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About a New Class of Integral Transforms in Hilbert Space

S. B. Yakubovich

Presented by Bl. Sendov

In this paper a new class of integral transforms is constructed by means of compositions of the general Fourier and Watson transforms [5]. This class involves both transforms with integration with respect to the arguments(the convolution type transforms [2], [4]) and to the parameters (the index transforms [7]-[9]) of special functions in the kernels. Mapping properties are investigated in the space $L_2(R+)$, where $L_2(R+)$ is the Hilbert space of square summable functions on $R+$. A Plancherel type theorem is proved. New pairs of index transforms involving the Lommel functions [1] and the Meijer G-functions [3] as kernels are discussed.

1. Introduction

In accordance with [5] the transform

$$[Kf](x) = g(x) = \lim_{N\to\infty} \int_0^N k(xy)f(y)dy \tag{1}$$

is said to be a general Fourier transform, where $f(x) \in L_2(R+)$, $k(x)$ is called a Fourier kernel and the integral (1) converges in L_2-sense. The operator (1) is bounded on the space $L_2(R+)$ and its inversion in the general case has the form

$$[\hat{K}g](x) = f(x) = \lim_{N\to\infty} \int_0^N \hat{k}(xy)g(y)dy \tag{2}$$

where $\hat{k}(x)$ is the conjugate Fourier kernel and there is the equality in terms of Mellin transforms [3] $k^*(s) = \mathcal{M}\{k(y); s\}$ and $\hat{k}^*(s) = \mathcal{M}\{\hat{k}(y); s\}$ of the kernels $k(x)$ ans $\hat{k}(x)$:

$$k^*(s)\hat{k}^*(1-s) = 1, \quad \Re(s) = 1/2, \tag{3}$$

where

$$\mathcal{M}\{f(y); s\} = \int_{R+} y^{s-1} f(y) dy. \tag{4}$$

In order to provide the existence of Mellin transforms (4) for the Fourier kernels mentioned above we need the following

Definition 1.1 [6]. Denote by $\mathcal{R}$ the set of kernels $k(x)$ for which the following conditions hold:

1) $k(x) \in L(\epsilon, E)$ for any ϵ, E such that $0 < \epsilon < E < \infty$;

2) The integral (4) for $k^*(s)$, $\Re(s) = 1/2$ is convergent and there exists a constant $C > 0$ such that for almost all $\epsilon, E > 0$ and $t \in R$ the following estimation is fulfilled:

$$\left| \int_{\epsilon}^{E} k(u) u^{it-1/2} du \right| < C. \tag{5}$$

If for the kernel $k(x) \in \mathcal{R}$ there exists a conjugate kernel $\hat{k}(x) \in \mathcal{R}$ so that the equality (3) holds almost everywhere on the line $\Re(s) = 1/2$, then we say that $k(x)$, $\hat{k}(x) \in \mathcal{R}^* \subset \mathcal{R}$.

For the K-transform and the $\hat{K}$-transform the relation of Parceval type is true

$$\int_0^\infty [Kf](u)[\hat{K}f](u) du = \int_0^\infty f^2(y) dy \tag{6}$$

with the inequalities

$$c_1 \int_0^\infty |f(y)|^2 dy < \int_0^\infty |[Kf](u)|^2 du < c_2 \int_0^\infty |f(y)|^2 dy, \tag{7}$$

$$\hat{c}_1 \int_0^\infty |f(y)|^2 dy < \int_0^\infty |[\hat{K}f](u)|^2 du < \hat{c}_2 \int_0^\infty |f(y)|^2 dy, \tag{8}$$

where c_i, $\hat{c}_i$-const, $i = 1, 2$, and the integrals are understood as improper.

In particular, if $k(x) = \hat{k}(x)$ and $|k^*(s)| = 1$, then the operator (1) is unitary on the space $L_2(R+)$, i.e.

$$\int_0^\infty |[Kf](u)|^2 du = \int_0^\infty |f(y)|^2 dy \tag{9}$$

and its inversion formula has the symmetric form

$$[Kg](x) = f(x) = \lim_{N \to \infty} \int_0^N k(xy) g(y) dy. \tag{10}$$

Many examples of kernels from $\mathcal{R}^*$ can be found in [1].It is natural to note the known pairs of symmetrical cos- and sin-Fourier transforms

$$[F_c f](x) = g(x) = \sqrt{2/\pi} \lim_{N\to\infty} \int_0^N \cos(xy) f(y) dy, \tag{11}$$

$$[F_s f](x) = g(x) = \sqrt{2/\pi} \lim_{N\to\infty} \int_0^N \sin(xy) f(y) dy. \tag{12}$$

Now we can introduce the transform (1) in terms of Watson kernels. A function $k_1(x)$ is called a Watson kernel if

$$k_1(x) = \int_0^x k(y) dy = \frac{x}{2\pi i} \lim_{N\to\infty} \int_{1/2-iN}^{1/2+iN} \frac{k^*(s)}{1-s} x^{-s} ds. \tag{13}$$

A function $\hat{k}_1(x)$ is called a conjugate Watson kernel if

$$\hat{k}_1(x) = \frac{x}{2\pi i} \lim_{N\to\infty} \int_{1/2-iN}^{1/2+iN} \frac{\hat{k}^*(s)}{1-s} x^{-s} ds \tag{14}$$

and the equality (4) takes place. Since the functions $k^*(s)$, $\hat{k}^*(s)$ are bounded on the line $\Re(s) = 1/2$ according to Definition 1.1, then the integrals (13)–(14) exist in L_2-sense and in accordance with the theory of the Mellin transform $k_1(x)/x$, $\hat{k}_1(x)/x$ belong to $L_2(R+)$.

The following improper integral

$$\int_0^\infty \frac{k_1(xu)\hat{k}_1(yu)}{u^2} du = \min(x, y), \quad x > 0, \quad y > 0 \tag{15}$$

is said to be a Watson condition which provides the boundedness in $L_2(R+)$ of the corresponding Watson transforms pairs

$$[Kf](x) = g(x) = \frac{d}{dx} \int_0^\infty k_1(xy) f(y) \frac{dy}{y}, \tag{16}$$

$$[\hat{K} f](x) = f(x) = \frac{d}{dx} \int_0^\infty \hat{k}_1(xy) g(y) \frac{dy}{y}. \tag{17}$$

Moreover as it is shown in [5] the Watson condition (15) is necessary for the existence of the dual transforms pair (16), (17).

2. New general transforms

Let the functions $k(x)$, $h(x)$ be arbitrary Fourier kernels for the set $\mathcal{R}^*$ and $[Kf](x)$, $[Hf](x)$ be the corresponding general Fourier transforms defined by formula (1). Let also $\mu(x)$ be a monotonically increasing positive function on $R+$, which is differentiable one and $\mu(0), \mu'(0) \neq 0$.

Let us consider the following composition of general Fourier transforms

$$g(x) = [K\sqrt{\mu'(t)}[Hf](\mu(t))](x)$$

(18)

$$\lim_{N\to\infty} \int_0^N k(xt)\sqrt{\mu'(t)} \lim_{N\to\infty} \int_0^M h(\mu(t)y)f(y)\,dy\,dt$$

and its conjugate

$$\hat{g}(x) = [\hat{K}\sqrt{\mu'(t)}[\hat{H}f](\mu(t))](x)$$

(19)

$$\lim_{N\to\infty} \int_0^N \hat{k}(xt)\sqrt{\mu'(t)} \lim_{N\to\infty} \int_0^M \hat{h}(\mu(t)y)f(y)\,dy\,dt.$$

It is evident that if $f(x) \in L_2(R+)$, then $\sqrt{\mu'(x)}[Hf](\mu(x))$, $\sqrt{\mu'(x)}[\hat{H}f](\mu(x)) \in L_2(R+)$ and after twice using the equality (6) we will have the corresponding Parseval relation for the compositions (18), (19), namely:

$$\int_0^\infty g(x)\hat{g}(x)dx = \int_0^\infty [Hf](\mu(t))[\hat{H}f](\mu(t))d\mu(t) = \int_0^\infty f^2(y)dy \tag{20}$$

with the inequalities

$$c_1\int_0^\infty |f(y)|^2dy < \int_0^\infty |g(u)|^2du < c_2\int_0^\infty |f(y)|^2dy, \tag{21}$$

$$\hat{c}_1\int_0^\infty |f(y)|^2dy < \int_0^\infty |\hat{g}(u)|^2du < \hat{c}_2\int_0^\infty |f(y)|^2dy, \tag{22}$$

c_i, $\hat{c}_i$-const, $i = 1, 2$.

Now we consider the composition (18) in the space of C^∞-functions with compact support on $(0,\infty)$ as follows

$$\begin{aligned} g(x) &= \frac{d}{dx}\int_0^\infty \frac{k_1(xt)}{t}\frac{1}{\sqrt{\mu'(t)}}\frac{d}{dt}\int_0^\infty \frac{h_1(\mu(t)y)}{y}f(y)dydt \\ &= \frac{d}{dx}\int_0^\infty \frac{k_1(xt)}{t}\frac{1}{\sqrt{\mu'(t)}}\frac{d}{dt}\int_0^\infty \frac{h_1(u)}{u}f(u/\mu(t))dudt \\ &= -\frac{d}{dx}\int_0^\infty \frac{k_1(xt)}{t}\frac{\sqrt{\mu'(t)}}{\mu(t)}\int_0^\infty h_1(\mu(t)y)f'(y)dydt. \end{aligned} \tag{23}$$

Since $k_1(xt)/t \in L_2(R+)$ for any $x > 0$, $\frac{\sqrt{\mu'(t)}}{\mu(t)} h_1(\mu(t)y) \in L_2(R+)$ for any $y > 0$,then the last iterated integral is absolute convergent one and after changing the order of integration and integration by parts the composition (18) will have the following representation

$$g(x) = \frac{d}{dx} \int_0^\infty f(y) \frac{\partial}{\partial y} \int_0^\infty \frac{k_1(xt)}{t} h_1(\mu(t)y) \frac{\sqrt{\mu'(t)}}{\mu(t)} dt dy, \tag{24}$$

where the integrated terms vanish according to the properties of the function $f(y)$ and the following estimate of the inside integral follows by the Schwartz inequality

$$\int_0^\infty \left| \frac{k_1(xt)}{t} h_1(\mu(t)y) \frac{\sqrt{\mu'(t)}}{\mu(t)} \right| dt \le$$
$$\left(\int_0^\infty \frac{|k_1(xt)|^2}{t^2} dt \right)^{1/2} \left(\int_0^\infty \frac{|h_1(yv)|^2}{v^2} dv \right)^{1/2} \le B\sqrt{xy},$$

B-const.

If we denote by

$$K^\mu_{kh}(x,y) = \frac{\partial}{\partial y} \int_0^\infty \frac{k_1(xt)}{t} h_1(\mu(t)y) \frac{\sqrt{\mu'(t)}}{\mu(t)} dt, \tag{25}$$

then we obtain almost for all $x > 0$ the following general transform

$$g(x) = \frac{d}{dx} \int_0^\infty K^\mu_{kh}(x,y) f(y) dy, \tag{26}$$

or analogously as for the composition (19) we have the conjugate index transform

$$\hat{g}(x) = \frac{d}{dx} \int_0^\infty K^\mu_{\hat{k}\hat{h}}(x,y) \overline{f}(y) dy. \tag{27}$$

Let now $f(y)$ be an arbitrary function from $L_2(R+)$. By substitution $\mu(t) = v$ in (25) and by the properties of Watson kernels $k_1(x)$, $h_1(x)$ it is not difficult to show that $K^\mu_{kh}(x,y)$ belongs to $L_2(R+)$ for almost all $x > 0$, and analogously $K^\mu_{\hat{k}\hat{h}}(x,y) \in L_2(R+)$. Therefore, from the inequalities (21), (22) the operators (26), (27) are bounded in $L_2(R+)$ and defined for any function $f(y) \in L_2(R+)$, since the set of C^∞-functions with compact support on $(0,\infty)$ is dense everywhere in $L_2(R+)$.

If we consider the equality (20) for two different functions

$f(y)$, $\hat{f}(y) \in L_2(R+)$ and their general transforms $g(x)$, $\hat{g}(x)$

$$\int_0^\infty g(u)\hat{g}(u)du = \int_0^\infty f(y)\hat{f}(y)dy \tag{28}$$

then taking $\hat{f}(y) = 1$, $0 < y \le x$, $\hat{f}(y) = 0$, $y > x$ and evaluating

$$\hat{g}(u) = K^{\mu}_{\hat{k}\hat{h}}(u, x) \in L_2(R+)$$

for almost all $x > 0$, where

$$\hat{K}^{\mu}_{\hat{k}\hat{h}}(u, x) = \frac{\partial}{\partial u}\int_0^\infty \frac{k_1(ut)}{t}\hat{h}_1(\mu(t)x)\frac{\sqrt{\mu'(t)}}{\mu(t)}dt, \tag{29}$$

we get immediately the dual formula

$$f(x) = \frac{d}{dx}\int_0^\infty \hat{K}^{\mu}_{\hat{k}\hat{h}}(u, x)g(u)du \tag{30}$$

or analogously from (27) we have

$$f(x) = \frac{d}{dx}\int_0^\infty \hat{K}^{\mu}_{\hat{k}\hat{h}}(u, x)\hat{g}(u)du. \tag{31}$$

Theorem 2.1 *Let $\mu(x)$ be a monotonically increasing positive function on $R+$ which is differentiable and $\mu(0) = 0$, $\mu'(0) \neq 0$. The general transforms (26) and (27) map $L_2(R+)$ onto $L_2(R+)$ and the Parseval equality (20) takes place with the inequalities (21)–(22). Moreover, the dual formulae (30), (31) are true almost everywhere on the $R+$.*

Now we can get analogues of Watson condition (15) for the general transforms (26), (27). Since (25), (29) yield that

$$K^{\mu}_{kh}(x, y) = \lim_{N\to\infty}\int_0^N \frac{k_1(xt)}{t}h(\mu(t)y)\sqrt{\mu'(t)}dt, \tag{32}$$

$$\hat{K}^{\mu}_{\hat{k}\hat{h}}(x, y) = \lim_{N\to\infty}\int_0^N \hat{k}(ut)\hat{h}_1(\mu(t)x)\frac{\sqrt{\mu'(t)}}{\mu(t)}dt, \tag{33}$$

then according to the Parseval equality (6) we have

$$\int_0^\infty K^{\mu}_{kh}(x, u)K^{\mu}_{\hat{k}\hat{h}}(y, u)du = \int_0^N \frac{k_1(xt)\hat{k}_1(yt)}{t^2}dt = \min(x, y), \tag{34}$$

$$(35)\quad \int_0^\infty \hat{K}^{\mu}_{kh}(u,x)\hat{K}^{\mu}_{\hat{k}\hat{h}}(u,y)du = \int_0^N \frac{h_1(x\mu(t))\hat{h}_1(y\mu(t))}{\mu^2(t)}d\mu(t) = \min(x,y).$$

Hence, as in Section 1 we conclude that the transforms pairs (26), (30) and (27), (31) are dual ones iff the conditions (34), (35) are fulfilled.

Now we apply the theory of the Mellin transform (4) in $L_2(R+)$ [5] to the pairs (26)-(30) and (27)-(31) in order to fill the set of new integral transforms.

Let $f(x) \in L_2(x^{2\gamma-1}; R+) \cap L_2(x^{1-2\gamma}; R+)$, $1/2 < \gamma < 1$, where the space $L_2(\mu(x); R+)$ denotes the space of square summable functions with a weight $\mu(x)$. From the Hölder inequality we find that $f(x) \in L_2(R+)$ in this case and moreover, from the Parceval equality (20) and the Mellin-Parseval equality [5] it follows that

$$(36)\qquad \int_0^\infty g(u)\hat{g}(u)du = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)f^*(1-s)ds,$$

where $f^*(s)$ denotes the Mellin transform (4) of the function $f(x)$ in $L_2(x^{2\gamma-1}; R+)$ and $f^*(s), f^*(1-s) \in L_2(\gamma - i\infty, \gamma + i\infty)$.

Let the kernel $h_1(x)/x \in L_2(x^{2\gamma-1}; R+) \cap L_2(R+)$, $1/2 < \gamma < 1$. Then in the space of smooth functions we use the representation (23) and according to the definitions (13), (14) of Watson kernels with the Mellin-Parseval equality and the changing of the order of integration by the property 1) from Definition 1.1 we will have

$$\begin{aligned} g(x) &= -\frac{d}{dx}\int_0^\infty \frac{k_1(xt)}{t}\frac{\sqrt{\mu'(t)}}{\mu(t)}\int_0^\infty h_1(\mu(t)y))f'(y)dydt \\ (37)\qquad &= -\lim_{N\to\infty}\int_{1/N}^N k(xt)\sqrt{\mu'(t)}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{h^*(s)}{1-s}F(1-s)(\mu(t))^{-s}ds = \\ &= -\lim_{N\to\infty}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{h^*(s)}{1-s}\mathcal{M}_k(x,1-s,N)F(1-s)ds, \end{aligned}$$

where $F(1-s) \in L_2(\gamma - i\infty, \gamma + i\infty)$ and

$$(38)\qquad F(s) = \int_0^\infty y^s f'(y)dy,$$

$$(39)\qquad \mathcal{M}_k(x,s,N) = \int_{1/N}^N k(xt)\sqrt{\mu'(t)}(\mu(t))^{s-1}dt.$$

But integration by parts in (38) for C^∞-functions with the compact supports gives $F(s) = -sf^*(s)$. So we get finally,

$$(40)\qquad g(x) = \lim_{N\to\infty}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} h^*(s)\mathcal{M}_k(x,1-s,N)f^*(1-s)ds$$

and similarly for $\hat{g}(x)$,

$$\hat{g}(x) = \lim_{N\to\infty} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \hat{h}^*(s)\mathcal{M}_{\hat{k}}(x, 1-s, N)f^*(1-s)ds, \tag{41}$$

if the kernel $\hat{h}_1(x)/x \in L_2(x^{2\gamma-1}; R+) \cap L_2(R+)$, $1/2 < \gamma < 1$.Substituting in (39) $\mu(t) = v$ we obtain

$$\mathcal{M}_k(x, 1-s, M) = \int_{1/M}^{M} \frac{k(x\mu^{-1}(v))}{\sqrt{\mu'(\mu^{-1}(v))}} v^{-s}dv \tag{42}$$

and from the condition $v^2\mu^{1-2\gamma}(v) < C$, C-const for all $v > 0$ it follows that for all $x > 0$ $\frac{k(x\mu^{-1}(v))}{\sqrt{\mu'(\mu^{-1}(v))}} \in L_2(v^{1-2\gamma}; R+)$. Thus, we get that

$$\lim_{N\to\infty} \mathcal{M}_k(x, 1-s, M) = \mathcal{M}_k(x, 1-s) \in L_2(\gamma - i\infty, \gamma + i\infty).$$

Now we modify the condition (5) on the Watson kernels, namely: let for the kernel $k(y)$ (similarly for the kernel $\hat{k}(y)$) the following inequality hold

$$\left| \int_{\epsilon}^{E} k(xu)\sqrt{\mu'(u)}(\mu(u))^{-\gamma-it}du \right| < C, \quad 1/2 < \gamma < 1, \tag{43}$$

where ϵ, E and C are some constants, $X > 0, T \in R$. So, in these cases according to the Lebesgue theorem we have that the integrals (40)–(41) converge uniformly in N for all $x > 0$ and the corresponding limits coincide almost everywhere on $R+$ with the limits in square and we get the following formulas

$$g(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} h^*(s)\mathcal{M}_k(x, 1-s)f^*(1-s)ds, \tag{44}$$

$$\hat{g}(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \hat{h}^*(s)\mathcal{M}_{\hat{k}}(x, 1-s)f^*(1-s)ds, \tag{45}$$

where

$$\mathcal{M}_k(x, 1-s) = \int_0^\infty k(xt)\sqrt{\mu'(t)}(\mu(t))^{-s}dt. \tag{46}$$

Now for the arbitrary function $f^*(s)$ such that $f^*(s)$, $f^*(1-s) \in L_2(\gamma - i\infty, \gamma + i\infty)$ from the Parseval equalities for the Mellin transform [5]:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |f^*(\gamma + it)|^2 dt = \int_0^\infty y^{2\gamma-1}|f(y)|^2 dy,$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|f^*(1-\gamma+it)|^2dt=\int_0^{\infty}y^{1-2\gamma}|f(y)|^2dy$$

it follows that $f(x)\in L_2(x^{2\gamma-1};R+)\cap L_2(x^{1-2\gamma};R+)\subset L_2(R+)$, $1/2<\gamma<1$ and from the estimates (21)–(22) we have

$$\int_0^{\infty}|g(t)|^2dt<A\int_{-\infty}^{\infty}|f^*(\gamma+it)|^2dt, \tag{47}$$

$$\int_0^{\infty}|\hat{g}(t)|^2dt<A_1\int_{-\infty}^{\infty}|f^*(\gamma+it)|^2dt,\quad A,A_1-const.$$

So, the equality (44) is true for all functions from $L_2(\gamma-i\infty,\gamma+i\infty)\cap L_2(1-\gamma-i\infty,1-\gamma+i\infty)$ and the integral transform (44) maps them in $L_2(R+)$. From (28) we have the analogue of equality (36) for two different functions $f^*(s)$, $\hat{f}^*(1-s)$. Setting $\hat{f}(1-\gamma-it)=1$, $-\infty<t\le\tau$, $\hat{f}(1-zg-it)=0$, $t>\tau$ from the Parseval equality (36) it follows that almost everywhere on R the dual formula holds valid

$$f^*(\gamma+i\tau)=\frac{d}{d\tau}\int_0^{\infty}g(u)\int_{-\infty}^{\tau}\hat{h}^*(\gamma+it)\mathcal{M}_{\hat{k}}(u,1-\gamma-it)dtdu. \tag{48}$$

Theorem 2.2 *Let $\mu(x)$ be a monotonically increasing positive function on R which is differentiable,$\mu(0)=0,\mu'(0)\neq 0$,and $x^2\mu^{1-2\gamma}(x)<C,C-const$, $x>0$, $1/2<\gamma<1$. Let also the Watson kernels $h_1(x)$, $\hat{h}_1(x)$ be so that $h_1(x)/x\in L_2(x^{2\gamma-1};R+)\cap L_2(R+)$, $1/2<\gamma<1$, $\hat{h}_1(x)/x\in L_2(x^{2\gamma-1};R+)\cap L_2(R+)$, $1/2<\gamma<1$, and for the kernels $k_1(x)$ and $\hat{k}_1(x)$ the inequality (43) holds. Then, for arbitrary function $f^*(s)$ such that $f^*(s)$, $f^*(1-s)\in L_2(\gamma-i\infty,\gamma+i\infty)$ the general transform (44) belongs to $L_2(R+)$ and the Parseval equality (36) is true, where $\hat{g}(x)$ is defined by (45) with the estimations (47). Moreover, the dual formula (48) takes place almost everywhere on R.*

3. Examples

Let us consider the cases of transforms (26), (27), (44), when $\mu(t)=e^t-1$. Moreover, we set $k(x)=\hat{k}(x)=\sqrt{2/\pi}\cos(x)$ or $k(x)=\hat{k}(x)=\sqrt{2/\pi}\sin(x)$. Then, after simple manipulations using the condition (5) for the kernel $h(x)$ from the representation (25) we get immediately that

$$\frac{\partial}{\partial x}K^{e^t-1}_{\left\{\cos \atop \sin\right\}}(x,y)=\sqrt{\frac{2}{\pi}}\left\{\Re \atop \Im\right\}\int_0^{\infty}\frac{h(yt)dt}{(t+1)^{\frac{1}{2}-ix}}=\left\{\Re \atop \Im\right\}S_h(y,\frac{1}{2}-ix), \tag{49}$$

where the integral in (49) is meant as improper according to condition (5). So we have the following pairs of integral transforms which map the space $L_2(R+)$ onto $L_2(R+)$:

$$g(x) = \lim_{N\to\infty} \int_0^N \left\{ \begin{smallmatrix} \Re \\ \Im \end{smallmatrix} \right\} S_h(y, 1/2 - ix) f(y) dy, \tag{50}$$

$$f(x) = \lim_{N\to\infty} \int_0^N \left\{ \begin{smallmatrix} \Re \\ \Im \end{smallmatrix} \right\} S_{\hat{h}}(x, 1/2 - iy) g(y) dy, \tag{51}$$

As an example of the pairs (50)-(51) we consider them when $h(y) = \hat{h}(y) = G^{m,n}_{2n,2m}\left(y \left| {(\alpha_{2n}) \atop (\beta_{2m})} \right.\right)$ is the Meijer's G-function [3], where $m, n \in N$, $m \neq n$, $(\alpha_{2n}) = \alpha_1, ..., \alpha_{2n}$, $(\beta_{2m}) = \beta_1, ..., \alpha_{2m}$, $\alpha_j \in R$, $\alpha_{n+j} = -\alpha_j$, $j = \overline{1,n}$ $\beta_j \in R$, $\beta_{m+j} = -\beta_j$, $j = \overline{1,m}$. As it is known from [6], these G-functions are Fourier kernels and the corresponding integral transforms pairs (50)-(51) are unitary in $L_2(R+)$, i.e. setting $\hat{f}(x) = \overline{f(x)}$, $\hat{g}(x) = \overline{g(x)}$, from (28) it follows the Parceval equality in the form

$$\int_0^\infty |g(x)|^2 dx = \int_0^\infty |f(y)|^2 dy. \tag{52}$$

The integral (49) can be evaluated according to [3], Section 2.24.2, formula (4) and we have

$$\begin{aligned} \left\{ \begin{smallmatrix} \Re \\ \Im \end{smallmatrix} \right\} \quad S_g(y, 1/2 - ix) &= \sqrt{\frac{2}{\pi}} \left\{ \begin{smallmatrix} \Re \\ \Im \end{smallmatrix} \right\} \int_0^\infty \frac{1}{(t+1)^{1/2-ix}} G^{m,n}_{2n,2m}\left(yt \left| {(\alpha_{2n}) \atop (\beta_{2m})} \right.\right) dt. \\ &= \sqrt{\frac{2}{\pi}} \left\{ \begin{smallmatrix} \Re \\ \Im \end{smallmatrix} \right\} \left[\frac{1}{\Gamma(1/2 - ix)} G^{m+1,n+1}_{2n+1,2m+1}\left(y \left| {0,(\alpha_{2n}) \atop -1/2-ix,(\beta_{2m})} \right.\right) \right]. \end{aligned} \tag{53}$$

So we get two new pairs of integral transforms (50)–(51) with Meijer's G-functions as kernels, namely:

$$\begin{aligned} g(x) &= \sqrt{\frac{2}{\pi}} \lim_{N\to\infty} \int_0^N \left\{ \begin{smallmatrix} \Re \\ \Im \end{smallmatrix} \right\} \left[\frac{1}{\Gamma(\frac{1}{2} - ix)} G^{m+1,n+1}_{2n+1,2m+1}\left(y \left| {0,(\alpha_{2n}) \atop -\frac{1}{2}-ix,(\beta_{2m})} \right.\right) \right] f(y) dy, \\ f(x) &= \sqrt{\frac{2}{\pi}} \lim_{N\to\infty} \int_0^N \left\{ \begin{smallmatrix} \Re \\ \Im \end{smallmatrix} \right\} \left[\frac{1}{\Gamma(\frac{1}{2} - iy)} G^{m+1,n+1}_{2n+1,2m+1}\left(x \left| {0,(\alpha_{2n}) \atop -\frac{1}{2}-iy,(\beta_{2m})} \right.\right) \right] g(y) dy. \end{aligned} \tag{54}$$

Further, we consider new pairs of integral transforms (50)–(51) with Lommel functions $S_{\mu,\nu}(z)$ [1] as kernels when a) $h(y) = \hat{h}(y) = \sqrt{2/\pi}\sin(y)$ and b)

$h(y) = \hat{h}(y) = \sqrt{2/\pi}\cos(y)$. From the Mellin transform formula (See [3], Section 8.4.27, formula (3)), the following representations follow:

$$\int_0^\infty \frac{\sin(yt)dt}{(t+1)^{1/2-ix}} = y^{-ix} S_{ix,1/2}(y),$$

$$\int_0^\infty \frac{\cos(yt)dt}{(t+1)^{1/2-ix}} = 4y^{-2}(1/2-ix)(y/4)^{ix} S_{1-ix,1/2}(y)$$

and we get immediately in the case a)

$$(55)\qquad \begin{aligned} g(x) &= 2/\pi \lim_{N\to\infty} \int_0^N \begin{Bmatrix}\Re\\ \Im\end{Bmatrix} \left[y^{-ix} S_{ix,1/2}(y)\right] f(y)dy,\\ f(x) &= 2/\pi \lim_{N\to\infty} \int_0^N \begin{Bmatrix}\Re\\ \Im\end{Bmatrix} \left[x^{-iy} S_{iy,1/2}(x)\right] g(y)dy,\end{aligned}$$

and in the case b)

$$(56)\qquad \begin{aligned} g(x) &= \frac{8}{\pi} \lim_{N\to\infty} \int_0^N \begin{Bmatrix}\Re\\ \Im\end{Bmatrix} \left[(1/2-ix)(y/4)^{ix} S_{-1+ix,1/2}(y)\right] y^{-2} f(y)dy,\\ f(x) &= \frac{8}{\pi} x^{-2} \lim_{N\to\infty} \int_0^N \left\{\Re\_Im\right\} \left[(1/2-iy)(x/4)^{iy} S_{-1+iy,1/2}(x)\right] g(y)dy.\end{aligned}$$

In both cases (55)–(56) the Parseval equality (52) is true.

Finally we get a particular case of transform (44), when $\mu(t) = e^t - 1$, $k(x) = \hat{k}(x) = \sqrt{2/\pi}\cos(x)$ or $k(x) = \hat{k}(x) = \sqrt{2/\pi}\sin(x)$ and $h(x)$, $\hat{h}(x)$ are arbitrary conjugate kernels so that the corresponding Watson kernels satisfy the conditions of the theorems mentioned above. From the known representation for the Beta-function, using (46) we obtain that

$$(57)\qquad \begin{aligned} \mathcal{M}_{\left\{\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\right\}}(x,1-s) &= \Gamma(1-s) \begin{Bmatrix}\Re\\ \Im\end{Bmatrix} \frac{\Gamma(s-1/2-ix)}{\Gamma(1/2-ix)}\\ &= \Gamma(1-s)\left[\frac{\Gamma(s-1/2-ix)}{\Gamma(1/2-ix)} \pm \frac{\Gamma(s-1/2-ix)}{\Gamma(1/2+ix)}\right].\end{aligned}$$

Thus the pair (44), (48) will be as follows

$$(58)\qquad g(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} h^*(s)\Gamma(1-s) \begin{Bmatrix}\Re\\ \Im\end{Bmatrix} \frac{\Gamma(s-1/2-ix)}{\Gamma(1/2-ix)} f^*(1-s)ds,$$

$$1/2 < \gamma < 1,$$

$$(59) \qquad f^*(\gamma + i\tau) = \frac{d}{d\tau}\int_0^\infty g(u)\int_{-\infty}^{\tau} \hat{h}^*(\gamma + it)\Gamma(1-\gamma-it) \times \times \left\{ \begin{matrix} \Re \\ \Im \end{matrix} \right\} \frac{\Gamma(\gamma + it) - 1/2 - iu)}{\Gamma(1/2 - iu)} dtdu.$$

If $h(x) = \hat{h}(x)$ and the kernels $h(x)$, $\hat{h}(x)$ are real functions, then from the Parceval equality (36) for the pair (58), (59) we have the following relation

$$(60) \qquad \int_0^\infty |g(x)|^2 dx = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} f^*(s)\overline{f^*(1-\bar{s})}ds.$$

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Department of Mathematics and Mechanics
Byelorussian State University
P.O. Box 385, Minsk-50, 220050
BELARUS

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