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Properties of the Contrajectory in Support Maximum Principle for Time - Delayed Systems with Functional Restrictions

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In the main part of published papers about systems with delay finite dimensional boundary conditions are considered, as in ordinary systems. The consideration of functional boundary conditions meets serious difficulties [1,2]. The present paper is connected with the results of [1,2], where, using the constructive approach [3], the support maximum principle is proved for a system with time-lag and functional terminal restrictions. The present research is devoted to the properties of the support contrajectory.

We consider the system

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + A_1x(t-h) + bu(t), t \in [0, t^* + h] = T, \\ x(\tau) &= x_0(\tau), \tau \in [-h, 0[, x(0) = x^0, \end{aligned} \tag{1}$$

on the trajectories of which the linear terminal criterion is to be maximized:

$$J(u) = c'x(t^* + h) \to max. \tag{2}$$

In (1)-(2) $x(t) \in R^n, u(t) \in R^1; c, b \in R^n; A_0, A_1$ are $n \times n$ - matrices; $h > 0$; $x_0(\tau)$ is a piece-wise continuous n - vector function.

Every piece-wise continuous function $u(t)$, $t \in T$, will be called an admissible control, if

$$|u(t)| \leq 1, t \in T, \tag{3}$$

and for the corresponding trajectory of (1) the next terminal restriction is satisfied

$$d'x(t) = y, t \in T^* = [t^*, t^* + h], \tag{4}$$

where d, y are given n-vector and a scalar. Here and further we shall use for transposition the sign$'$.

The basic means to give an account of the constraints in the constructive approach is support [3]. When applying the constructive approach for problems with finite dimensional boundary conditions the support is concentrated, consists of limited number of points. The functional constraints for a time-delayed system considerably complicate the problem, causing the necessity of principle generalization [1,2] of concepts of the method for ordinary systems.

Let us chose the segment $T^* = [t^*, t^* + h]$ points $\mu_i, i = \overline{0, \rho + 1}, \rho \geq 0, \mu_0 = t^*, \mu_{\rho+1} = t^* + h, \mu_i < \mu_{i+1}, i = \overline{0, \rho}$. To every segment $T_i = [\mu_i, \mu_i + 1]$ let us put in correspondence an integer k_i ("depth" of T_i), $k_i \leq m$, $(m \leq [t^*/h])$, $k_i \neq k_{i+1}$, $i = \overline{0, \rho - 1}$. For convenience we put $k_{-1} = k_{\rho+1} = -1$. Let $I = \{\overline{0, \rho}\}$. We introduce also the segments

$$\tilde{T}_i = [\mu_i, \mu_{i+1}], i \in I^{++} = \{i \in I : k_i > k_{i-1}, k_i > k_{i+1}\},$$

$$\tilde{T}_i = [\mu_i, \mu_{i+1}[, i \in I^{+-} = \{i \in I : k_i > k_{i-1}, k_i < k_{i+1}\},$$

$$\tilde{T}_i =]\mu_i, \mu_{i+1}], i \in I^{-+} = \{i \in I : k_i < k_{i-1}, k_i > k_{i+1}\},$$

$$\tilde{T}_i =]\mu_i, \mu_{i+1}[, i \in I^{--} = \{i \in I : k_i < k_{i-1}, k_i < k_{i+1}\},$$

and the sets of indexes

$$I^+ = I^{++} \cup I^{+-}, I^- = I^{-+} \cup I^{--}, I_p = \{i \in I : k_i = p\}.$$

In intervals $\tilde{T}_i$ we chose points $\tau_{ij}, j = \overline{1, s_i}, \tau_{ij} < \tau_{i,j+1}, j = \overline{1, s_i - 1}$. Denote

$$\mu_i^s = \mu_i - sh, \tau_{ij}^s = \tau_{ij} - sh, T_i^s = [\mu_i^s, \mu_{i+1}^s],$$

$$T_0 = \bigcup_{i=0}^{\rho} \bigcup_{s=0}^{k_i - 1} T_i^s, T_{sup} = \bigcup_{i=0}^{k} T_i^{k_i}, T_n = T \setminus T_0,$$

$$T_{nn} = T_n \setminus T_{sup};$$

$$S_i = \begin{cases} \emptyset, & i \in I^-, \\ \{k_{i-1} + 1, k_{i-1} + 2, ..., k_i\}, & i \in I^+, \end{cases}$$

$$K_0 = \sum_{i=0}^{\rho} (|S_i| + s_i).$$

The physical sense of the "depth" k_i of T_i is that for keeping the identity (4) on T_i the control $u(t)$ is found on $T_i^{k_i}$ in form of a feedback [1].

On the set T_0 we introduce functions $f(t)$, $t \in T^*$, $\Delta u_*(t)$, $t \in T_i^s$, $s = \overline{0, k_i - 1}, i = \overline{0, \rho}$, with following properties:

1. $f(t) \in C^{k_i+1}, t \in T_i, i = \overline{0, \rho}$, i.e. $f(t), t \in T_i$, is continuous and has continuous derivatives up to the k_i - order inclusive and piece wise continuous derivative of $(k_i + 1)$ - order;

2. $f(t) \in C^{k_i+2}$ in neighbourhoods of the points $t \in \{\tau_{ij}, j = \overline{1, s_i}\} \setminus \{\mu_i, \mu_{i+1}\}, i = \overline{0, \rho}, f(t) \in C^{min\{k_i+1, k_{i+1}+1\}}$ in neighbourhoods of the points $\mu_{i+1}, i = \overline{0, \rho}$.

3. $\Delta u_*(t) \in C^{k_i - s+1}, t \in T_i^s, s = \overline{0, k_i - 1}, i = \overline{0, \rho}$;

4. a) if $s < min\{k_i + 1, k_{i+1} + 1\}$, then $\Delta u * (t) \in C^{min\{k_i - s+1, k_{i+1} - s+1\}}$ in neighbourhoods of the points $\mu_{i+1}^s, i = \overline{0, \rho}$;

b) if $s < k_i$, then $\Delta u * (t) \in C^{k_i - s+2}$ in neighbourhoods of points $t \in \{\tau_{ij}^s, j = \overline{1, s_j}\} \setminus \{\mu i^s, \mu_{i+1}^s\}\ i = \overline{0, \rho}$.

In points $\tau_{ij}^{k_i}$ we assign numbers Δu_{ij}, $j = \overline{1, s_i}$, $i = \overline{0, \rho}$ ($\tau_{i_1}^{k_i} = \mu_i^{k_i} + 0$, if $\tau_{i_1} = \mu_i$ and $\tau_{i,s_i}^{k_i} = \mu_{i+1}^{k_i} - 0$, if $\tau_{i,s_i} = \mu_{i+1}$).

We introduce the support in connection with controlability of time-delayed systems. Together with the system (I) consider the system

$$\Delta \dot{x}(t) = A_0 \Delta x(t) + A_1 \Delta x(t-h) + b \Delta u(t), t \in T, \quad (5)$$
$$\Delta x(t) \equiv 0, t \in [-h, 0].$$

Definition 1. The system (5) is called c o n t r o l l a b l e with respect to the set T_0 and moments $\tau_{i\nu}^{k_i}, \nu = \overline{1, s_i}, i = \overline{0, \rho}$, if for arbitrary functions $f(t), t \in T^*, \Delta u_* t), t \in T_0$, satisfying 1.-4., and numbers $\Delta u_{i\nu}, i = \overline{0, \rho}, \nu = \overline{1, s_i}$, it is possible to find $\Delta u(t), t \in T_n$, such that for the control $\Delta \overline{u}(t) = \Delta u_*(t), t \in T_0; \Delta \overline{u}(t) = \Delta u(t), t \in T_n$, and corresponding trajectory $\Delta x(t)$ of the system (5) the next identity

$$d' \Delta x(t) \equiv f(t), t \in T^*. \quad (6)$$

and equalities

$$\Delta u(\tau_{ij}^{ki}) = \Delta u_{ij}, i = \overline{0, \rho}, i = \overline{1, s_i} \quad (7)$$

are satisfied.

So if we define arbitrary $\Delta u_*(t)$ (submitted to 3.–4.) on T_0, find $\Delta u(t)$ in a form of a feedback on T_{sup}, then for keeping the equalities (6)–(7) it is necessary and sufficient to solve in relation to the function $\Delta u(t), t \in T_{nn} = T_n \setminus T_{sup}$ the finite dimensional boundary problem (6),(7). For the solvability of this problem it is necessary and sufficient [1] that there exist such different moments $\{t_j, j = \overline{1, K_0}\}, t_j \in T_{nn}$, for which the support matrix $Psup$ [1] is invertible.

Definition 2. The totality $S_{sup} = \{T_i, k_i, \tau_{ij}, j = \overline{1, s_i}, i = \overline{0, \rho}, t_k, k = \overline{1, K_0}\}$ is called a support of the problem (1)–(4), if $det P_{sup} \neq 0$ (P_{sup} is a finite dimensional matrix, made up of the parameters of the problem and S_{sup} [1]).

Definition 3. The pair $\{u, S_{sup}\}$ of an admissible control $u(\cdot)$ and a support S_{sup} of the problem is called a support control.

For the deviation of the criterion (2) the next formula is worked out [2].

$$
\begin{aligned}
\Delta J(u) = &\int_0^{t^*+h} \Psi'(t) b \Delta u(t) dt \\
&+ \sum_{i \in I^{+0}} \sum_{p=0}^{k_i} \sum_{l=0}^{p-1} \sum_{s=0}^{p-l-1} v_i^p d_s'(p-l-1) b \Delta u^{(l)}(\mu_i - sh + 0) \\
&+ \sum_{i \in I^{-0}} \sum_{p=0}^{k_{i-1}} \sum_{l=0}^{p-1} \sum_{s=0}^{p-l-1} v_i^p d_s'(p-l-1) b \Delta u^{(l)}(\mu_i - sh - 0) \\
&+ \sum_{i=0}^{\rho} \sum_{j=1}^{s_i} \sum_{p=1}^{k_i+1} \sum_{l=0}^{p-1} \sum_{s=0}^{p-l-1} v_{ij}^p d_s'(p-l-1) b \Delta u^{(l)}(\tau_{ij} - sh),
\end{aligned}
\tag{8}
$$

where $I^{+0} = \{i \in I^+ : \mu_i \neq \tau_{i1}\}, I^{-0} = \{i \in I^- \cup \{\rho + 1\} : \mu_i \neq \tau_{i-1, s_{i-1}}\}$ $I^0 = I^{+0} \cup I^{-0}, v_i^p, v_{ij}^p$ are constants, n-vectors $d_s(p)$ are calculated by the recurrent formulae:

$$d_0(0) = d, d_0'(1) = d' A_0, d_1(1) = d' A_1, d_0'(p) = d_0'(p-1) A_0,$$

$$d_p'(p) = d_{p-1}'(p-1) A_1, d_s'(p) = d_s'(p-1) A_0 + d_{s-1}'(p-1) A_1,$$

$$s = 1, 2, ..., p-1, d_s(p) = 0, s < 0, s > p.$$

The support contrajectory $\Psi(t), t \in T$, is a n-vector -solution of the conjugate system

$$\begin{aligned}(9)\qquad &\dot{\Psi}(t) = -A_0'\Psi(t) - A_1'\Psi(t+h) + \xi(t)d, t \in T,\\ &\xi(t) \equiv 0, t \notin T^*, \Psi(t) \equiv 0, t > t^* + h,\end{aligned}$$

with a boundary condition

$$(10)\qquad \Psi(t^* + h - 0) = c + \sum_{p=0}^{k_\rho+1} \tilde{v}_{\rho+1}^p d_0(p),$$

and jumps

$$\begin{aligned}(11)\qquad &\Psi(t^* - sh - 0) = \Psi(t^* - sh + 0) + \sum_{p=s}^{k_0+1} \tilde{v}_0^p d_s(p)\\ &+ \sum_{p=s+1}^{k_\rho+1} \tilde{v}_{\rho+1}^p d_{s+1}(p), s = \overline{0, max\,\{k_0+1, k_\rho+1\}},\end{aligned}$$

$$\begin{aligned}(12)\qquad &\Psi(\mu_i - sh - 0) = \Psi(\mu_i - sh + 0) + \sum_{p=s}^{m_i} v_i^p d_s(p),\\ &i \in I^0 \setminus \{0, \rho+1\}, s = \overline{0, m_i}, m_i = max\,\{k_i, k_{i-1}\},\end{aligned}$$

$$\begin{aligned}(13)\qquad &\Psi(\tau_{ij} - sh - 0) = \Psi(\tau_{ij} - sh + 0) + \sum_{p=s}^{k_i+1} v_{ij}^p d_s(p), i = \overline{0, \rho},\\ &j = \overline{1, s_i}, s = \overline{0, k_i+1}, \tau_{0,1} \neq t^*, \tau_{\rho,s_\rho} \neq t^* + h,\end{aligned}$$

$$(14)\qquad \tilde{v}_{\rho+1}^p = \begin{cases} v_{\rho+1}^p, & p = \overline{0, k_\rho},\\ 0, & p = k_\rho + 1, \rho + 1 \in I^{-0};\\ v_{\rho,s_\rho}^p, & p = \overline{0, k_\rho + 1}, \rho + 1 \notin I^{-0};\end{cases}$$

$$(15)\qquad \tilde{v}_0^p = \begin{cases} v_0^p, & p = \overline{0, k_0},\\ 0, & p = k_0 + 1, 0 \in I^{+0};\\ v_{0,1}^p, & p = \overline{0, k_0 + 1}, 0 \notin I^{+0};\end{cases}$$

Definition 4. The scalar product $\Delta(t) = \Psi'(t)b, t \in T$, will be called a co-control.

We shall prove the next theorem, describing the properties of the support contrajectory.

Theorem 1. *If a support of the problem (1)–(4) exists, then there exists an unique totality of numbers* $v_i^p, i \in I^0, p = \overline{0, m_i}, v_{ij}^p, i = \overline{0, \rho}, j = \overline{1, s_i}, p = \overline{0, k_i + 1}$, *and a function* $\xi(t), t \in T^*, \xi(t) \equiv 0, t \notin T^*$, *such that the solution of the system (9)–(13) satisfies the conditions:*

$$\Delta(t) = \Psi'(t)b \equiv 0, t \in intT_i^{k_i}, i = \overline{0, \rho}, \tag{16}$$

$$\Delta(t_k) = \Psi'(t_k)b = 0, k = \overline{1, K_0}, \tag{17}$$

Proof. We shall prove the theorem by derivation in another way the formula for the deviation of the criterion (2).

The problem (6)–(7) can be written in a vector form. Let us chose $\Delta u(t), t \in Tnn$, in the form:

$$\Delta u(t) = \Delta u_n(t) + \sum_{k=1}^{K_0} v_k \delta(t - t_k),$$

where $\Delta u_n(t), t \in T_{nn}$, is an arbitrary function, $v_k, k = \overline{1, K_0}$, are uncertain coefficients, $\delta(t)$ is Dyrak's function. Let

$$v = (z_0', z_1', ..., z_{k_*-1}', v_k, k = \overline{1, K_0})',$$

where $z_j = \Delta x(t^* - jh), j = \overline{0, k_* - 1}, k_* = max_{i=\overline{0,\rho}} k_i$. Denote

$$\bar{g}_{k_i+1}(t) = \sum_{j=1}^{k_i} \sum_{s=0}^{k_i - j} d_s'(k_i - j) b \Delta u_*^{(j)}(t - sh) \tag{18}$$

$$+ \sum_{s=0}^{k_i - 1} d_s'(k_i) b \Delta u_*(t - sh), t \in T_i, i = \overline{0, \rho};$$

$$\bar{f}_i(t) = \frac{1}{\alpha_i}(f^{(k_i+1)}(t) - \bar{g}_{k_i+1}(t)), t \in T_i, i = \overline{0, \rho}. \tag{19}$$

Assume

$$\alpha_i \overset{Def}{=} d'_{k_i}(k_i)b = d'A_1^{k_i}b \neq 0, i = \overline{0,\rho}. \tag{20}$$

If

$$\overline{f}(\tau) = \overline{f}_i(\tau + k_i h), \tau \in T_i^{k_i}, i = \overline{0,\rho}, \tag{21}$$

then the equalities (6)–(7) can be written [1] in the form

$$P_{sup}v + \int_{T_{sup}} p(t)\overline{f}(t)dt + \int_{T_0} p(t)\Delta u_*(t)dt + \int_{T_{nn}} p(t)\Delta u_n(t)dt = \eta, \tag{22}$$

where

$$p(t) = \begin{pmatrix} G_{p+1}\Omega(t^* + h, t)b \\ p = \overline{0, k_* - 1} \\ \tau_p'\Omega(\mu_i, t)b \\ i = \overline{0,\rho}, p \in S_i \\ \tau_{k_i+1}'\Omega(\tau_{ij}, t)b \\ i \in I \setminus I_{k_*}, j = \overline{1, s_i} \\ \tau_{k_*+1}'\Omega(\tau_{ij}, t)b + d'_{k_*+1}(k_* + 1)F(\tau_{ij}^{k_*+1}, t)b \\ i \in I_{k_*}, j = \overline{1, s_i} \end{pmatrix}, t \in T,$$

$$\eta = \begin{pmatrix} \left.\begin{matrix} \overline{O} \\ \cdot \\ \cdot \\ \cdot \\ \overline{O} \end{matrix}\right\} k_* \\ \eta_{ip} \\ i = \overline{0,\rho}, p \in S_i \\ \alpha_i \xi_{ij} \\ i = \overline{0,\rho}, j = \overline{1, s_i} \end{pmatrix},$$

G_p is $n \times n(k_* + 1)$ - matrix,

$$G_p = \left(\underbrace{O, \ldots, O}_{p}, E, O, \ldots, O\right),$$

O and E are $n \times n$ -zero and identity matrix. $\Omega(t,\tau) - n(k_* + 1) \times n$ -matrix function, defined in [1], $r_p, \overline{r}_{k_*+1} - n(k_* + 1)$ -vectors

$$r_p' = (d_0'(p), d_1'(p), \ldots, d_p'(p), \overline{O}', \ldots, \overline{O}'),$$

$$\overline{r}'_{k_*+1} = (d'_0(k_*+1), \ldots, d'_{k_*}(k_*+1)),$$

$\overline{O}-$ n-vector,

$$\eta_{ip} = f^{(p)}(\mu_i+0) - \sum_{j=0}^{p-1}\sum_{s=0}^{p-j-1} d^i_s(p-j-1)b\Delta u_*^{(j)}(\mu_i - sh + 0),$$

$$\xi_{ij} = \overline{f}_i(\tau_{ij}) - u_{ij}, i = \overline{0,\rho}, j = \overline{1,s_i}, p \in S_i.$$

From (22) assuming the support S_{sup} exists we get

$$(23)\quad v = P^{-1}_{sup}[\eta - \int_{T_{sup}} p(t)\overline{f}(t)dt - \int_{T_0} p(t)\Delta u_*(t)dt - \int_{T_{nn}} p(t)\Delta u_n(t)dt].$$

The deviation of the criterion (2), using [1], can be written in the form:

$$(24)\qquad \Delta J(u) = c'\Delta x(t^*+h) - \int_{T_0} c'\Omega_0(t^*+h,t)b\Delta u_*(t)dt$$

$$+\int_{T_{sup}} c'\Omega_0(t^*+h,t)b\overline{f}(t)dt + \int_{T_{nn}} c'\Omega_0(t^*+h,t)b\Delta u_n(t)dt + \overline{c}'_{sup}v.$$

Here $\Omega_s(t,\tau), s = \overline{0,k_*}$, is $n\times n$ -matrix, constituted from n-rows of the matrix $\Omega(t,\tau)$,

$$\Omega = \begin{pmatrix} \Omega_0 \\ \Omega_1 \\ \cdot \\ \cdot \\ \cdot \\ \Omega_{k_*} \end{pmatrix}$$

$$\overline{c}'_{sup} = (c'\Omega_0(t^*+h, t^*-ph), p = \overline{0,k_*-1}; c'\Omega_o(t^*+h,t_k)b, k = \overline{1,K_0}).$$

We denote

$$\overline{\delta}(t) = c(t) - \nu' p(t), t \in T; c(t) = c'\Omega_0(t^*+h,t)b, t\in T; \nu' = \overline{c}'_{sup}P^{-1}_{sup},$$

and components of ν respectively

$$\nu' = (\nu'_s, s = \overline{0,k_*-1}; \nu^p_i, i = \overline{0,\rho}, p\in S_i, \nu_{ij}, i = \overline{0,\rho}, j = \overline{1,s_i}),$$

where ν_s are n-vectors, ν_i^p, ν_{ij} -numbers. Putting (23) in (24) and using introduced symbols, we get

$$(25)\quad \Delta J(u) = \int_{T_{nn}} \overline{\delta}(t)\Delta u_n(t)dt + \int_{T_0} \overline{\delta}(t)\Delta u_*(t)dt + \int_{T_{sup}} \overline{\delta}(t)\overline{f}(t)dt + \nu'\eta.$$

As for the admissible control $\overline{u}(t) = u(t) + \Delta u(t), t \in T$, the identity $f(t) \equiv 0, t \in T^*$, holds, then if we write (25) in detail using the definition of $\eta, \nu, \overline{\delta}(t), p(t), G_p$ and r_p, we can express $\overline{\delta}(t)$ as following:

$$\overline{\delta}(t) = \overline{\Psi}'(t)b, t \in T,$$

where

$$(26)\quad \begin{aligned} \overline{\Psi}'(t) &= c'\Omega_0(t^* + h, t) - \sum_{s=0}^{k_*-1} \nu_s'\Omega_{s+1}(t^* + h, t) \\ &- \sum_{i=0}^{\rho}\sum_{p\in S_i} \nu_i^p \sum_{s=0}^{p} d_s'(p)\Omega_s(\mu_i, t) \\ &- \sum_{i\in I\setminus I_{k_*}} \sum_{j=1}^{s_i} \nu_{ij} \sum_{s=0}^{k_i+1} d_s'(k_i + 1)\Omega_s(\tau_{ij}, t) \\ &- \sum_{i\in I_{k_*}} \sum_{j=1}^{s_i} \nu_{ij} \sum_{s=0}^{k_*} d_s'(k_* + 1)\Omega_s(\tau_{ij}, t) \\ &- \sum_{i\in I_{k_*}} \sum_{j=1}^{s_i} \nu_{ij}\alpha_{k_*+1}'(k_* + 1)F(\tau_{ij}^{k_*+1}, t), t \in T, \end{aligned}$$

$F(\tau, t)$ is the fundamental matrix of the homogeneous system, corresponding to (1). Besides, from (21), (19), (18) we obtain

$$\begin{aligned} \overline{f}(t) &= \overline{f}_i(t + k_ih) = -\frac{1}{\alpha_i}\overline{g}_{k_i+1}(t + k_ih) \\ &= -\frac{1}{\alpha_i}\left(\sum_{p=1}^{k_i}\sum_{s=0}^{k_i-p} d_s'(k_i - p)b\Delta u_*^{(p)}(t + (k_i - s)h)\right. \\ &\left. + \sum_{s=0}^{k_i-1} d_s'(k_i)b\Delta u_*(t + (k_i - s)h)\right), t \in T_i^{k_i}, i = \overline{0, \rho}; \end{aligned}$$

$$\begin{aligned}
\nu'\eta &= \sum_{i=0}^{\rho}\sum_{p\in S_i}\nu_i^p\eta_{ip}+\sum_{i=0}^{\rho}\sum_{j=1}^{s_i}\alpha_i\nu_{ij}\xi_{ij}\\
&=-\sum_{i=0}^{\rho}\sum_{p\in S_i}\nu_i^p g_p(\mu_i+0)+\sum_{i=0}^{\rho}\sum_{j=1}^{s_i}\alpha_i\nu_{ij}\left(\overline{f}(\tau_{ij})-u_{ij}\right)\\
&=-\sum_{i=0}^{\rho}\sum_{p\in S_i}\nu_i^p\sum_{l=0}^{p-1}\sum_{s=0}^{p-l-1}d_s'(p-l-1)b\Delta u_*^{(l)}(\mu_i-sh+0)\\
&-\sum_{i=0}^{\rho}\sum_{j=1}^{s_i}\nu_{ij}\sum_{l=1}^{k_i}\sum_{s=0}^{k_i-l}d_s'(k_i-l)b\Delta u_*^{(l)}(\tau_{ij}-sh)\\
&-\sum_{i=0}^{\rho}\sum_{j=1}^{s_i}\nu_{ij}\sum_{s=1}^{k_i-1}d_s'(k_i)b\Delta u_*^{(l)}(\tau_{ij}-sh)-\sum_{i=0}^{\rho}\sum_{j=1}^{s_i}\nu_{ij}\alpha_i u_{ij},
\end{aligned}$$

where

$$g_p(t)=\sum_{j=0}^{p-1}\sum_{s=0}^{p-j-1}d_s'(p-j-1)b\Delta u_*^{(j)}(t-sh),p=\overline{1,k_i},t\in T_i,i=\overline{0,\rho}.$$

As $S_i=\emptyset$ when $i\in I^-$ and from (7), (20) we obtain (25) in the form:

$$\begin{aligned}
\Delta J(u) &= \int_{T_{nn}}\overline{\Psi}'(t)b\Delta u_n(t)dt+\int_{T_0}\overline{\Psi}'(t)b\Delta u_*(t)dt\\
&-\sum_{i=0}^{\rho}\int_{T_i^{k_i}}\sum_{p=1}^{k_i}\sum_{s=0}^{k_i-p}\frac{\overline{\Psi}'(t)b}{\alpha_i}d_s'(k_i-p)b\Delta u_*^{(p)}(t+(k_i-s)h)dt\\
&-\sum_{i=0}^{\rho}\int_{T_i^{k_i}}\sum_{s=0}^{k_i-1}\frac{\overline{\Psi}'(t)b}{\alpha_i}d_s'(k_i)b\Delta u_*(t+(k_i-s)h)dt\\
&-\sum_{i\in I^+}\sum_{p=k_{i-1}+1}^{k_i}\nu_i^p\sum_{l=0}^{p-1}\sum_{s=0}^{p-l-1}d_s'(p-l-1)b\Delta u_*^{(l)}(\mu_i-sh+0)\\
&-\sum_{i=0}^{\rho}\sum_{j=1}^{s_i}\nu_{ij}\sum_{l=0}^{k_i}\sum_{s=0}^{k_i-l}d_s'(k_i-l)b\Delta u_*^{(l)}(\tau_{ij}-sh).
\end{aligned}\tag{27}$$

From the definition (26) of the function $\overline{\Psi}(t)$ and the differential equation for the matrix function $\Omega(t,\tau)$ [1] we obtain the equations:

$$(28)\quad \dot{\overline{\Psi}}'(t) = \begin{cases} -\overline{\Psi}'(t)A_0 - \overline{\Psi}'(t+h)A_1 + \\ \quad + \overline{\Psi}'(t+(p-k_i)h)bd'_p(k_i+1), & t\in T_i^p, i=\overline{0,\rho}, p=\overline{0,k_i}; \\ -\overline{\Psi}'(t)A_0 - \overline{\Psi}'(t+h)A_1 + \\ \quad + \overline{\Psi}'(t+h)\frac{bd'_{k_i+1}(k_i+1)}{\alpha_i}, & t\in T_i^{k_i+1}, i=\overline{0,\rho}; \\ -\overline{\Psi}'(t)A_0 - \overline{\Psi}'(t+h)A_1, & t\in T_i^p, p>k_i+1, i=\overline{0,\rho}; \end{cases}$$

$$(29)\qquad \overline{\Psi}(t) \equiv 0, t > t^* + h;$$

$$(30)\qquad \overline{\Psi}(t^*+h-0) = c - \begin{cases} 0, & \tau_{\rho,s_\rho} \neq t^*+h, \\ \nu_{\rho,s_\rho} d_0(k_\rho+1), & \tau_{\rho,s_\rho} = t^*+h; \end{cases}$$

$$(31)\quad \overline{\Psi}(\mu_i - sh - 0) = \overline{\Psi}(\mu_i - sh + 0) -$$

$$- \sum_{\substack{p=k_{i-1}1 \\ p\geq s}}^{k_i} \nu_i^p d_s(p) - \begin{cases} 0, & \tau_{i1} \neq \mu_i \\ \nu_i d_s(k_i+1), & \tau_{i1} = \mu_i i \in T^+, s=\overline{0,k_i+1}; \end{cases}$$

$$\overline{\Psi}(\tau_{ij} - sh - 0) = \overline{\Psi}(\tau_{ij} - sh + 0) - \nu_{ij} d_s(k_i+1), i \in \overline{0,\rho}, j=\overline{1,s_i},$$

$$(32)\qquad s = \overline{0,k_i+1}, \tau_{0,1} \neq t^*, \tau_{\rho,s_\rho} \neq t^*+h, k_i+1 \leq k_*.$$

Using the definition of $\overline{\Psi}'(t)$, it follows:

$$(33)\qquad \overline{\Psi}'(t_k)b = 0, k=\overline{1,K_0}$$

Let us introduce the vector function

$$\tilde{\Psi}(t) \equiv \overline{\Psi}(t), t \in T_{nn},$$

$$(34)\qquad \tilde{\Psi}(t) = \overline{\Psi}'(t) - \sum_{p=0}^{k_i-s}(-1)^p \frac{\overline{\Psi}^{(p)'}(t+(s-k_i)h)b}{\alpha_i} d'_s(k_i-s),$$

$$t\in T_i^s, s=\overline{0,k_i}, i=\overline{0,\rho}.$$

As $t_k \in T_{nn}, k = \overline{1, K_0}$, then from (33), (34) we have the equalities (17). From the definition of $\tilde{\Psi}(t)$ we get

$$\tilde{\Psi}'(t) = \overline{\Psi}'(t)\left(E - \frac{bd'_{k_i}(k_i)}{\alpha_i}\right), t \in T_i^{k_i}, i = \overline{0, \rho}, \tag{35}$$

and as $d'_{k_i}(k_i)b = \alpha_i$, then from (34) we obtain $\tilde{\Psi}'(t)b = 0$, $t \in intT_i^{k_i}, i = \overline{0, \rho}$, i.e. (16) holds. If $\Delta u(t) = \Delta u_n(t), t \in T_{nn}, \Delta u(t) = \Delta u_*(t), t \in T_0$, then using (16), (34) and after some transformations in the right part of (27), we get

$$\Delta J(u) = \int_0^{t^*+h} \tilde{\Psi}'(t)b\Delta u(t)dt \tag{36}$$

$$-\sum_{i\in I^+}\sum_{p=k_{i-1}+1}^{k_i} \nu_i^p \sum_{l=0}^{p}\sum_{s=0}^{p-l-1} d'_s(p-l-1)b\Delta u^{(l)}(\mu_i - sh + 0)$$

$$-\sum_{i=0}^{\rho}\sum_{j=1}^{si} \nu_{ij}\sum_{l=0}^{k_i}\sum_{s=0}^{k_i-l} d'_s(k_i-l)b\Delta u^{(l)}(\tau_{ij}-sh)$$

$$-\sum_{i\in I^0}\sum_{s=0}^{k_i-1}\sum_{p=1}^{k_i-s}\sum_{l=0}^{p-1}(-1)^{p-l-1}\frac{\overline{\Psi}^{(p-l-1)'}(\tau_{i,1}-k_ih-0)b}{\alpha_i}d'_s(k_i-p)b\Delta u^{(l)}(\tau_{i,1}-sh)$$

$$+\sum_{\substack{i\in I^0\\ i\neq\rho+1}}\sum_{s=0}^{k_i-1}\sum_{p=1}^{k_i-s}\sum_{l=0}^{p-1}(-1)^{p-l-1}\frac{\overline{\Psi}^{(p-l-1)'}(\mu_i-k_ih+0)b}{\alpha_i}\cdot$$

$$\cdot\, d'_s(k_i-p)b\Delta u^{(l)}(\mu_i-sh+0)$$

$$-\sum_{i\in I^0}\sum_{s=0}^{k_{i-1}-1}\sum_{p=1}^{k_{i-1}-s}\sum_{l=0}^{p-1}(-1)^{p-l-1}\frac{\overline{\Psi}^{(p-l-1)'}(\mu_i-k_{i-1}h-0)b}{\alpha_{i-1}}d'_s(k_{i-1}-p)b\Delta u^{(l)}(\mu_i-sh-0)$$

$$+\sum_{i\in I^0}\sum_{s=0}^{k_{i-1}-1}\sum_{p=1}^{k_{i-1}-s}\sum_{l=0}^{p-1}(-1)^{p-l-1}\frac{\overline{\Psi}^{(p-l-1)'}(\tau_{i-1,s_{i-1}}-k_ih+0)b}{\alpha_{i-1}}\cdot$$

$$\cdot\, d'_s(k_{i-1}-p)b\Delta u^{(l)}(\tau_{i-1,s_{i-1}}-sh)$$

$$+\sum_{i=0}^{\rho}\sum_{s=0}^{k_{i-1}-1}\sum_{p=1}^{k_i-s}\sum_{l=0}^{p-1}(-1)^{p-l-1}\frac{\overline{\Psi}^{(p-l-1)'}(\tau_{i,1}-k_ih+0)b}{\alpha_i}\cdot$$

$$\cdot\, d'_s(k_i-p)b\Delta u^{(l)}(\tau_{i,1}-sh+0)$$

$$-\sum_{i=0}^{\rho}\sum_{s=0}^{k_i-1}\sum_{p=1}^{k_i-s}\sum_{l=0}^{p-1}(-1)^{p-l-1}\frac{\overline{\Psi}^{(p-l-1)'}(\tau_{i,s_i}-k_ih-0)b}{\alpha_i}\cdot$$

$$\cdot\, d'_s(k_i-p)b\Delta u^{(l)}(\tau_{i,s_i}-s_ih-0)$$

$$-\sum_{i=0}^{\rho}\sum_{j=2}^{s_i-1}\sum_{s=0}^{k_i-1}\sum_{p=1}^{k_i-s}\sum_{l=0}^{p-1}(-1)^{p-l-1}\left[\frac{\overline{\Psi}^{(p-l-1)'}(\tau_{i,j}-k_ih-0)}{\alpha_i}\right.$$

$$\left.-\frac{\overline{\Psi}^{(p-l-1)'}(\tau_{i,j}-k_ih+0)}{\alpha_i}\right]bd'_s(k_i-p)b\Delta u^{(l)}(\tau_{i,j}-sh)$$

We shall obtain a differential equation for $\tilde{\Psi}(t)$. First, for every ρ we will find $\overline{\Psi}^{(p)}$, $t\in T_i^{k_i}, i=\overline{0,\rho}$. The relation

$$\overline{\Psi}'^{(p)}(t)=(-1)^p\sum_{s=0}^{p}\overline{\Psi}'(t+sh)A_{i,s}^{(p)},\quad p=1,2,\dots,t\in T_i^{k_i},i=\overline{0,\rho}, \tag{37}$$

is true, where

$$\begin{aligned}
&A_{i,0}^{(0)}=E\\
&A_{i,s}^{(p)}=A_0A_{i,0}^{(p-1)}-\sum_{s=0}^{p-1}\frac{bd'_{k_i-s}(k_i+1)}{\alpha_i}A_{i,s}^{(p-1)},p=1,2,\dots\\
&A_{i,s}^{(p)}=O,\quad s>p,\\
&A_{i,s}^{(p)}=A_0A_{i,s}^{(p-1)}+A_1A_{i,s-1}^{(p-1)},p=1,2,\dots;\ s=1,2,\dots,p.
\end{aligned}\tag{38}$$

The proof of (37) can be done by induction. Then instead of (34) we get

$$\tilde{\Psi}'(t)\ =\ \overline{\Psi}'(t)-\sum_{p=0}^{k_i-s}\sum_{k=0}^{p}\frac{\overline{\Psi}'(t+(k+s-k_i)h)A_{i,k}^{(p)}b}{\alpha_i}d'_s(k_i-p)$$

$$= \overline{\Psi}'(t) - \sum_{k=0}^{k_i-s} \overline{\Psi}'(t + (k + s - k_i)h) \sum_{p=k}^{k_i-s} \frac{A_{i,k}^{(p)} b d_s'(k_i - p)}{\alpha_i}$$

or

$$(39)\quad \tilde{\Psi}'(t) = \begin{cases} \overline{\Psi}'(t) - \sum_{k=0}^{k_i-s} \overline{\Psi}'(t + (k + s - k_i)h) B_{s,k}^{(i)}, & t \in T_i^s, i = \overline{0, \rho}, \\ & s = \overline{0, k_i}; \\ \overline{\Psi}'(t), & t \in T_{nn}, \end{cases}$$

where

$$(40)\qquad B_{s,k}^{(i)} = \sum_{p=k}^{k_i-s} \frac{A_{i,k}^{(p)} b d_s'(k_i - p)}{\alpha_i}, s = \overline{0, k_i}, k = \overline{0, k_i - s}, i = \overline{0, \rho}.$$

Using the form (39) of the function $\tilde{\Psi}(t)$ and equations (28) for $\overline{\Psi}(t)$, we shall obtain the equations (9) for $\tilde{\Psi}(t)$, where $\xi(t)$ has the form

$$(41)\qquad \xi(t) = \sum_{k=0}^{k_i} \overline{\Psi}'(t + (k - k_i)h) \frac{A_{i,k}^{(k_i+1)} b}{\alpha_i}, t \in T_i.$$

We will consider, that if $t > t^* + h$, then $t \in T_i^{-1}$ and $B_{-1,k}^{(i)} = O$. Therefore from the property (29) of the function $\overline{\Psi}(t)$ and the definition of $\tilde{\Psi}(t)$ we get, that $\tilde{\Psi}(t) \equiv 0, t > t^* + h$. Let us go back to the relation (36). The formula (39) can be written in another form, using the change $k_1 = -k - s + k_i$:

$$(42)\tilde{\Psi}'(t) = \begin{cases} \overline{\Psi}'(t) - \sum_{k=0}^{k_i-s} \overline{\Psi}'(t - kh) B_{s,k_i-k-s}^{(i)}, & t \in T_i^s, i = \overline{0, \rho}, s = \overline{0, k_i}; \\ \overline{\Psi}'(t), & t \in T_{nn} \end{cases}$$

moreover, according to (40)

$$(43)\qquad B_{s,k_i-k-s}^{(i)} = \sum_{p=k_i-k-s}^{k_i-s} \frac{A_{i,k_i-k-s}^{(p)} b d_s'(k_i - p)}{\alpha_i},$$
$$i = \overline{0, \rho}, \quad s = \overline{0, k_i}, \quad k = \overline{0, k_i - s}.$$

Let us look through every of the sums in the relation (36), in which the derivative $\overline{\Psi}^{(p-l-1)}$ is included, taking into consideration (37). For the first sum we obtain:

$$\sum_{i\in I^0}\sum_{s=0}^{k_i-1}\sum_{p=1}^{k_i-s}\sum_{l=0}^{p-1}(-1)^{p-l-1}\frac{\overline{\Psi}^{(p-l-1)'}(\tau_{i,1}-k_ih-0)b}{\alpha_i}d'_s(k_i-p)b\Delta u^{(l)}(\tau_{i,1}-sh)$$
$$=\sum_{i\in I^0}\sum_{p=1}^{k_i}\sum_{l=0}^{p-1}\sum_{s=0}^{p-l-1}(-1)^{k_i-p}\frac{\overline{\Psi}^{(k_i-p)'}(\tau_{i,1}-k_ih-0)b}{\alpha_i}d'_s(p-l-1)b\Delta u^{(l)}(\tau_{i,1}-sh)$$

For the rest of the sums analogous transformations can be done. At last, formula (36) takes the form:

$$\Delta J(u)=\int_0^{t^*+h}\Psi'(t)b\Delta u(t)dt \tag{44}$$
$$-\sum_{i\in I^*}\sum_{p=k_{i-1}+1}^{k_i}\sum_{l=0}^{p}\sum_{s=0}^{p-l-1}\nu_i^p d'_s(p-l-1)b\Delta u^{(l)}(\mu_i-sh+0)$$
$$-\sum_{i-0}^{\rho}\sum_{j=1}^{s_i}\sum_{l=0}^{k_i}\sum_{s=0}^{k_i-l}\nu_{ij}d'_s(k_i-l)b\Delta u^{(l)}(\tau_{ij}-sh)$$
$$-\sum_{i\in I^0}\sum_{p=1}^{k_i}\sum_{l=0}^{p-1}\sum_{s=0}^{p-l-1}(-1)^{k_i-p}\frac{\overline{\Psi}^{(k_i-p)'}(\tau_{i,1}-k_ih-0)b}{\alpha_i}\cdot$$
$$\cdot\, d'_s(p-l-1)b\Delta u^{(l)}(\tau_{i,1}-sh)$$
$$+\sum_{\substack{i\in I^0\\ i\neq\rho+1}}\sum_{p=1}^{k_i}\sum_{l=1}^{p-1}\sum_{s=0}^{p-l-1}(-1)^{k_i-p}\frac{\overline{\Psi}^{(k_i-p)'}(\mu_i-k_ih+0)b}{\alpha_i}\cdot$$
$$\cdot\, d'_s(p-l-1)b\Delta u^{(l)}(\mu_i-sh+0)$$
$$-\sum_{\substack{i\in I^0\\ i\neq 0}}\sum_{p=1}^{k_{i-1}}\sum_{l=0}^{p-1}\sum_{s=0}^{p-l-1}(-1)^{k_{i-1}-p}\frac{\overline{\Psi}^{(k_{i-1}-p)'}(\mu_i-k_{i-1}h-0)b}{\alpha_{i-1}}\cdot$$

$$\cdot\, d'_s(p-l-1)b\Delta u^{(l)}(\mu_i - sh - 0)$$

$$+\sum_{i\in I^0}\sum_{p=1}^{k_{i-1}}\sum_{l=0}^{p-1}\sum_{s=0}^{p-l-1}(-1)^{k_{i-1}-p}\frac{\overline{\Psi}^{(k_{i-1}-p)'}(\tau_{i-1,s_{i-1}} - k_i h + 0)b}{\alpha_{i-1}}\cdot$$

$$\cdot\, d'_s(p-l-1)b\Delta u^{(l)}(\tau_{i-1,s_{i-1}} - sh)$$

$$+\sum_{i=0}^{\rho}\sum_{p=0}^{k_i}\sum_{l=0}^{p-1}\sum_{s=0}^{p-l-1}(-1)^{k_i-p}\frac{\overline{\Psi}^{(k_i-p)'}(\tau_{i,1} - k_i h + 0)b}{\alpha_i}\cdot$$

$$\cdot\, d'_s(p-l-1)b\Delta u^{(l)}(\tau_{i,1} - sh)$$

$$-\sum_{i=0}^{\rho}\sum_{p=0}^{k_i}\sum_{l=0}^{p-1}\sum_{s=0}^{p-l-1}(-1)^{k_i-p}\frac{\overline{\Psi}^{(k_i-p)'}(\tau_{i,s_i} - k_i h - 0)b}{\alpha_i}\cdot$$

$$\cdot\, d'_s(p-l-1)b\Delta u^{(l)}(\tau_{i,s_i} - sh)$$

$$-\sum_{i=0}^{\rho}\sum_{j=2}^{s_i-1}\sum_{p=0}^{k_i}\sum_{l=0}^{p-1}\sum_{s=0}^{p-l-1}(-1)^{k_i-p}$$

$$\left\{\left[\overline{\Psi}^{(k_i-p)}(\tau_{ij} - k_i h - 0) - \overline{\Psi}^{(k_i-p)}(\tau_{ij} - k_i h + 0)\right]' b\right\}\cdot$$

$$\cdot\, d'_s(p-l-1)b\Delta u^{(l)}(\tau_{ij} - sh)/\alpha_i\ .$$

Let us denote:

$$\begin{aligned}\gamma_i^{p^+} &= (-1)^{k_i-p}\overline{\Psi}^{(k_i-p)'}(\mu_i - k_i h + 0)b/\alpha_i\\ &= \frac{1}{\alpha_i}\sum_{k=0}^{k_i-p}\overline{\Psi}'(\mu_i - (k_i-k)h + 0)A_{i,k}^{(k_i-p)}b,\\ & i\in I^0\setminus\{\rho+1\}, p=\overline{0,k_i};\end{aligned}\tag{45}$$

$$\begin{aligned}\gamma_i^{p^-} &= -\frac{1}{\alpha_i}\sum_{k=0}^{k_i-p}\overline{\Psi}'(\mu_i - (k_i-k)h - 0)A_{i,k}^{(k_i-p)}b,\\ & i\in I^0\setminus\{0\}, p=\overline{0,k_{i-1}};\end{aligned}\tag{46}$$

$$
v_{ij}^p = \begin{cases}
-\frac{1}{\alpha_i}\sum\limits_{k=0}^{k_i-p}\left[\overline{\Psi}(\tau_{ij}-(k_i-k)h-0)-\right. & \\
\left.\overline{\Psi}(\tau_{ij}-(k_i-k)h+0)\right]' A_{i,k}^{(k_i-p)}b, & j=\overline{2,s_i-1}, \\
 & p=\overline{0,k_i},\ i=\overline{0,\rho}; \\
-\frac{1}{\alpha_i}\sum\limits_{k=0}^{k_i-p}\left[\overline{\Psi}(\tau_{i,1}-(k_i-k)h-0)-\right. & \\
\left.\overline{\Psi}(\tau_{i,1}-(k_i-k)h+0)\right]' A_{i,k}^{(k_i-p)}b, & j=1,\tau_{i,1}\neq\mu_i, \\
 & p=\overline{0,k_i},\ i=\overline{0,\rho}; \\
-\frac{1}{\alpha_{i-1}}\sum\limits_{k=0}^{k_{i-1}-p}\overline{\Psi}'(\tau_{i,1}-(k_{i-1}-k)h-0)A_{i-1,k}^{(k_{i-1}-p)}b & \\
+\frac{1}{\alpha_i}\sum\limits_{k=0}^{k_i-p}\overline{\Psi}'(\tau_{i,1}-(k_i-k)h+0)A_{i,k}^{(k_i-p)}b, & j=1,p=\overline{0,k_{i-1}}, \\
 & \tau_{i,1}=\mu_i,\ i\in I^+\setminus\{0\}; \\
\frac{1}{\alpha_i}\sum\limits_{k=0}^{k_i-p}\overline{\Psi}'(\tau_{i,1}-(k_i-p)h+0)A_{i,k}^{(k_i-p)}b-\nu_i^p, & \\
 & j=1,p=\overline{k_{i-1}+1,k_i}, \\
 & \tau_{i,1}=\mu_i,\ i\in I^+; \\
-\frac{1}{\alpha_i}\sum\limits_{k=0}^{k_i-p}\left[\overline{\Psi}(\tau_{i,s_i}-(k_i-k)h-0)-\right. & \\
\left.\overline{\Psi}(\tau_{i,s_i}-(k_i-k)h+0)\right]' A_{i,k}^{(k_i-p)}b, & j=s_i,\tau_{i,s_i}\neq\mu_{i+1}, \\
 & p=\overline{0,k_i},\ i=\overline{0,\rho}; \\
-\frac{1}{\alpha_i}\sum\limits_{k=0}^{k_i-p}\overline{\Psi}'(\tau_{i,s_i}-(k_i-k)h-0)A_{i,k}^{(k_i-p)}b & \\
+\frac{1}{\alpha_{i+1}}\sum\limits_{k=0}^{k_{i+1}-p}\overline{\Psi}'(\tau_{i,s_i}-(k_{i+1}-k)h+0)A_{i+1,k}^{(k_{i+1}-p)}b, & j=s_i,\ i+1\in I^-\setminus\{\rho+1\} \\
 & p=\overline{0,k_{i+1}},\ \tau_{i,s_i}=\mu_{i+1} \\
-\frac{1}{\alpha_i}\sum_{k=0}^{k_i-p}\overline{\Psi}'(\tau_{i,s_i}-(k_i-k)h-0)A_{i,k}^{(k_i-p)}b, & \\
 & j=s_i,p=\overline{k_{i+1}+1,k_i}, \\
 & \tau_{i,s_i}=\mu_{i+1},\ i+1\in I^-; \\
v_{ij}^p=-\nu_{ij},\ i=\overline{0,\rho},\ j=\overline{1,s_i},\ p=k_i+1; &
\end{cases}
\tag{47}
$$

$$v_i^p = \begin{cases} \gamma_i^{p^+} + \gamma_i^{p^-}, & p = \overline{0, k_{i-1}}, i \in I^{+0} \setminus \{0\}; \\ \gamma_i^{p^+} - \gamma_i^p, & p = \overline{k_{i-1}, k_i}, i \in I^{+0}; \\ \gamma_i^{p^+} + \gamma_i^{p^-}, & p = \overline{0, k_i}, i \in I^{-0} \setminus \{\rho + 1\}; \\ \gamma_i^{p^-}, & p = \overline{k_i + 1, k_{i-1}}, i \in I^{-0}. \end{cases} \tag{48}$$

With the help of (45)-(48) the relation (44) takes the form (8) (with respect to $\tilde{\Psi}(t)$). We will show, that the jumps of $\tilde{\Psi}(t)$ have the form (10)-(13). After (42), (43) we have

$$\begin{aligned} \tilde{\Psi}'(t) &= \overline{\Psi}'(t) - \sum_{k=0}^{k_i - s} \overline{\Psi}(t - kh) B_{s,k_i-k-s}^{(i)} \\ &= \overline{\Psi}'(t) - \sum_{k=0}^{k_i-s} \sum_{p=k_i-k-s}^{k_i-s} \frac{\overline{\Psi}'(t-kh) A_{i,k_i-k-s}^{(p)} b}{\alpha_i} d_s'(k_i - p) \\ &= \overline{\Psi}'(t) - \sum_{p=0}^{k_i-s} \left(\sum_{k=k_i-s-p}^{k_i-s} \frac{\overline{\Psi}'(t-kh) A_{i,k_i-k-s}^{(p)} b}{\alpha_i} \right) d_s'(k_i - p) \\ &= \overline{\Psi}'(t) - \sum_{p=s}^{k_i} \left(\sum_{k=p-s}^{k_i-s} \frac{\overline{\Psi}'(t-kh) A_{i,k_i-k-s}^{(k_i-p)} b}{\alpha_i} \right) d_s'(p) \\ &= \overline{\Psi}'(t) - \sum_{p=s}^{k_i} \left(\sum_{k=0}^{k_i-p} \frac{\overline{\Psi}'(t-(k_i-k-s)h) A_{i,k}^{(k_i-p)} b}{\alpha_i} \right) d_s'(p) \\ & t \in T_i^s, i = \overline{0, \rho}, s = \overline{0, k_i}; \end{aligned}$$

or

$$\begin{aligned} \tilde{\Psi}'(t - sh) &= \overline{\Psi}'(t - sh) - \sum_{p=s}^{k_i} \left(\sum_{k=0}^{k_i-p} \frac{\overline{\Psi}'(t-(k_i-k)h) A_{i,k}^{(k_i-p)} b}{\alpha_i} \right) d_s'(p) \\ & t \in T_i, s = \overline{0, k_* + 1}, i = \overline{0, \rho}. \end{aligned} \tag{49}$$

Thus, according to (45)-(49), (30)-(32), we get

$$\begin{aligned} \tilde{\Psi}'(t^* + h - 0) &= \overline{\Psi}'(t^* + h - 0) \\ &- \sum_{p=0}^{k_\rho} \left(\sum_{k=0}^{k_\rho - p} \frac{\overline{\Psi}'(t^* + h - (k_\rho - k)h - 0)}{\alpha_\rho} A_{\rho,k}^{(k_\rho-p)} b \right) d_0'(p) \end{aligned}$$

$$
\begin{aligned}
&= c' + \sum_{p=0}^{k_\rho} \gamma_{\rho+1}^{p^-} d'_0(p) \\
&= c' + \sum_{p=0}^{k_\rho} v_{\rho+1}^{p} d'_0(p), \quad \rho+1 \in I^{-0};
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Psi}'(t^*+h-0) &= \overline{\Psi}'(t^*+h-0) \\
&- \sum_{p=0}^{k_\rho}\left(\sum_{k=0}^{k_\rho-p} \frac{\overline{\Psi}'(t^*+h-(k_\rho-k)h-0)}{\alpha_\rho} A_{\rho,k}^{(k_\rho-p)} b\right) d'_0(p) \\
&= c' - \nu_{\rho,s_\rho} d'_0(k_\rho+1) + \sum_{p=0}^{k_\rho} v_{\rho,s_\rho} d'_0(p) \\
&= c' - \sum_{p=0}^{k_\rho+1} v_{\rho,s_\rho} d'_0(p), \quad \rho+1 \notin I^{-0};
\end{aligned}
$$

i.e. after (15) the equality (10) holds. Also we get

$$
\tilde{\Psi}(\mu_i - sh - 0) - \tilde{\Psi}(\mu_i - sh + 0) = \sum_{p=s}^{k_i} v_i^p d_s(p), i \in I^{+0}, s = \overline{0, k_i}
$$

$$
\tilde{\Psi}(\mu_i - sh - 0) - \tilde{\Psi}(\mu_i - sh + 0) = \sum_{p=s}^{k_{i-1}} v_i^p d_s(p), i \in I^{-0}, \setminus \{\rho+1\},
$$

$s = \overline{0, k_{i-1}}$,

Which according to the definition of I^{+0}, I^{-0} and m_i is the equality (12). By analogy we obtain the equalities (11) and (13). The theorem is proved.

Definition 5. The support control $\{u, S_{sup}\}$ is not degenerated, if:

$$
\lim_{t \to t\pm} u(\tau) \neq \pm 1 \text{ when } t \in \tilde{T}_i^{k_i} \setminus \bigcup_{j=1}^{s_i} \tau_{ij}^{k_i}, i = \overline{0, \rho};
$$

$$
|\left(u(t_k+0) + u(t_k-0)\right)/2| \neq 1, k = \overline{1, K_0};
$$

$$
\dot{u}(\mu_i^{k_i} + 0) \neq 0 \text{ when } i \in I^- \text{ and } |u(\mu_i^{k_i} + 0)| = 1,
$$

$$\dot{u}(\mu_{i+1}^{k_i}-0) \neq 0 \text{ when } i+1 \in I^+ \text{ and } |u(\mu_{i+1}^{k_i}-0)| = 1.$$

For the case $d'b \neq 0$ the next theorem is true (2):

Theorem 2 (Support maximum principle). *Let $\{u, S_{sup}\}$ be a support control. For optimality of the admissible control $u(t), t \in T$, it is sufficient that for $u(t)$, $t \in T$, and the solutions $x(t), \Psi(t), t \in T$, of the systems (1) and (9)-(13) respectively Hamilton function $H(x, z, \Psi, u, t) = \Psi'(A_0x + A_1z + bu)$ reaches the maximum:*

$$H(x(t), x(t-h), \Psi(t), u(t), t) = \max_{|u|\leq 1} H(x(t), x(t-h), \Psi(t), u, t), \quad t \in T. \tag{50}$$

and the next conditions of co-ordination are fulfilled:

$$\begin{gathered} sign\ v_i^1 = sign\ (d'bu(\mu_i))\ when\ |u(\mu_i)| = 1, \\ v_i^1 = 0\ when\ |u(\mu_i)| < 1 \\ v_i^p = 0, \quad p = \overline{2, m_i}, \quad i = \overline{0, \rho+1}; \end{gathered} \tag{51}$$

$$\begin{gathered} sign\ v_{ij}^1 = sign\ (d'bu(\tau_{ij}))\ when\ |u(\tau_{ij})| = 1, \\ v_{ij}^1 = 0\ when\ |u(\tau_{ij})| < 1 \\ v_{ij}^p = 0, \quad p = \overline{2, k_i+1}, \quad j = \overline{1, s_i} \quad i = \overline{0, \rho}; \end{gathered} \tag{52}$$

Let the support control $\{u, S_{sup}\}$ is not degenerated. For optimality of the admissible control $u(t), t \in T$, conditions for maximum (50) and for co-ordination (51), (5'2) are necessary.

With a view to the theorem 1, the specificity of obtained maximum principle is, that it follows from it, that every optimal control has parts of singularity (16), the total length of which is not less than the delay h and these parts can appear in arbitrary moments of the process of control. Therefore, in the problem considered the relay principle in general case is not fulfilled.

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Errata

"Properties of the Cotrajectory in Support Maximum Principle for Time - Delayed Systems with Functional Restrictions"

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	Printed	To read
Everywhere	Contrajectory	Cotrajectory
p.218 l.11 from above	close the	close on the
p.220 l.3 from below	$d_1(1) = d'A_1$	$d'_1(1) = d'A_1$
p.223 l.10,12,14 from above	τ'	r'
p.224 l.3 from above	d^i_s	d'_s
p.225 l.6 from below	α'	d'
p.226 l.5 from above	$s = 1$ $u_*^{(l)}$	$s = 0$ u_*
p.227 l.12 from above	$p = K_{i-1} 1$ $i \in T^+$	$p = K_{i-1} + 1$ $i \in I^+$
p.234 l.2 from above	$-\gamma_i^p$	$-\nu_i^p$
p.236 l.2 from above	(-2)	[2]
p.237 l.3 from above	(in Russian)	