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Fixed Points of Maps with a Contractive Iterate at a Point

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Fixed point theorems are obtained for some contractive type maps on a complete metric space. Examples are given to show that our main theorem does improve upon some recent results of J. Kincses and V. Totik [6]. A case of maps on an arbitrary metric space is also discussed.

1. Introduction

Let T be a selfmap of a metric space (X, d). For a positive integer p, the pth iterate of T is denoted by T^p. We consider some fixed point theorems of contractive type in which the basic hypothesis is an estimate of $d(T^px, T^py)$ in terms of x, y and, eventually, T^ix and T^jy. For example, a classical contraction principle of Banach states that if (X, d) is complete and T satisfies the inequality,

$$d(Tx, Ty) \leq hd(x, y),$$

for some h in $[0, 1)$ and all x, y in X then T has a unique fixed point. Obviously, such a map T is continuous. The following relaxation of Banach's condition was given by V. Bryant [2]:

$$d(T^px, T^py) \leq hd(x, y),$$

for some positive integer p and h in $[0, 1)$, and all x, y in X. This condition does not force a continuity of T. In our paper we examine a contractive type condition involving the term $d(T^px, T^py)$ in which the integer p may depend on a point x. The first result in this direction is that of V. M. Sehgal [9] who proved the following.

Theorem S. *Let T be a selfmap of a complete metric space (X,d) such that for some h in $[0,1)$ and for each x in X there exists a positive integer $p(x)$ such that for each y in X,*

$$d(T^{p(x)}x, T^{p(x)}y) \leq hd(x,y).$$

Then T has a unique fixed point.

A number of generalizations of Th.S have been established by various authors. We have selected here two results obtained recently by J. Kincses and V. Totik [6].

Theorem KT1 ([6], Th.4). *Let T be a selfmap of a complete meatric space (X,d) such that for some h in $[0,1)$ and for each x in X there exists a positive integer $p(x)$ such that for each y in X,*

$$(h)\quad d(T^{p(x)}x, T^{p(x)}y) \leq h\max\{d(x,y), d(x, T^{p(x)}y), d(y, T^{p(x)}x)\}.$$

Then T has a fixed point z, and $T^n \to z$ for all x in X.

Theorem KT2 ([6], Th.6). *Let T be a selfmap of a metric space (X,d). Assume there exists a nonincreasing function $\alpha : (0,\infty) \to [0,1)$ such that for each x in X there is a positive integer $p(x)$ such that for each y in X with $y \neq x$,*

$$(\alpha)\quad d(T^{p(x)}x, T^{p(x)}y) \leq \alpha[d(x,y)]d(x,y).$$

Then T has a fixed point z, and $T^n x \to z$ for all x in X.

The purpose of this paper is to unify and generalize the above two theorems (see section 3). In section 4 two examples are given to demonstrate that our main theorem does improve Th.KT1 as well as Th.KT2. Finally, in section 5 we extend the recent results of M. T e l c i and K. T a s [10] concerning fixed points of maps on an arbitrary metric space.

Following S. L e a d e r [7] we say that a point z in X is a contractive fixed point of T if $z = Tz$ and, for all x in X, $T^n x \to z$ as $n \to \infty$. For $x \in X$, the set $\{x, Tx, T^2x, \ldots, T^n x, \ldots\}$ is called an orbit of T at x.

Following W. W a l t e r [11] we consider a class [W] of contractive gauge functions Φ from $\mathbb{R}_+$, the nonnegative reals, into $\mathbb{R}_+$ defined as follows. $\Phi \in [W]$ iff Φ is continuous, nondecreasing and such that

$\Phi(t) < t$ for all $t > 0$, and $\lim_{t\to\infty}(t - \Phi(t)) = \infty$.

The letters $\mathbb{N}$ and $\mathbb{Z}_+$ denote the sets of positive integers and nonnegative integers, respectively.

2. Boundedness of orbits

Proposition 1. *Let T be a selfmap of a metric space (X, d) and let $p : X \to \mathbb{N}$ be a function such that for all x, y in X,*

$$d(T^{p(x)}x, T^{p(x)}y) \leq \Phi[\max\{d(x, T^i y), d(T^{p(x)}x, T^i y), d(T^i x, T^j x) : i, j = 0, 1, \ldots, p(x)\}], \tag{1}$$

where a function Φ is nondecreasing and such that $\Phi(t) < t$ for all $t > 0$, and $\lim_{t\to\infty}(t - \Phi(t)) = \infty$. Then all orbits of T are bounded.

Proof. Fix an $x \in X$. Denote $p := p(x)$ and $a := \max\{d(T^i x, T^j x) : i, j = 0, 1, ..., p\}$. We show that for all $n \in \mathbb{N}$

$$d(T^p x, T^n x) \leq \max\{d(x, T^i x), a : i = 1, 2, ..., n\}.$$

To do it, we apply induction with respect to n to prove that for all $n \in \mathbb{N}$, ■

$$d(T^p x, T^{np+k} x) \leq \max\{d(x, T^i x), a : i = 1, 2, \ldots, np + k\}, \tag{2}$$

for $k = 0, 1, \ldots, p-1$. By the definition of the real a, (2) is obvious when $n = 0$. Assuming (2) to hold for some $n \in \mathbb{Z}_+$, we shall prove it for $n+1$. In case when $k = 0$ we obtain using (1) that

$$d(T^p x, T^{(n+1)p} x) \leq \Phi[\max\{d(x, T^{np+i} x), d(T^p x, T^{np+i} x, a : i = 0, 1, ..., p\}]. \tag{3}$$

By the induction hypothesis, for $i = 0, 1, \ldots, p-1$,

$$d(T^p x, T^{np+i} x) \leq \max\{d(x, T^j x), a : j = 1, 2, \ldots, np + i\}. \tag{4}$$

Observe we can omit the term $s := d(T^p x, T^{np+p} x)$ occurring in the right side of (3) : if the maximum in (3) is equal to s then, by (3), we get that $s \leq \Phi(s)$ and hence $s = 0$ since $\Phi(t) < t$ for $t > 0$. Thus using (3), (4) and the monotonicity of Φ, we obtain that (2) holds for $n+1$ and $k = 0$. We leave it to the reader to verify that (2) holds for other k (apply here induction with respect to (k). Thus (2) is proved.

Now define $b_n := \max\{d(x, T^{np+k}x) : k = 0, 1, \ldots, p-1\}$ for $n \in \mathbb{Z}_+$. Since $\lim_{t\to\infty}(t - \Phi(t)) = \infty$, there exists a $t_0 > 0$ such that $t - \Phi(t) > a$ for all $t \geq t_0$. Without loss of generality we may assume that $t_0 > a$. We shall show that $b_n < t_0$ for all $n \in \mathbb{Z}_+$. Suppose not. Then we may define the integer l by

$$l := \min\{k \in \mathbb{Z}_+ : b_k \geq t_0\}.$$

Since $b_0 \leq a < t_0$, we have that $l \geq 1$. By the definition of $\{b_n\}$, there exists an integer $q, 0 \leq q \leq p-1$ such that $b_l = d(x, T^{lp+q}x)$. Then, by the definition of l and q, we obtain

$$d(x, T^i x) \leq b_l, \tag{5}$$

for $i = 1, 2, \ldots, lp + q$. Next, by the triangle inequality, (1), (2) and the monotonicity of Φ,

$$b_l \leq d(x, T^p x) + d(T^p x, T^{l(p+q)}x) \leq a+$$

$$\Phi[\max\{d(x, T^{(l-1)p+q+i}x), d(T^p x, T^{(l-1)p+q+i}x), a : i = 0, 1, \ldots, p\}]$$

$$\leq a + \Phi[\max\{d(x, T^i x), a : i = 1, 2, \ldots, lp + q\}].$$

Hence, applying (5), the inequality $b_l > a$ and the monotonicity of Φ we obtain that $b_l - \Phi(b_l) \leq a$ which since $b_l \geq t_0$, contradicts the definition of t_0. Thus $b_n < t_0$ for all $n \in \mathbb{Z}_+$ which means that the sequence $\{T^n x\}$ is bounded.

R e m a r k 1. For $A \subseteq X$, denote $diam A := \sup\{d(x, y) : x, y \in A\}$. In case when a function p occurring in (1) is constant, the right side of (1) can be increased to $\Phi[diam(\{x, Tx, \ldots, T^p x\} \cup \{y, Ty, \ldots, T^p y\})]$ and then the orbits are bounded (see Lemma 3 of J. J a c h y m s k i [5]). We do not know if that is possible when $p(x)$ can vaery as x. The following proposition gives a partial answer to that problem.

Proposition 2. *Let T be a selfmap of a metric space (X, d) and let $p : X \to \mathbb{N}$ be a function such that for all $x, y \in X$,*

$$d(T^{p(x)}x, T^{p(x)}y) \leq h \quad diam(\{x, Tx, \ldots, T^{p(x)}x\} \cup \{y, Ty, \ldots, T^{p(x)}y\}), \tag{6}$$

where $h \in [0, 1/2)$. Then all orbits are bounded.

Proof. By the triangle inequality, we have $d(T^ix, T^jy) \le d(x, T^ix) + d(x, T^jy) \le 2\max\{d(x, T^ix), d(x, T^jy)\}$ and similarly

$$d(T^iy, T^jy) \le 2\max\{d(x, T^iy), d(x, T^jy)\},$$

for all $i, j = 1, 2, \dots, p(x)$. Hence and by (6), one can obtain that (1) is satisfied with a function Φ defined as $\Phi(t) := 2ht$ for $t \in \mathbb{R}_+$. So a boundednes of orbits follows from Prop.1. ∎

3. Main theorem and corollaries

The following theorem extends and unifies Th.KT1 and Th.KT2, and a theorem of K. I s e k i [4].

Theorem. *Let T be a selfmap of a complete metric space (X, d) and let $p : X \to \mathbb{N}$ be a function such that for all $x, y \in X$,*

$$d(T^{p(x)}x, T^{p(x)}y) \le \Phi[\max\{d(x, T^iy), d(T^{p(x)}x, T^iy) : i = 0, 1, \dots, p(x)\}], \tag{7}$$

where $\Phi \in [W]$. Then T has a contractive fixed point.

Proof. Fix an $x_0 \in X$, and define $x_{n+1} := T^{r(n)}x_n$ for $n \in \mathbb{Z}_+$, where $r(n) := p(x_n)$. By putting in (7)$x := x_n$ and $y := T^kx_0$ for some k, $n \in \mathbb{Z}_+$, we obtain

$$d(x_{n+1}, T^{r(n)+k}x_0 \le \Phi[\max\{d(x_n, T^{k+i}x_0), d(x_{n+1}, T^{k+i}x_0) : i = 0, 1, \dots, r(n)\}]. \tag{8}$$

Denote $a_n := \lim_{k\to\infty} \sup d(x_n, T^kx_0)$. By Prop.1, the sequence $\{T^nx_0\}$ is bounded so each a_n is finite. We show that $\lim_{n\to\infty} a_n = 0$. Fix an $n \in \mathbb{Z}_+$. There exists a sequence $\{k(j)\}_{j=1}^{\infty}$ of positive integers such that

$\lim_{j\to\infty} d(x_{n+1}, T^{r(n)+k(j)}x_0) = a_{n+1}$. By passing to subsequences if necessary, we may assume that for each $i = 0, 1, \dots, r(n)$, the sequences $\{d(x_n, T^{k(j)+i}x_0)\}_{j=1}^{\infty}$ and $\{d(x_{n+1}, T^{k(j)+i}x_0)\}_{j=1}^{\infty}$ are convergent as $j \to \infty$. Denote its limits by b_i and c_i, respectively, for $i = 0, 1, \dots, r(n)$. Then, for such i,

$$b_i \le a_n \quad and \quad c_i \le a_{n+1}. \tag{9}$$

By putting in (8) $k = k(j)$ and letting $j \to \infty$, we obtain using a continuity of Φ that

$$a_{n+1} \le \Phi[\max\{b_i, c_i : i = 0, 1, \dots, r(n)\}].$$

Hence, by (9) and the monotonicity of Φ, we get that

$$a_{n+1} \leq \Phi[\max\{a_n, a_{n+1}\}].$$

We may assume that $\max\{a_n, a_{n+1}\} = a_n$; for otherwise, we get that $a_{n+1} \leq \Phi(a_{n+1})$ and hence $a_{n+1} = 0$. Thus we can infer that $a_{n+1} \leq \Phi(a_n)$, for each $n \in \mathbb{Z}_+$. That easily implies,

$$a_n \leq \Phi^n(a_0), \tag{10}$$

for all $n \in \mathbb{N}$. But for each continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\gamma(t) < t$ for $t > 0$, we have $\lim \gamma^n(t) = 0$ for all $t \in \mathbb{R}_+$ (see [8], Rem.1.). In particular, this is the case for the function Φ. Thus, by (10), we can infer that $\lim_{n\to\infty} a_n = 0$.

We show that $\{T^n x_0\}$ is a Cauchy sequence. Fix an $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = 0$, there exists an $m \in \mathbb{N}$ such that $a_m < \varepsilon$, i.e. $\lim_{k\to\infty} \sup d(x_m, T^k x_0) < \varepsilon$. Hence, there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0, d(x_m, T^k x_0) < \varepsilon$. That implies $\{T^n x_0\}$ is Cauchy.

By completeness, $T^n x_0 \to z$ as $n \to \infty$, for some $z \in X$. By putting in (7) $x = z$ and $y = T^n x_0$, we get

$$d(T^{p(z)}z, T^{p(z)+n}x_0) \leq \Phi[\max\{d(z, T^{i+n}x_0), d(T^{p(z)}z, T^{i+n}x_0) :$$

$$i = 0, 1, \ldots, p(z)\}].$$

Letting here $n \to \infty$, we obtain using a continuity of Φ that $d(T^{p(z)}z, z) \leq \Phi[d(T^{p(z)}, z)]$ and hence $z = T^{p(z)}z$, i.e. z is a periodic point of T. Then however it must be $z = Tz$; for otherwise, $\{T^n z\}$ would not be a Cauchy sequence. The uniqueness of a fixed point follows easily from (7) and the property $\Phi(t) < t$ for $t > 0$.

As an immediate consequence we obtain the following

Corollary 1. *Let T be a selfmap of a complete metric space (X, d) and let $p : X \to \mathbb{N}$ be a function such that for all $x, y \in X$,*

$$d(T^{p(x)}x, T^{p(x)}y) \leq \Phi[\max\{d(x, y), d(x, T^{p(x)}y, d(y, T^{p(x)}x)\}] \tag{11}$$

where $\Phi \in [W]$. Then T has a contractive fixed point.

Corollary 2. *Let T be a selfmap of a complete metric space (X, d) and let $p : X \to \mathbb{N}$ be a function such that for all $x, y \in X$,*

$$\begin{aligned} d(T^{p(x)}x, T^{p(x)}y) \leq\ & a[d(x, T^{p(x)}x) + d(y, T^{p(x)}y)] + \\ & b[d(x, T^{p(x)}y) + d(y, T^{p(x)}x)] + c\, d(x, y), \end{aligned} \tag{12}$$

where $a, b, c \in \mathbb{R}_+$ and $4a + 2b + c < 1$. Then T has a contractive fixed point.

Proof. (12) easily implies that for all $x, y \in X$,

$$d(T^{p(x)}x, T^{p(x)}y) \le (4a + 2b + c)\max\{d(x, y),$$

$$\frac{1}{2}[d(x, T^{p(x)}y) + d(y, T^{p(x)}x)],\ \ \frac{1}{4}[d(x, T^{p(x)}x) + d(y, T^{p(x)}y)].$$

Next we have

$$\frac{1}{2}[d(x, T^{p(x)}y) + d(y, T^{p(x)}x)] \le \max\{d(x, T^{p(x)}y), d(y, T^{p(x)}x)\}$$

and

$$\frac{1}{4}[d(x, T^{p(x)}x) + d(y, T^{p(x)}y)] \le \frac{1}{4}[d(x, y)+$$

$$d(y, T^{p(x)}x) + d(x, y) + d(x, T^{p(x)}y)] \le \max\{d(x, y), d(x, T^{p(x)}y, d(y, T^{p(x)}x)\},$$

which means that (11) holds with Φ defined as
$\Phi(t) := (4a + 2b + c)t$ for $t \in \mathbb{R}_+$. Thus the result follows from Cor.1.

Remark 2. Cor.2 improves a result of K. Iseki [4] who assumed that(12) was fulfilled with $4a + 4b + c < 1$.

Now observe that the assumptions of Cor.2 are satisfied in particular if for all $x, y \in X$,

$$d(T^{p(x)}x, T^{p(x)}y) \le h[d(x, T^{p(x)}x) + d(y, T^{p(x)}y)], \tag{13}$$

where $h \in [0, \frac{1}{4})$.

On the other hand, J. Matkowski ([8], Th.3) obtained a fixed point theorem for maps T stisfying (12) with $3a + 3b + 3c < 1$. Hence a map T which satisfies (13) with $h \in [0, \frac{1}{3})$ has a contractive fixed point. In case when a function p occuring in (13) is constant, a fixed point theorem can be obtained for maps satisfying (13) with $h \in [0, \frac{1}{2})$ (see the "truth table" in [6]). However such an extension is not possible when $p(x)$ can vary as x. This fact was noticed by J. Kincses and V. Totik ([6], Ex.1) however they missed in their "truth table" the case when $h \in [0, \frac{1}{3})$. Moreover, they did not give an exact value of a constant h in their Ex.1 showing only that $h \in [0, \frac{1}{2})$. We leave it to the reader to verify that a h in Ex.1 [6] cannot be less than $\frac{16}{33}$. So we would welcome either an example of a fixed point free map T for which (13) holds with $h = \frac{1}{3}$ or a better estimation of a constant h in a fixed point theorem if such an example does not exist.

Now we present that our Cor.1 subsumes Th.KT2. We have at our disposal the following lemma which is due to W. Walter ([11], Lemma2).

Lemma. *Given a nonincreasing function $\alpha : (0, \infty) \to [0, 1)$, there exists a function $\Phi \in [W]$ such that for all $t > 0$,*

$$t\alpha(t) \leq \Phi(t).$$

Using this lemma we can infer that if a map T satisfies condition (α) of Th.KT2 then there exists $\Phi \in [W]$ such that

$$d(T^{p(x)}x, T^{p(x)}y) \leq \Phi[d(x, y)]. \tag{14}$$

By the monotonicity of Φ, (14) implies (11). That means Th.KT2 is subsumed by our Cor.1.

Remark 3. In case when a function p occurring in (14) is constant, the condition $\lim_{t\to\infty}(t - \Phi(t)) = \infty$ is unnecessary in a fixed point theorem involving (14) (for more general results, see [5]). It is interesting that if $p(x)$ can vary as x then this condition cannot be omitted (see Ex.1 of Z. Benedykt and J. Matkowski [1]).

4. Examples

The following example shows that our Theorem does improve upon Th.KT even if a space (X, d) is compact and convex, and a map T is continuous.

Example 1. Let $X := [0, 1]$ be endowed with the euclidean metric, $Tx := x/(x + 1)$ for $x \in X$, and $\Phi(t) := t/(t + 1)$ for $t \in \mathbb{R}_+$. It is easy to verify that $\Phi \in [W]$.

Now fix points $x, y \in X$. We have that

$$d(Tx, Ty) = |\frac{x}{x+1} - \frac{y}{y+1}| = \frac{|x-y|}{(x+1)(y+1)},$$

and

$$\Phi[d(x,y)] = \frac{|x-y|}{1+|x-y|}.$$

Since $(1+x)(1+y) \geq 1+|x-y|$ we get that $d(Tx,Ty) \leq \Phi[d(x,y)]$. Thus the assumptions of our Theorem are fulfilled but that is not the case for Th.KT1. To see it, suppose the condition (h) of Th.KT1 holds. Denote $r := p(0)$. For all $y \in X, T^r y = \frac{y}{1+ry.}$ Thus putting in (h) $x = 0$ one can obtain that $\frac{y}{1+ry} \leq hy$ and hence, for $y \in (0,1], \frac{1}{1+ry} \leq h$. This implies $\lim_{y\to 0} \frac{1}{1+ry} = 1 \leq h < 1$, a contradiction.

Our next example shows that our Theorem is stronger than Th.KT2. We involve here so called non-Archimedean metric (for a general definition, see [3], p.504).

E x a m p l e 2. Let $X := \mathbb{R}_+$ and, for $x, y \in X, d(x,y) = 0$ if $x = y$, and $d(x,y) = \max\{x,y\}$ if $x \neq y$. One can verify that (X,d) is a complete metric space. Define the map T by

$$Tx := x/(x+1) \quad \text{if } x \in [0,1], \quad \text{and } Tx := x - \sqrt{x} + \frac{1}{2} \ \text{ if } x > 1.$$

Further, let $\Phi := T$. Then it is easy to check that $\Phi \in [W]$. Fix points $x, y \in X, x \neq y$. Then if follows from the definition of T, Φ and d that

$$d(Tx,Ty) = \max\{Tx,Ty\} = \max\{\Phi(x), \Phi(y)\} = \Phi[\max\{x,y\}] = \Phi[d(x,y)].$$

That means the assumptions of our Theorem are fulfilled. Now suppose Th.KT2 is applicable. Define inductively the sequence $\{x_n\}$ as follows

$$(15) \qquad x_1 := 1 \quad \text{and } x_{n+1} := \left[\frac{1+\sqrt{4x_n - 1}}{2}\right]^2 \quad \text{for } n \in \mathbb{N}.$$

Then, $x_{n+1} = x_n + \sqrt{x_n - 0.25}$ for $n \in \mathbb{N}$. Hence, by an easy induction, one can obtain that $x_n \geq 1$ so $x_{n+1} \geq x_n + \frac{\sqrt{3}}{2}$ for $n \in \mathbb{N}$, and thus $\lim_{n\to\infty} x_n = \infty$. Moreover, it follows from (15) that for $n \in \mathbb{N}$ $Tx_{n+1} = x_{n+1} - \sqrt{x_{n+1}} + \frac{1}{2} = x_n$.

Now put in (α) $x = 0$ and $y = x_n$, and denote $r = p(0)$. Then we get

$$(16) \qquad d(0, T^r x_n) \leq \alpha[d(0,x_n)]d(0,x_n).$$

Since $Tx_{n+1} = x_n$, we can infer that for $n > r, T^r x_n = x_{n-r}$. Thus, by (16), we get that for all $n > r, x_n - r \leq \alpha(x_n)x_n$. Hence, since $x_n \geq 1$ and α is nonincreasing, we have

$$\frac{x_{n-r}}{x_n} \le \alpha(1), \tag{17}$$

for all $n > r$. On the other hand, we have that

$\frac{x_n}{x_{n+1}} = \frac{x_n}{x_n+\sqrt{x_n-0.25}} \to 1$ as $n \to \infty$ since $\lim_{n\to\infty} x_n = \infty$.

Thus $\frac{x_{n-r}}{x_n} = \frac{x_{n-r}}{x_{n-r+1}} \cdot \frac{x_{n-r+1}}{x_{n-r+2}} \cdots \frac{x_{n-1}}{x_n} \to 1$ as $n \to \infty$.

But simultaneously, by 17, we get that $\lim_{n\to\infty} \frac{x_{n-r}}{x_n} \le \alpha(1) < 1$, a contradiction. That means Th.KT2 cannot be applied in this case.

5. A case of an arbitrary metric space

Our results here are in the spirit if those obtained recently by M. T e l c i and K. T a s [10]. They have established two fixed point thei rems for a selfmap T of an arbitrary metric space (X, d) (not necessarily complete) under the hypothesis that a function $x \to d(x, Tx), (x \in X)$, attains its minimum on X. The following simple result will enable us to extend their results.

Proposition 3. *Let T be a selfmap of metric space (X, d) such that given $x \in X$ with $d(x, Tx) > 0$, there exists a positive integer $p(x)$ such that*

$$d(T^{p(x)}x, T^{p(x)+1}x) < d(x, Tx). \tag{18}$$

Assume further there exists a point $u \in X$ such that

$$f(u) = \inf\{f(x) : x \in X\}, \tag{19}$$

where $f(x) := d(x, Tx)$ for $x \in X$. Then u is a fixed point of T.

P r o o f. Suppose $u \neq Tu$. By (18), we get that for positive integer $p = p(u), d(T^p u, T^{p+1}u) < d(u, Tu)$ which means that $f(T^p u) < f(u)$. That contradicts (19). Thus $u = Tu$. ■

R e m a r k 4. It is easy to verify that (18) is fulfilled whenever T satisfies one of the following conditions.

(20) All orbits of T are equivalent,

i.e. for all $x, y \in X, d(T^n x, T^n y) \to 0$ as $n \to \infty$;

(21) T is iteratively contractive,

i.e. (see [?]) given $x, y \in X$ with $x \neq y$, there exists a $p \in \mathbb{N}$ such that $d(T^p x, T^p y) < d(x, y)$.

Next, it follows from Lemma 2 and 3 [5] that (20) holds in particular when T is Ciric's contraction, i.e. there exists a $h \in [0,1)$ such that for all $x, y \in X$,

$$d(Tx,Ty) \leq h \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}. \tag{22}$$

Thus we obtain the following

Corollary 3. *Each Ciric's contraction T of a metric space (X,d) satisfying* (19) *has a contractive fixed point.*

R e m a r k 5. Cor.3 extends Th.2 of M. T e l c i and K. T a s [10] who have required a constant h in (22) to be less than $\frac{1}{2}$. Moreover, we have strengthened their Th.2 by showing a contractive character of a fixed point.

R e m a r k 6. It is interesting that a unique fixed point of maps satisfying (19) and (21) need not be contractive even if a map is continuous and a space is compact (see [7], p.157).

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