, \mathcal{G})$ of such spaces is a function $f : A \to B$ such that $f(C) \in \mathcal{F}$ if $C \in \mathcal{F}$ i.e f is continuous. We denote by PBorn and Born, respectively, the categories so formed and by PBorn* the full subcategory of PBorn determined by those spaces $(A, \mathcal{F})$ with $\Phi \notin \mathcal{F}$ (see [3] or [2]). These categories are topological over sets. Moreover, PBorn* and Born are normalized, but PBorn is not [2].

Let $U\ E \to$ Sets, the category of sets, be topological and X an object in E with $UX = B$. Let F be a nonempty subset of B. We denote by X/F the final lift of the epi U-sink $q\ U(X) = B \to B/F$, where q is the epi map identifying F to a point $*$.

Definitions:

1. X is PT_2 iff X is separated [11] and $Pre\overline{T}_2$.

2. X is RT_2 iff X is separated [11] and $PreT_2'$.

3. $F \subset X$ is strongly closed iff X/F is T_1 at * [1] p. 337.

4. $F \subset X$ is closed iff * is closed in X/F [1] p.337.

5. X is ST_3' iff X is T_1 and X/F is $PreT_2'$ for all strongly closed $F \neq \Phi$ in $U(X)$ [1] p. 340.

6. X is $S\overline{T}_3$ iff X is T_1 and X/F is $Pre\overline{T}_2$ for all strongly closed $F \neq \Phi$ in $U(X)$ [1] p. 340.

7. X is $\overline{T}_3$ iff X is T_1 and X/F is $Pre\overline{T}_2$ for all strongly closed $F \neq \Phi$ in $U(X)$ [1] p. 340.

8. X is T_3' iff X is T_1 and X/F is $PreT_2'$ for all closed $F \neq \Phi$ in $U(X)$ [1] p. 340.

9. X is T_4' iff X is T_1 and X/F is $S\overline{T}_3'$ for all strongly closed $F \neq \Phi$ in $U(X)$ [1] p. 340.

10. X is $S\overline{T}_4'$ iff X is T_1 and X/F is ST_3 for all strongly closed $F \neq \Phi$ in $U(X)$ [1] p. 340.

11. X is $\overline{T}_4$ iff X is T_1 and X/F is $\overline{T}_3$ for all strongly closed $F \neq \Phi$ in $U(X)$ [1] p. 340.

12. X is T_4' iff X is T_1 and X/F is T_3' for all strongly closed $F \neq \Phi$ in $U(X)$ [1] p. 340.

13. X is T_5' (i.e completely normal) iff every subspace of X is T_4'.

14. X is $\overline{T}_5$ (i.e completely normal) iff every subspace of X is $\overline{T}_4$.

15. X is ST_5' (i.e completely normal) iff every subspace of X is ST_4'.

16. X is $S\overline{T}_5$ (i.e completely normal) iff every subspace of X is $S\overline{T}_4$.

Remark. Note that for the category of topological spaces, TOP all of T_i's reduce the usual T_i, $i = 2, 3, 4, 5$, topological spaces. This follows from [1], [4], [11], and Definition 1.1.

2. Separation Properties.

We now give the characterizations of each of the separation properties defined in 1.1 for PBorn, PBorn*, and Born as well as investigate some of their invariance properties.

Theorem 2.1.

1. All objects (A, Φ) in PBorn, PBorn, or Born are T_0', $\overline{T}_0$, T_1, and ST_2 [2] and [3].*

2. $X = (A, \mathcal{F})$ in PBorn or PBorn is $Pre\overline{T}_2$, $\overline{T}_2$, and KT_2 iff X is strictly hereditary closed i.e. if $\Phi \neq V \subset U$ and $U \in \mathcal{F}$, then $V \in \mathcal{F}$ [3] and [4].*

3. $X = (A, \mathcal{F})$ in PBorn or PBorn is $PreT_2'$, T_2', and LT_2 iff X is hereditary closed i.e. $X \in$ Born [3] or [4].*

4. $X = (A, \mathcal{F})$ in PBorn, PBorn, or Born is ΔT_2 iff X is a point of the empty set [3] or [4].*

5. Every object in Born is $PreT_2'$, T_2', $Pre\overline{T}_2$, $\overline{T}_2$, KT_2, and LT_2. It follows from parts (2) and (3).

6. Let $X = (A, \mathcal{F})$ be in PBorn, PBorn or Born. $\Phi \neq F \subset A$ is closed iff $F = A$ [2], and F is always strongly closed [2].*

Theorem 2.2. *$X = (A, \mathcal{F})$ be in PBorn, PBorn* or Born is T_0, separated, MT_2, NT_2, PT_2 or RT_2 iff A is a point or the empty set.*

Proof. Since each two-element (pre)bornological space is always indiscrete, it follows that the T_0-objects of PBorn, PBorn* or Born are those with at most one point. The result for the separated objects see [11] p.322. The proof for the others follows from these observations and definitions in 1.1 and [4]. ■

Theorem 2.3. *All objects $X = (A, \mathcal{F})$ in PBorn, PBorn* or Born are T_3', $\overline{T}_3$, T_4', $\overline{T}_4$, T_5', and $\overline{T}_5$.*

Proof. Note that, by 2.1, X is T_1. If $\Phi \neq F \subset A$ is closed, then by 2.1, $F = A$ and consequently $A/F = \{*\}$. It follows that, by 2.1, X is $PreT_2'$ and $Pre\overline{T}_2$. Hence, by definition 1.1, X is T_3' and $\overline{T}_3$. The proof for the rest follows from these and definition 1.1. ■

Theorem 2.4. *All objects $X = (A, \mathcal{F})$ in Born are "T_i" for $i = 3, 4, 5$*

Proof. It follows easily from 1.1 and 2.1. ■

Theorem 2.5. *$X = (A, \mathcal{F})$ in PBorn or PBorn* is ST_3' ($S\overline{T}_3$) iff X is T_2' (resp. $\overline{T}_2$).*

Proof. Suppose X is ST_3' ($S\overline{T}_3$). Since one-element sets are strongly closed (see 2.1), it follows easily that $X/\{a\}$ is isomorphic to X, where $\{a\}$ is one point set, and consequently X is T_2' (resp. $\overline{T}_2$).

Conversely, suppose X is T_2' ($\overline{T}_2$). By 2.1, X is T_1. It remains to show that $X/F = (A/F, \mathcal{F}')$ is $PreT_2'$ (resp $Pre\overline{T}_2$) for all nonempty subsets F of A, where $\mathcal{F}'$ is the quotient structure on A/F. Suppose $V \subset U$ ($\Phi \neq V \subset U$) and $U \in \mathcal{F}'$. Then $U = q(W)$ for some $W \in \mathcal{F}$ since q is the quotient map (see

[3] or [7]). If $* \in V$, then $q^{-1}(V - \{*\}) \subset W - F \subset W \in \mathcal{F}$ and consequently $q^{-1}(V - \{*\}) \in \mathcal{F}$ since X is T_2' (resp. $\overline{T}_2$). Hence, $qq^{-1}(V - \{*\}) = V - \{*\} \in \mathcal{F}'$ and $V = V - \{*\} \cup \{*\} \in \mathcal{F}'$. If $* \notin V$, then $q^{-1}V \subset W - F \subset W \in \mathcal{F}$ and consequently $qq^{-1}(V\}) = V \in \mathcal{F}'$. Therefore X/F is $PreT_2'$ (resp. $Pre\overline{T}_2$). This completes the proof. ∎

Theorem 2.6. *$X = (A, \mathcal{F})$ in PBorn or PBorn* is ST_4' ($S\bar{T}_4$) iff X is T_2' (resp. $\overline{T}_2$).*

Proof. Combine 1.1, 2.1, and 2.5. ∎

Lemma 2.7. *In all of the above categories, 1. If $(A, \mathcal{F})$ is "T_k" space for $k = 0, 1, 2, 3, 4$, then every subspace of $(A, \mathcal{F})$ is "T_k". 2. The cartesian product $(X = \{\prod A_i | \ i \in I\}, \mathcal{F})$ is "T_k" for $k = 1, 1, 2, 3, 4$ iff each $(A_i, \mathcal{F}_i)$ is "T_k for $k = 0, 1, 2, 3, 4$ (for each i, $A_i \neq \Phi$).*

Proof. It follows easily from 2.1-2.6 and definition of subspace and product space in these categories (see [9] for initial lifts). ∎

Theorem 2.8. *$X = (A, \mathcal{F})$ in PBorn or PBorn* is ST_5' ($S\bar{T}_5$) iff X is T_2' (resp. $\overline{T}_2$).*

Proof. Combine 1.1, 2.6, and 2.7 (1). ∎

We can infer the following relationships among the various forms of the separation properties in PBorn, PBorn*, and Born.

Remark 2.9. (1) In Born, (i) All of T_0, separatedness, MT_2, NT_2, PT_2, RT_2, and ΔT_2 are equivalent.

(ii) All of T_0', $\overline{T}_0$, T_1, ST_2, LT_2, $\overline{T}_2$, T_2', KT_2, T_3', $\overline{T}_3$, ST_3', $S\overline{T}_3$, T_4', $\overline{T}_4$, ST_4', $S\overline{T}_4$, T_5', $\overline{T}_5$, ST_5' and $S\overline{T}_5$ are equivalent. Note also that (i) implies (ii), but this is not true in TOP, the category of topological spaces. Furthermore, $MT_2 = NT_2 = PT_2 = RT_2 = \Delta T_2$ implies $ST_2 = \overline{T}_2 = T_2' = KT_2 = LT_2$ but the converse is not true, in general.

(2) In PBorn and PBorn*, (i) All of T_0, separatedness, MT_2, NT_2, PT_2, RT_2, and ΔT_2 are equivalent.

(ii) All of T_0', $\overline{T}_0$, T_1, ST_2, T_3', $\overline{T}_3$, T_4', $\overline{T}_4$, T_5' and $\overline{T}_5$ are equivalent.

(iii) All of $\bar{T}_2$, KT_2, $S\overline{T}_3$, $S\overline{T}_4$, and $S\overline{T}_5$ are equivalent.

(iv) All of T_2', LT_2, ST_3', ST_4' and $S\overline{T}'_5$ are equivalent.

(v) (i) implies (iii), (iii) implies (iv), and (iv) implies (ii) but the converse of each implication is not true, in general.

(vi) T_0 = separatedness implies $T_0' = \overline{T}_0$ but the converse is not true. In [2], it is shown that all of these "T_0" structures are different in an arbitrary topological category. Moreover, $MT_2 = NT_2 = PT_2 = RT_2 = \Delta T_2$ implies $T_2' = LT_2$, LT_2 implies $\bar{T}_2 = KT_2$, KT_2 implies ST_2, ST_i' implies $S\overline{T}_i$, $S\overline{T}_i$ implies $T_i' = \overline{T}_i$ for

each $i = 3,4,5$, and PT_2 implies ST_i' for $i = 3,4,5$, but the converse of each implication is not true, in general.
(3) Lemma 2.6 does not hold in TOP for $k = 4$, in general.
(4) If $(A, \mathcal{F})$, in all of the above categories, is "T_i", then $(A/F, \mathcal{F}')$ is also "T_i" for each $i = 0,1,2,3,4,5$ for all nonempty subset F of A. However, this is not true in TOP, in general.

We next define the notation of (strongly) closure, $\bar{B}$ $(S(B))$ of a nonempty subset B of A, where $X = (A, \mathcal{F})$ in PBorn, PBorn* or Born. $\bar{B}(S(B)) = \cap\{D D$ is closed (strongly closed) and $D \supset B\}$. Note that $B \subset \bar{B}(S(B) = B)$ and $\bar{B} = B$ iff B is closed. Furthermore,
1. If $\Phi \neq D \subset B$, then $\bar{D} = \bar{B}$ $(S(D) \subset S(B))$.
2. $\bar{B}$ is always closed ($S(B)$ is always strongly closed).
3. $\cup_{i\in I}\overline{B}_i = \overline{\cup_{i\in I}B_i}$, $(\cup_{i\in I}S(B_i) = S(\cup_{i\in I}B_i))$.
4. $\cap_{i\in I}\bar{B}_i = \cap_{i\in I}\bar{B}_i$ if $\cap_{i\in I}B_i \neq \Phi$, $(\cap_{i\in I}S(B_i) = S(\cap_{i\in I}B_i))$.
Note that, by [2], Φ is (strongly) closed, and (3) and (4) do not hold in TOP, in general.

Remark 2.10. Let $X = (A, \mathcal{F})$ be in PBorn, PBorn* or Born and $\Phi \neq F \subset A$.
(1) If F is closed, then F is strongly closed but the converse is not true. For example, take $F = \{a\}$, one-point set, and A to be any infinite set. This example also shows that T_1 does not imply points are closed.
(2) Suppose X or X/F is T_0 or PT_2. Then F is closed iff F is strongly closed. However, in TOP, this result holds if X is T_1.
(3) F is closed iff X/F is T_0 or PT_2. But X is T_1 iff all subsets of X are strongly closed. This does not hold in TOP.

Theorem 2.11. *Let $X = (A, \mathcal{F})$ and $Y = (B, \mathcal{G})$ be in PBorn, PBorn* or Born. For every closed $F \subset A$, each morphism (i.e. continuous) $h(F, \mathcal{G}) \to Y$ has a continuous extension H $X \to Y$.*

Proof. Since F is closed, $F = A$, by 2.1, and take H to be h. ■

Remark 2.12. In 2.11, if we take X to be any of the "T_4" and Y to be $R, \mathcal{G})$, where R is the set of real numbers, then Theorem 2.11 is a generalization of the "Tietze Eztension Theorem in case of TOP.

Theorem 2.13. *Let $X = (A, \mathcal{F})$ and $Y = (B, \mathcal{G})$ be in PBorn, PBorn* or Born. If X is T_0 or PT_2, then for every strongly closed $F \subset A$, each morphism h $(F, \mathcal{B} \to Y$ has a continuous extension H $X \to Y$.*

Proof. X is T_0 or PT_2 iff, by 2.2, A is a point or the empty set. The result follows. ■

Theorem 2.14. *Let $X = (A, \mathcal{F})$ be in PBorn, PBorn* or Born. Every pair of parallel morphism to X has a closed equilizer iff X is T_0 or PT_2.*

Proof. Note that, by 2.9, T_0 and PT_2 are equivalent. IF X is T_0, then by 2.2, A is a point or the empty set. It follows easily that every pair of parallel morphism $f, g\, Y \to X$ must be equal and consequently their equilizer E, as a set, is equal to Y, as a set. Hence, by 2.1, E is closed.

Conversely, let $Y = X^2$, the product space of X, $f = \pi_1$, $g = \pi_2$, the projections. It is clear that the equilizer of π_1, $\pi_2\, X^2 \to X$, as a set, is equal to Δ, the diagonal, in X^2. Since Δ is closed (by assumption), it follows, by 2.1, that $\Delta = A$ which implies A is a point or the empty set. Hence, by 2.2, X is T_0. ∎

Remark 2.15. Theorem 2.14 holds in TOP when X is T_2 (Hausdorff) topological space.

Finally, we consider the subcategories T_iPBorn, T_iPBorn* and T_iBorn of PBorn, PBorn* and Born, respectively, for $i = 0, 1, 2, 3, 4, 5$. Objects of these subcategories are the T_i-objects of PBorn, PBorn* and Born, respectively, for $i = 0, 1, 2, 3, 4, 5$. It follows easily from Theorems 2.1-2.8 and the results 1.13, 1.14 and 2.5 of [9] that all of these subcategories are initially structured categories; in particular, all are monotopological [11] and closed under formation of powers in PBorn, PBorn* and Born, i.e., powers of T_i-objects are T_i-objects for each $i = 0, 1, 2, 3, 4, 5$. Moreover, each T_iBorn is cartesian closed since Born is [9].

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