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On Fixed Point Theorems in Banach Spaces Over Topological Semifields

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1. Introduction

The notion of topological semifield has been introduced by the mathematicians M. Antonovski, V. Boltjanski and Ṫ. Sarymsakov in [1]. Let E be a topological semifield and K the set of all its positive elements. Take any two elements x, y in E. If $y - x$ is in $\overline{K}$ (in K), this is denoted by $x \ll y$ ($x < y$). As proved in [1], every topological semifield E contains a subsemifield isomorphic to the field R of real numbers.

Linear spaces considered in this paper are defined on the field R. Let X be a linear space. The ordered triple $(X, \| \|, E)$ is called the feeble normed space over the topological semifield if there exists a mapping $\| \| \, X \mapsto E$ satisfying the usual axioms for a norm (see [1] and [3]).

Let $(X, \| \|, E)$ be a feeble normed space over a topological semifield E and let $d(x, y) = \|x - y\|$ for all x, y in X. A space $(X, \| \|, E)$ is said to be Banach over topological semifield E if (X, d, E) is sequentially complete metric space over a topological semifield E.

The results of this paper are inspired by the results of Ghosh [2] and Rhoades [4]. We prove some theorems on fixed points for mapping in Banach space over topological semifield.

2. Main Results

Theorem 1. *Let X be a Banach space over a topological semifield E and $T : X \mapsto X$ be a mapping satisfying*

$$\|x - Tx\| + \|y - Ty\| \ll p\|x - y\| \tag{1}$$

for all x, y in X, where p, t are in R, $0 < t < 1$ and $1 \leq pt < 2$. Then the sequence $\{x_n\}_{n=0}^{\infty}$, the members of which are

$$x_{n+1} = (1-t)x_n + tTx_n, \quad n = 0, 1, 2, \ldots; \; x_0 \in X \tag{2}$$

converges to the fixed point of T in X.

Proof. Let x_0 in X be an arbitrary point. From (2), we get

$$\|x_{n+1} - x_n\| = t\|Tx_n - x_n\|. \tag{3}$$

If in (1) we put $x = x_{n-1}$ and $y = x_n$, then by (3) we have

$$t^{-1}(\|x_n - x_{n-1}\| + \|x_{n+1} - x_n\|) \ll p\|x_{n-1} - x_n\|$$

and, hence,

$$\|x_{n+1} - x_n\|) \ll (pt - 1)\|x_n - x_{n-1}\|.$$

Since $0 \leq pt - 1 < 1$ it follows that $\{x_n\}$ is a Cauchy sequence in X. Because X is a Banach space over the topological semifield E we deduce that $\{x_n\}$ converges to a point u in X.

Now putting $x = u$ and $y = x_n$ in (1) we have

$$\|u - Tu\| + \|x_n - Tx_n\| \ll p\|u - x_n\|.$$

If now n tends to infinity one has $\|u - Tu\| \ll 0$, which implies $Tu = u$. ∎

Theorem 2. *Let X be a Banach space over a topological semifield E and T_1, T_2 $X \mapsto X$ two maps satisfying the condition*

$$\|x - T_1x\|^2 + \|y - T_2y\|^2 \ll p\|x - y\|^2 \tag{4}$$

for all x, y in X, where p, t are in R, $0 < t < 1$ and $1 \leq pt^2 < 2$. Then the sequence $\{x_n\}_{n=0}^{\infty}$, the members of which are

$$(5) \qquad \begin{aligned} x_{2n+1} &= (1-t)x_{2n} + tT_1x_{2n}, \\ x_{2n+2} &= (1-t)x_{2n+1} + tT_2x_{2n+1}, \qquad n = 0,1,2,\ldots, x_0 \in X, \end{aligned}$$

converges to the common fixed point of T_1 and T_2 in X.

Proof. Let x_0 in X be an arbitrary point. From (5), we get

$$(6) \qquad \begin{aligned} \|x_{2n+1} - x_{2n}\| &= t\|T_1x_{2n} - x_{2n}\|, \\ \|x_{2n+2} - x_{2n+1}\| &= t\|T_2x_{2n+1} - x_{2n+1}\|. \end{aligned}$$

If in (4) we put $x = x_{2n}$ and $y = x_{2n+1}$, then by (6) we have

$$t^{-2}(\|x_{2n+1} - x_{2n}\|^2 + \|x_{2n+2} - x_{2n+1}\|^2) \ll p\,\|x_{2n} - x_{2n+1}\|^2$$

and, hence,

$$(7) \qquad \|x_{2n+2} - x_{2n+1}\| \ll (pt^2 - 1)^{\frac{1}{2}}\|x_{2n} - x_{2n+1}\|$$

for all n. Now, if we put in (4) $x = x_{2n+2}$ and $y = x_{2n+1}$, and use (6), we get

$$t^{-2}(\|x_{2n+3} - x_{2n+2}\|^2 + \|x_{2n+2} - x_{2n+1}\|^2) \ll p\|x_{2n+2} - x_{2n+1}\|^2$$

and, hence,

$$(8) \qquad \|x_{2n+3} - x_{2n+2}\| \ll (pt^2 - 1)^{\frac{1}{2}}\|x_{2n+2} - x_{2n+1}\|$$

for all n. From (7) and (8) then we obtain

$$\|x_n - x_{n+1}\| \ll (pt^2 - 1)^{\frac{1}{2}}\|x_{n-1} - x_n\|$$

which implies

$$\|x_n - x_{n+1}\| \ll (pt^2 - 1)^{\frac{n}{2}}\|x_0 - x_1\|.$$

Since $0 \leq pt^2 - 1 < 1$ it follows that $\{x_n\}$ is a Cauchy sequence in X. Because X is a Banach space over the topological semifield E we deduce that $\{x_n\}$ converges to a point u in X.

Now putting $x = u$ and $y = x_{2n+1}$ in (4) we have

$$\|u - T_1u\|^2 + \|x_{2n+1} - T_2x_{2n+1}\|^2 \ll p\|u - x_{2n+1}\|^2.$$

If now n tends to infinity one has $\|u - T_1u\|^2 \ll 0$, which implies $T_1u = u$. Hence, u is a fixed point for T_1. Similarly, $T_2u = u$. So u is a common fixed point of T_1 and T_2. This completes the proof. ■

References

1. M. Antonovski, V. Boltjanski and T. Sarymsakov. Topological semifields, Tashkent, 1960.
2. K.M. Ghosh. *A note on a theorem of Rhoades*, Math. Sem. Notes, **8**, 505-507, (1980).
3. S. Kasahara. *On formulations of topological linear spaces by topological semifields*, Math. Sem. Notes, **1**, 11-29, (1973).
4. B.E. Rhoades. *Extensions of some fixed point theorems of Ćirić, Maiti and Pal*, Math. Sem. Notes, **6**, 41-46, (1978).

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