Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol.9, 1995, Fasc.4

Some Aspects of Rigid Rod Dynamics

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A mathematical truth is neither simple
nor complicated in itself, it is.
Émile Lemoine

A *modus operandi* in rigid rod dynamics is proposed affording opportunities to reduce any particular dynamical problem concerning those extravagant solids to the general dynamics scheme proposed by Euler in 1750 and known as Euler's dynamical equations of the motion of the mass-center of any solid and of the solid itself around its mass-center. The advantages of such a reduction are obvious: Euler's dynamical equations represent, in point of fact, a mathematical cliché adopted to the solution of any problem of solid dynamics. The pith of matter of the proposed method may be described as follows. An inertial orthonormal right-hand orientated Cartesian system of reference being given, another also orthonormal right-hand orientated Cartesian system of reference may be defined invariably connected with the rigid rod in such a manner that an axis of the first and the second system turn out to be perpendicular; then the instantaneous angular velocity of the second system with respect to the first one may be defined playing, as it is well known, a cardinal role in Eulerian dynamical approach. Various curcumstances concerning this *modus operandi* are discussed, including 10 standard cases when single mechanical liaisons are imposed on rigid bodies, including naturally rigid rods, too.

Rigid rods are a quite special kind of rigid bodies — as pecular as to seem eccentric. And yet, they are one of the oldest mechanical devices — statical if not dynamical. Already Archimedes used them in order to represent levers mathematically. Later authors also made use of rigid rods — dynamically adequately at that, after Euler at least. And though, there is still room for improvements: the traditional mechanical literary sources leave much to be desired, at least as regards strict mathematical formulations related to this mechanical notion, as well as the most effective computational approaches to the dynamical behaviour of these solids in certain kinetical curcumstances. The present paper contains some observations in this respect, founded on a sustained professional experience in analitycal mechanical praxis in the course of some decades.

As it is well known, classical rigid bodies are traditionally divided into three categories depending on the analytical nature of that inalienable characteristic of the rigid body concept which is described as its density. A rigid body is called one-dimensional, if its density is zero everywhere but along a curve line, which is then called the circumference of that body. Similarly, a rigid body is called two-dimensional, if its density is zero everywhere but upon a surface, which is then called the periphery of that body. At last, a rigid body is called three-dimensional, if its density is zero everywhere but within a volume, which is then called the room of that body. Now a rigid rod is an one-dimensional rigid body the curcumference of which is a straight line, called the directrix of the rod.

At that it is presupposed that a vested property of the rigid body concept consists in the mathematical possibility to ascribe in a clear-cut manner to any rigid body S at least one (infinitely many as a matter of fact) orthonormal right-hand orientated Cartesian system of reference $\Omega\xi\eta\zeta$, so that the motion of S with respect to any orthonormal right-hand orientated Cartesian system of reference $Oxyz$ may be described by means of the movement of $\Omega\xi\eta\zeta$ with respect to $Oxyz$. Any such system of reference $\Omega\xi\eta\zeta$ attached or related to S is a fundamental mathematical characteristic of S and is called invariably connected with S.

In order to fix the ideas, let $\mathbf{i},\mathbf{j},\mathbf{k}$ and $\bar{\xi}^0,\bar{\eta}^0,\bar{\zeta}^0$ be the unit vectors of the axes Ox, Oy, Oz and $\Omega\xi, \Omega\eta, \Omega\zeta$ of $Oxyz$ and $\Omega\xi\eta\zeta$ respectively. Then by hypothesis

$$\mathbf{i}^2 = \mathbf{j}^2 = 1,\ \mathbf{ij} = 0,\ \mathbf{k} = \mathbf{i}\times\mathbf{j}, \tag{1}$$

$$(\bar{\xi}^0)^2 = (\bar{\eta}^0)^2 = 1,\ \bar{\xi}^0\bar{\eta}^0 = 0,\ \bar{\zeta}^0 = \bar{\xi}^0\times\bar{\eta}^0, \tag{2}$$

whence

$$\mathbf{k}^2 = 1,\ \mathbf{ik} = \mathbf{jk} = 0,\ \mathbf{i} = \mathbf{j}\times\mathbf{k},\ \mathbf{j} = \mathbf{k}\times\mathbf{i}, \tag{3}$$

$$(\bar{\zeta}^0)^2 = 1,\ \bar{\xi}^0\bar{\zeta}^0 = \bar{\eta}^0\bar{\zeta}^0 = 0,\ \bar{\xi}^0 = \bar{\eta}^0\times\bar{\zeta}^0,\ \bar{\eta}^0 = \bar{\zeta}^0\times\bar{\xi}^0, \tag{4}$$

$$\mathbf{i}\times\mathbf{j}\cdot\mathbf{k} = 1,\ \bar{\xi}^0\times\bar{\eta}^0\cdot\bar{\zeta}^0 = 1. \tag{5}$$

If

$$\mathbf{k}\times\bar{\zeta}^0 \neq \mathbf{0}, \tag{6}$$

then let the elementary angle θ and the orientated angles ψ and ϕ be defined as follows:

$$(7) \qquad \cos\theta = \mathbf{k}\bar{\zeta}^0 \ (0<\theta<\pi),$$

$$(8) \qquad \bar{\gamma}^0 = \frac{\mathbf{k}\times\bar{\zeta}^0}{\sin\theta},$$

$$(9) \qquad \sin\psi = \mathbf{j}\bar{\gamma}^0,\ \cos\psi = \mathbf{i}\bar{\gamma}^0,\ (0\le\psi<2\pi),$$

$$(10) \qquad \sin\phi = -\bar{\eta}^0\bar{\gamma}^0,\ \cos\phi = \bar{\xi}^0\bar{\gamma}^0,\ (0\le\phi<2\pi).$$

Then obviously

$$(11) \qquad (\bar{\gamma}^0)^2 = 1,\ \bar{\gamma}^0 = \cos\psi\, i + \sin\psi\, j = \cos\phi\,\bar{\xi}^0 - \sin\phi\,\bar{\eta}^0.$$

If by definition $\mathbf{r}_\Omega = \overline{O\Omega}$ and

$$(12) \qquad \mathbf{r}_\Omega = x_\Omega\,\mathbf{i} + y_\Omega\,\mathbf{j} + z_\Omega\,\mathbf{k},$$

then the quantities

$$(13) \qquad x_\Omega, y_\Omega, z_\Omega, \psi, \phi, \theta$$

are called the c a n o n i c p a r a m e t e r s of the rigid body S (with regard to the systems of reference $Oxyz$ and $\Omega\xi\eta\zeta$), and the motion of S with respect to $Oxyz$ is described mathematically by means of the sets of functions

$$(14) \qquad x_\Omega = x_\Omega(t),\ y_\Omega = y_\Omega(t),\ z_\Omega = z_\Omega(t),$$

$$(15) \qquad \psi = \psi(t),\ \phi = \phi(t),\ \theta = \theta(t)$$

of the time t.

Now such a construction (invented by Euler and comparable by its significance for solid dynamics to Lavoisier's idea of elements for chemistry) is denied rigid rods, and rigid rods only, in its original variant at least.

In order to fathom the fact, let d be the directrix of the rigid rod L and let d be chosen in the capacity of axis $\Omega\xi$ of a Cartesian system of reference $\Omega\xi\eta\zeta$. Now $\Omega\xi$ being fixed, the axes $\Omega\eta$ and $\Omega\zeta$ cannot be determined unequivocally, since their rotations around $\Omega\xi$ are indiscernible for an observer invariably connected with $\Omega\xi$ solely. This is the meaning of the statement that no Cartesian system of reference $\Omega\xi\eta\zeta$ may be connected invariably with a rigid rod in a mathematically clear-cut way.

And yet, there is a way — a bit artifical maybe, at first sight at least, nevertheless technically most advantageous — to break that deadlock.

In order to present it, let Ω be an arbitrary fixed point on the directrix d of the rigid rod L and let $\Omega\xi$ be directed along d in one way or another by means of the unit vector $\bar{\xi}^0$. The system of reference $Oxyz$ being chosen with axis Oz non-parallel to d, let l be the intersecting line of the plane Oxy with the plane through O perpendicular to d; then let $\Omega\zeta$ be chosen parallel to l, directed in one way or another by means of the unit vector $\bar{\zeta}^0$; at last, let $\bar{\eta}^0$ (and consequently the axis $\Omega\eta$) be defined by means of $\bar{\eta}^0 = \bar{\zeta}^0 \times \bar{\xi}^0$. All this settled, the system of reference $\Omega\xi\eta\zeta$ invariably connected with the rod L is defined univocally.

This procedure is qualified above as "a bit artifical"; let us specify the meaning of this reservation. If S is any solid other than a rigid rod, then the possibility to connect invariably with S a Cartesian system of reference $\Omega\xi\eta\zeta$ is unconditional; in other words, completely independent of any other Cartesian system of reference $Oxyz$ whatever fixed in space. On the contrary, the definition of the system $\Omega\xi\eta\zeta$ invariably connected with L and properly described by the aid of the above construction, depends essentially on the choice of the system $Oxyz$. Anyhow, $Oxyz$ being arbitrarily chosen (subordinate to the only condition that the directrix d of L is non-parallel to Oz), the system $\Omega\xi\eta\zeta$ may be defined in one and only way (as far as *der Sinn* of the vectors $\bar{\xi}^0$ and $\bar{\zeta}^0$ is already chosen in one way or another).

The vector $\bar{\zeta}^0$ being by definition perpendicular to the vector $\mathbf{k}$, the definition (7) at once implies

(16) $$\theta = \frac{\pi}{2}$$

so that the number of the canonic parameters (13) is reduced to 5:

(17) $$x_\Omega,\ y_\Omega,\ z_\Omega,\ \psi,\ \phi$$

Moreover, the cosine directors $a_{\mu,\nu}$ $(\mu, \nu = 1,2,3)$ being defined by means of the relations

(18) $$\begin{cases} \mathbf{i} = a_{11}\,\bar{\xi}^0 + a_{12}\,\bar{\eta}^0 + a_{13}\,\bar{\zeta}^0, \\ \mathbf{j} = a_{21}\,\bar{\xi}^0 + a_{22}\,\bar{\eta}^0 + a_{23}\,\bar{\zeta}^0, \\ \mathbf{k} = a_{31}\,\bar{\xi}^0 + a_{32}\,\bar{\eta}^0 + a_{33}\,\bar{\zeta}^0, \end{cases}$$

the condition (16) is leading to the following relations:

(19) $$\begin{cases} a_{11} = \cos\psi\cos\phi,\ a_{12} = -\cos\psi\sin\phi,\ a_{13} = \sin\psi, \\ a_{21} = \sin\psi\cos\phi,\ a_{22} = -\sin\psi\sin\phi,\ a_{23} = -\cos\psi, \\ a_{31} = \sin\phi,\ a_{32} = \cos\phi,\ a_{33} = 0 \end{cases}$$

between $a_{\mu,\nu}$ $(\mu, \nu = 1,2,3)$, on the one hand, and Eulerian angles ψ and ϕ, on the other hand.

Last not least, if

(20) $$\bar{\omega} = \frac{1}{2}(\bar{\xi}^0 \times \dot{\bar{\xi}}^0 + \bar{\eta}^0 \times \dot{\bar{\eta}}^0 + \bar{\zeta}^0 \times \dot{\bar{\zeta}}^0)$$

(dots denoting, as it has become a custom in rational mechanics, derivatives with respect to the time t, with regard to the system of reference $Oxyz$) denotes by definition the instantaneous angular velocity of $\Omega\xi\eta\zeta$ with respect to $Oxyz$, and if by definition

$$(21) \qquad \bar{\omega} = \omega_\xi \bar{\xi}^0 + \omega_\eta \bar{\eta}^0 + \omega_\zeta \bar{\zeta}^0,$$

then (16) implies that Eulerian kinematical equations take the form

$$(22) \qquad \omega_\xi = \dot{\psi} \sin\phi, \; \omega_\eta = \dot{\psi}\cos\phi, \; \omega_\zeta = \dot{\phi}.$$

Having arrived at that point, an uninitiated may quite reasonably get puzzled: Why all those tomfooleries? Wouldn't it be much simpler to choose an arbitrary origin Ω on the directrix d of the rod L and a unit vector $\bar{\xi}^0$ along d; if

$$(23) \qquad \bar{\xi}^0 = \cos\phi\cos\psi\, i + \cos\phi\sin\psi\, j + \sin\phi\, k$$

represents the decomposition of $\bar{\xi}^0$ with respect to $Oxyz$, then the parameters (17) of L are defined at once, without the cockabondy (16).

The truth is that this modus operandi is not only acceptable, but even prefered by the dominant majority of the authors of treatises, textbooks and books of problems of analytical mechanics. And yet, preference should be given to the construction by means of (16). The reason is a rather simple one: this approach proposes *ein Schablon* for solving dynamical problems concerning rigid rods. Indeed, whereas working through the medium of (23) without taking into consideration (16) all calculations must be made *ad hoc* — in other words especially accomodated to the particular dynamical problem to be solved — proceeding by means of (16) one has at his disposal — just little short of lying in the store for him — a beforehand prepared mathematical computational apparatus, at that in its as easy as ABC form.

Since this apparatus, when all is said and done, consists in Euler's dynamical equations, let us recall some curcumstances connected with the letters.

If P is any point in space and by definition $\bar{\rho} = \overline{\Omega P}$ and

$$(24) \qquad \bar{\rho} = \xi\bar{\xi}^0 + \eta\bar{\eta}^0 + \zeta\bar{\zeta}^0,$$

$\Omega\xi\eta\zeta$ being any system of reference invariably connected with the rigid body S, then P is called a p o i n t o f S if and only if

$$(25) \qquad \dot{\xi} = \dot{\eta} = \dot{\zeta} = 0 \; (\forall\, t).$$

It is easily seen that the set of all points of S constitutes a real standard vector space; let us denote it by V_s.

Let by definition

$$\kappa : V_s \to [0, \infty) \tag{26}$$

be the density of S ; its value

$$\kappa(\bar{\rho}) \quad (\forall\, \bar{\rho} \in V_s) \tag{27}$$

is called the density of S at the point P of S.

Let $d\mu$ denote an infinitesimal length, or infinitesimal area, or infinitesimal volume according to the dimensions of S, i.e. whether S is 1-dimensional, or 2-dimensional, or 3-dimensional respectively; then the quantity

$$dm = \kappa(\bar{\rho})\, d\mu \quad (\forall\, \bar{\rho} \in V_s) \tag{28}$$

is by definition the elementary mass of S around $\bar{\rho}$.

In the definition of the density (26) of the rigid body S it is hypothesized that the integral

$$m = \int \kappa(\bar{\rho})\, d\mu > 0 \tag{29}$$

exists; it is called the mass of S. At that, the record

$$m = \int dm \tag{30}$$

in view of (28) and (29) might not be dangerous owing to a collision of symbols since the meaning of m in the left-hand and right-hand sides of (30) becomes clear by the context.

Besides, for the sake of definiteness and simplicity let us note that the integral in (30) (as well as all integrals to appear later) is taken *im Riemannschen Sinne.*

The point G defined by

$$\bar{\rho}_G = \frac{1}{m} \int \bar{\rho}\, dm \quad (\forall\, \bar{\rho} \in V_s) \tag{31}$$

provided $\bar{\rho}_G = \overline{\Omega G}$ is called the mass-center of S. The definition (31), together (24), (25) and

$$\bar{\rho}_G = \xi_G \bar{\xi}^0 + \eta_G \bar{\eta}^0 + \zeta_G \bar{\zeta}^0 \tag{32}$$

implies

$$\dot{\xi}_G = \dot{\eta}_G = \dot{\zeta}_G = 0 \quad (\forall\, t), \tag{33}$$

i.e. G is a point of S.

If by definition

(34) $A = I_{\xi\xi} - J_{\xi\xi},\quad B = I_{\eta\eta} - J_{\eta\eta},\quad C = I_{\zeta\zeta} - J_{\zeta\zeta},$

(35) $D = I_{\eta\zeta} - J_{\eta\zeta},\quad E = I_{\zeta\xi} - J_{\zeta\xi},\quad F = I_{\xi\eta} - J_{\xi\eta}$

provided

(36) $I_{\xi\xi} = \int (\eta^2 + \zeta^2)\, dm,\ I_{\eta\eta} = \int (\zeta^2 + \xi^2)\, dm,\ I_{\zeta\zeta} = \int (\xi^2 + \eta^2)\, dm,$

(37) $I_{\eta\zeta} = \int \eta\zeta\, dm,\ I_{\zeta\xi} = \int \zeta\xi\, dm,\ I_{\xi\eta} = \int \xi\eta\, dm,$

(38) $J_{\xi\xi} = m\,(\eta^2 + \zeta^2),\ J_{\eta\eta} = m\,(\zeta^2 + \xi^2),\ J_{\zeta\zeta} = m\,(\xi^2 + \eta^2),$

(39) $J_{\eta\zeta} = m\,\eta_G\zeta_G,\ J_{\zeta\xi} = m\,\zeta_G\xi_G,\ J_{\xi\eta} = m\,\xi_G\eta_G,$

then A, B, C are called the moments of inertia and D, E, F are called the moments of deviation of S with respect to $\Omega\xi\eta\zeta$.

Let S be under the action of the active forces (i.e. completely determined by the conditions of the particular dynamical problem under consideration)

(40) $\vec{F}_\mu = (\mathbf{F}_\mu, \mathbf{M}_\mu) \quad (\mu = 1, \ldots, m),$

all moments $\mathbf{M}_\mu$ $(\mu = 1, \ldots, m)$ being taken with respect to O, and let by definition

(41) $$\mathbf{F} = \sum_{\mu=1}^{m} \mathbf{F}_\mu,\quad \mathbf{M} = \sum_{\mu=1}^{m} \mathbf{M}_\mu .$$

A most important dynamical hypothesis reads as follows:

Ax R. *Any geometrical constraint imposed on a rigid body generates a force acting on the body, the directrix of which runs through the point of contact of the solid and the constraint.*

The force generated by a geometrical constraint according to Ax R is called the reaction of the constraint.

The reactions of the geometrical constraints imposed on a solid S are called passive forces as well, in contrast to the active forces applied on S.

Let now S be subjugated to n geometrical constraints γ_ν with points of contact C_ν $(\nu = 1, \ldots, n)$ respectively and let

(42) $\vec{R}_\nu = (\mathbf{R}_\nu, \mathbf{N}_\nu) \quad (\nu = 1, \ldots, n),$

all moments N_ν $(\nu = 1, \ldots, n)$ being taken with respect to O too, be the corresponding reactions of the constraints. Besides, let by definition

$$R = \sum_{\nu=0}^{n} R_\nu, \quad N = \sum_{\nu=0}^{n} N_\nu. \tag{43}$$

The equations of the directrices of $\vec{R}_\nu$ $(\nu = 1, \ldots, n)$ are

$$r \times R_\nu = N_\nu \quad (\nu = 1, \ldots, n), \tag{44}$$

$\mathbf{r} = \mathbf{OP}$ denoting the radius-vector with respect to O of any point P in space. If by definition $\mathbf{r}_{m+\nu} = \mathbf{OC}_\nu$ $(\nu = 1, \ldots, n)$, then (44) and Ax R imply

$$\mathbf{N}_\nu = r_{m+\nu} \times \mathbf{R}_\nu \quad (\nu = 1, \ldots, n), \tag{45}$$

Now (45) and the second equation (43) imply

$$\mathbf{N} = \sum_{\nu=1}^{n} \mathbf{r}_{m+\nu} \times \mathbf{R}_\nu \quad (\nu = 1, \ldots, n). \tag{46}$$

If by definition $\mathbf{r}_G = \mathbf{OG}$, then the identity $\mathbf{OG} = \overline{O\Omega} + \overline{\Omega G}$ implies

$$\mathbf{r}_G = \mathbf{r}_\Omega + \bar{\rho}_G. \tag{47}$$

Let by definition

$$\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}, \tag{48}$$

$$\mathbf{R} = R_x \mathbf{i} + R_y \mathbf{j} + R_z \mathbf{k} \tag{49}$$

and let

$$\mathbf{M}_G = \mathbf{M} + \mathbf{F} \times \mathbf{r}_G, \quad \mathbf{N}_G = \mathbf{N} + \mathbf{R} \times \mathbf{r}_G \tag{50}$$

be the moments with regard to G of the system of forces (40) and (42) respectively. Besides, let by definition

$$\mathbf{M}_G = M_{G\xi}\,\bar{\xi}^0 + M_{G\eta}\,\bar{\eta}^0 + M_{G\zeta}\,\bar{\zeta}^0, \tag{51}$$

$$\mathbf{N}_G = N_{G\xi}\,\bar{\xi}^0 + N_{G\eta}\,\bar{\eta}^0 + N_{G\zeta}\,\bar{\zeta}^0, \tag{52}$$

$$\mathbf{r}_G = x_G \mathbf{i} + y_G \mathbf{j} + z_G \mathbf{k}. \tag{53}$$

After all this heavy artilery preparation we are already in a key position to write down Euler's dynamical equations for the mechanical behaviour of the rigid body S under the action of the forces (40) and (42). Namely, they are

$$m\ddot{x}_G = F_x + R_x, \quad m\ddot{y}_G = F_y + R_y, \quad m\ddot{z}_G = F_z + R_z \tag{54}$$

and

$$\left\{\begin{aligned} A\dot{\omega}_\xi - (B-C)\omega_\eta\omega_\zeta &- D(\omega_\eta^2 - \omega_\zeta^2) - E(\dot{\omega}_\zeta + \omega_\xi\omega_\eta) \\ &- F(\dot{\omega}_\eta - \omega_\zeta\omega_\xi) = M_{G\xi} + N_{G\xi}, \\ B\dot{\omega}_\eta - (C-A)\omega_\zeta\omega_\xi &- E(\omega_\zeta^2 - \omega_\xi^2) - F(\dot{\omega}_\xi + \omega_\eta\omega_\zeta) \\ &- D(\dot{\omega}_\zeta - \omega_\xi\omega_\eta) = M_{G\eta} + N_{G\eta}, \\ C\dot{\omega}_\zeta - (A-B)\omega_\xi\omega_\eta &- F(\omega_\xi^2 - \omega_\eta^2) - D(\dot{\omega}_\eta + \omega_\zeta\omega_\xi) \\ &- E(\dot{\omega}_\xi - \omega_\eta\omega_\zeta) = M_{G\zeta} + N_{G\zeta}. \end{aligned}\right. \tag{55}$$

As regards L a g r a n g e's d y n a m i c a l e q u a t i o n s, there is still a step to be taken before we could formulate them. As it is well known, the kinetic energy of the rigid body S with respect to the system of reference $Oxyz$ is defined by

$$T = \frac{1}{2}\int v^2\, dm, \tag{56}$$

v denoting the velocity with respect to $Oxyz$ of any point P of S. According to a renowned kinematical theorem of Euler, a necessary and sufficient condition the point P to belong to S is

$$\mathbf{v} = \mathbf{v}_\Omega + \bar{\omega} \times \bar{\rho} \tag{57}$$

provided $\bar{\rho} = \overline{\Omega P}$ and $\mathbf{v}_\Omega = \dot{\mathbf{r}}_\Omega$. By virtue of (33) the mass-center G of S is a point of S, whence

$$\mathbf{v}_G = \mathbf{v}_\Omega + \bar{\omega} \times \bar{\rho}_G \tag{58}$$

provided $\mathbf{v}_G = \dot{\mathbf{r}}_G$. Now (57), (58) imply

$$\mathbf{v} = \mathbf{v}_G + \bar{\omega} \times (\bar{\rho} - \bar{\rho}_G). \tag{59}$$

The definition (56) in conjuction with (59) implies

$$2T = \int v_G^2\, dm + \int (\bar{\omega} \times \bar{\rho} - \bar{\omega} \times \bar{\rho}_G)^2 dm + 2v_G \times \bar{\omega} \cdot \int (\bar{\rho} - \bar{\rho}_G) dm. \tag{60}$$

Now (31) implies that the third integral in the right-hand side of (60) is zero; again (31) implies

$$\int (\bar{\omega} \times \bar{\rho} - \bar{\omega} \times \bar{\rho}_G)^2 dm = \int (\bar{\omega} \times \bar{\rho})^2 dm - m(\bar{\omega} \times \bar{\rho}_G)^2 \tag{61}$$

The decompositions (21) and (24), along with the definitions (36), (37), imply

$$(62)\int(\bar{\omega}\times\bar{\rho})^2dm = I_{\xi\xi}\omega_\xi^2 + I_{\eta\eta}\omega_\eta^2 + I_{\zeta\zeta}\omega_\zeta^2 - 2(I_{\xi\eta}\omega_\xi\omega_\eta + I_{\eta\zeta}\omega_\eta\omega_\zeta + I_{\zeta\xi}\omega_\zeta\omega_\xi).$$

Similarly

$$(63) m(\bar{\omega}\times\bar{\rho}_G)^2 = J_{\xi\xi}\omega_\xi^2 + J_{\eta\eta}\omega_\eta^2 + J_{\zeta\zeta}\omega_\zeta^2 - 2(J_{\xi\eta}\omega_\xi\omega_\eta + J_{\eta\zeta}\omega_\eta\omega_\zeta + J_{\zeta\xi}\omega_\zeta\omega_\xi).$$

Now (60)–(63) and (34), (35) imply

$$(64)\ T = \frac{1}{2}mv_G^2 + \frac{1}{2}(A\omega_\xi^2 + B\omega_\eta^2 + C\omega_\zeta^2) - (F\omega_\xi\omega_\eta + D\omega_\eta\omega_\zeta + E\omega_\zeta\omega_\xi).$$

Let now P_μ $(\mu = 1,\dots,m)$ be any points of the solid S lying in the moment of time t on the directrices of the corresponding active forces (40) applied to the rigid body, i. e.

$$\mathbf{r}_\mu \times \mathbf{F}_\mu = \mathbf{M}_\mu \quad (\mu = 1,\dots,m) \tag{65}$$

provided $\mathbf{r}_\mu = \mathbf{OP}_\mu$ $(\mu = 1,\dots,m)$ and let

$$q_\lambda \quad (\lambda = 1,\dots,l) \tag{66}$$

be mutually independent parameters of S, i. e. quantities necessary as well as sufficient in their totality for the clear-cut determination of any position of S in space with respect to $Oxyz$, compatible with the geometrical constraints imposed on S. These prerequisites accepted, let by definition

$$Q_\lambda = \sum_{\mu=1}^{m} F_\mu \frac{\partial r_\mu}{\partial q_\lambda} \quad (\lambda = 1,\dots,l) \tag{67}$$

Now all premises for the formulation of Lagrange's dynamical equations are at hand: the geometrical constraints γ_ν $(\nu = 1,\dots,n)$ imposed on S being smooth by hypothesis (in other words, the corresponding reactions (42) being perpendicular to the constraints), the equations are

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\lambda} - \frac{\partial T}{\partial q_\lambda} - Q_\lambda = 0 \qquad (\lambda = 1,\dots,l). \tag{68}$$

After the promulgation of Euler's and Lagrange's dynamical equations (54), (55), and (68) respectively, the next step that should be taken is to accomodate them to the special case of rigid rods.

The first thing to do in this direction is to supplement the kinematical sketch of the rigid rod concept drawn above with some dynamical strokes. They reduce to the following hypothesis.

Ax L . *The directrix of any force acting on a rigid rod intersects (or coincides with) the directrix of the rod.*

The range of action of Ax L will become clear after some preparatory calculations.

If by definition

$$\kappa(\xi,\eta,\zeta) = \kappa(\bar{\rho}) \qquad (\bar{\rho} \in V_L) \tag{69}$$

provided (24), then the choice of $\Omega\xi\eta\zeta$ implies

$$\kappa(\xi,\eta,\zeta) = 0 \tag{70}$$

if

$$\eta^2 + \zeta^2 \neq 0. \tag{71}$$

For the sake of brevity let by definition

$$\kappa(\xi) = \kappa(\xi,0,0). \tag{72}$$

Now (69)–(72), (28) and the definition of $d\mu$ imply

$$dm = \begin{cases} \kappa(\xi)\,d\xi & (\eta^2+\zeta^2=0), \\ 0 & (\eta^2+\zeta^2\neq 0), \end{cases} \tag{73}$$

and (73), (36), (37) imply

$$I_{\xi\xi} = 0, \quad I_{\eta\eta} = I_{\zeta\zeta} = I, \quad I_{\xi\eta} = I_{\eta\zeta} = I_{\zeta\xi} = 0 \tag{74}$$

provided

$$I = \int \xi^2\kappa(\xi)\,d\xi. \tag{75}$$

The definition (31) implies

$$m\bar{\rho}_G = \int (\xi\bar{\xi}^0 + \eta\bar{\eta}^0 + \zeta\bar{\zeta}^0)\,dm \tag{76}$$

according to (24), whence

$$\bar{\rho}_G = \frac{1}{m}\int \xi\kappa(\xi)\,d\xi\,\bar{\xi}^0 \tag{77}$$

in view of (73). Now (77), (32) imply

$$\xi_G = \frac{1}{m} \int \xi \kappa(\xi)\, d\xi\,, \quad \eta_G = \zeta_G = 0, \tag{78}$$

i. e. the mass-center G of rod L lies on the ξ-axis, i. e. on the directrix of L. This fact suggests the idea to choose G in the capacity of the origin Ω of the system of reference $\Omega\xi\eta\zeta$ invariably connected with L. This procedure is generally accepted in dynamical problems concerning rigid rods for the sake of simplicity of calculations. *Exampli gratia*, in such a case

$$\rho_G = \mathbf{0}, \tag{79}$$

as (47) with $\Omega = G$ at once displays; now (79), (32) and (38), (39) immediately imply

$$J_{\xi\xi} = J_{\eta\eta} = J_{\zeta\zeta} = J_{\xi\eta} = J_{\eta\zeta} = J_{\zeta\xi} = 0, \tag{80}$$

and (80), (74), (34), (35) imply

$$A = 0, \ B = C = I, \ D = E = F = 0, \tag{81}$$

I being defined by (75).

The result (81) implies that the last two equations (55) are reduced to

$$I(\dot\omega_\eta - \omega_\zeta\omega_\xi) = M_{G\eta} + N_{G\eta}, \ I(\dot\omega_\zeta + \omega_\xi\omega_\eta) = M_{G\zeta} + N_{G\zeta} \tag{82}$$

respectively. As regards the first equation (55), it is annihilated to the identity

$$0 = 0. \tag{83}$$

The left-hand side of (83) is verified immediately. As regards its right-hand side, let us recall Ax L: it states that the directrix of any force acting on a rod L intersects or coincides with the directrix $G\xi$ of L. In the second case its moment with respect to G is trivially zero; in the first case this moment is perpendicular to $G\xi$, so that its projection on $G\xi$ is obviously zero. Hence

$$M_{G\xi} = N_{G\xi} = 0, \tag{84}$$

and (84) at once display that the right-hand side of the first equation (55) is zero, too. *Q. E. D.*

Summing up one may now state that the dynamical behaviour of any rigid rod L is governed by the equations (54) and (82). At that, $G\xi\eta\zeta$ being a system of reference invariably connected with L, the canonic parameters (13) of L become

(85) $$x_G,\ y_G,\ z_G,\ \psi,\ \phi,\ \theta = \frac{\pi}{2}$$

instead of (17).

As regards Lagrange's dynamical equations (68), their application is facilitated by incomparably simpler form the kinetic energy (64) takes in the rigid rod case. Indeed, (64) and (81) imply

(86) $$T = \frac{1}{2} m v_G^2 + \frac{1}{2} I(\omega_\eta^2 + \omega_\zeta^2),$$

I being defined by (75), or, by virtue of (53) and (22)

(87) $$T = \frac{m}{2}(\dot{x}_G^2 + \dot{y}_G^2 + \dot{z}_G^2) + \frac{I}{2}(\dot{\psi}^2 \cos^2\phi + \dot{\phi}^2).$$

The last specification that should be made in connection with the general rigid rod dynamics concerns the particular case, most frequently used in mechanical praxis, of homogeneous rods. A rigid rod L is qualified as such one if there exist points X and Y on the directrix d of L such that, Z being any point of d, the density $\kappa(Z)$ of L at Z is zero when Z lies outside the segment $[X, Y]$ and it is equal to $\sigma \neq 0$ when Z lies inside $[X, Y]$; the points X and Y are then called the extremities, or the end points, or simply the ends of L; and the distance XY is called the length of L (usually denoted by the doubled $2a$ of some quantity a).

Let α and $\beta > \alpha$ be the abscisses of the ends X and Y respectively of the homogeneous rigid rod L with the density σ and let $2a = \beta - \alpha$. Then (28) and (73) imply that the mass of L is

(88) $$m = \int_\alpha^\beta \sigma\, d\xi = \sigma(\beta - \alpha) = 2a\sigma.$$

On the other hand, the relations (31), (76), (73), (88) imply that the mass-center G of L is defined by

(89) $$\bar{\rho}_G = \frac{1}{2a\sigma}\int (\xi\bar{\xi}^0 + \eta\bar{\eta}^0 + \zeta\bar{\zeta}^0)\, dm = \frac{1}{2a}\int_\alpha^\beta \xi\, d\xi\, \bar{\xi}^0,$$

i.e.

(90) $$\bar{\rho}_G = \frac{1}{2}(\alpha + \beta)\bar{\xi}^0.$$

Hence G halves the segment $[X, Y]$, so that, G being chosen in the capacity of the origin of the axis $\Omega\xi$ of L, the abscisses of X and Y are $-a$ and a respectively.

At last, (75) implies that for a homogeneous rigid rod L with mass m and length $2a$, the quantity I, reffered to the system of reference $G\xi\eta\zeta$, is given by

$$I = \frac{1}{3} ma^2. \tag{91}$$

Before proceeding further, we shall expose some general considerations in connection with the term *constraint* imposed on a rigid body, repeatedly used above.

Working latterly on Hilbert's Sixth Problem concerning the axiomatical consolidation of the logical foundations of Newton-Eulerian mass-point and solid mechanics, we have been permanently and inevitably faced with one of its inalienable ingredients, namely the concept of geometrical restrictions or liaisons rigid bodies may be submitted to. The logical *niveau* of expositions one may come across in recent mechanical literary sources is that the reader is not entirely unjustified if he qualifies it by means of the paraphrase *mechanicum est, non legitur*. The roots of matter of the failure of any attempt to provide the liaison-concept with a clear-cut mathematical definition (failures bordering on genuine collapses menacing to degrade this notion to the level of *mésliaisons*, if there is such a word) are surface deep. They are concealed in the still not realized fact that the notion of geometrical constraints imposed on a rigid body is not a mathematical concept in the proper sense of the word: it is only an abridged *façon de parler* — a stenographical way, if one may say so, to reduce to a common denominator an infinite variety of particilar cases; that only an *inductio per enumerationem simplicem* is the proper approach to this notion; and that ergo the only mathematically acceptable alternative in this situation consists in a *definitio per enumerationem simplicem*, when all is said and done.

The age-old mechanical experience has approbated 10 particular cases of rigid body contacts that may be justifiably qualified as geometrical constraints imposed on solids. Here they are:

Liaison No 1. A point A fixed in the rigid body is constrained to remain on a surface given in space (possibly changeable in the course of time).

Liaison No 2. A surface fixed in the rigid body is constrained to pass through a point A given in space (possibly changeable in the course of time).

Liaison No 3. A point A fixed in the rigid body is constrained to remain on a curve line given in space (possibly changeable in the course of time).

Liaison No 4. A curve line fixed in the rigid body is constrained to pass through a point A given in space (possibly changeable in the course of time).

Liaison No 5. A point A fixed in the rigid body is constrained to coincide with a point given in space (possibly changeable in the course of time).

Liaison No 6. A surface fixed in the rigid body is constrained to touch a surface given in space (possibly changeable in the course of time), the common point A of the surfaces being in the general case variable on both of them.

Liaison No 7. A surface fixed in the rigid body is constrained to touch a curve line given in space (possibly changeable in the course of time), the common point of A the surface and the curve line being in the general case variable on both of them.

Liaison No 8. A curve line fixed in the rigid body is constrained to touch a surface given in space (possibly changeable in the course of time), the common point of A the curve line and the surface being in the general case variable on both of them.

Liaison No 9. A curve line fixed in the rigid body is constrained to intersect a curve line given in space (possibly changeable in the course of time), the common point of A the curve lines being in the general case variable on both of them.

Liaison No 10. A curve line fixed in the rigid body is constrained to touch a curve line given in space (possibly changeable in the course of time), the common point A of the curve lines being in the general case variable on both of them.

In any of those 10 cases the point A is called the point of contact of the rigid body with the corresponding constraint or liaison.

At that, the geometrical constraint or liaison imposed on the rigid body is, by definition, a *point* in the cases of Liaisons Nos 2, 4 and 5; a *curve line* in the cases of Liaisons Nos 3, 7, 9 and 10; and a *surface* in the cases of Liaisons Nos 1, 6 and 8.

Besides, a liaison is called rheonomic if it is a changeable in the course of time, and scleronomic otherwise.

All these "definitions" require some specifications.

Let by definition $\mathbf{r}_A = \mathbf{OA}$ and

$$\mathbf{r}_A = x_A \mathbf{i} + y_A \mathbf{j} + z_A \mathbf{k}. \tag{92}$$

Let also $\bar{\rho}_A = \overline{\Omega A}$ and

$$\bar{\rho}_A = \xi_A \bar{\xi}^0 + \eta_A \bar{\eta}^0 + \zeta_A \bar{\zeta}^0. \tag{93}$$

Now (92), (93), (12), (18) and the identity $\mathbf{OA} = \overline{O\Omega} + \overline{\Omega A}$ imply

$$\begin{cases} x_A = x_\Omega + a_{11}\xi_A + a_{12}\eta_A + a_{13}\zeta_A\,, \\ y_A = y_\Omega + a_{21}\xi_A + a_{22}\eta_A + a_{23}\zeta_A\,, \\ z_A = z_\Omega + a_{31}\xi_A + a_{32}\eta_A + a_{33}\zeta_A\,. \end{cases} \tag{94}$$

First of all, let us note that the expression "point A given in space (possibly changeable in the course of time)" of Liaisons Nos 2, 4 and 5 means that completely determined functions

$$x_A = x_A(t),\ y_A = y_A(t),\ z_A = z_A(t) \tag{95}$$

of the time t are prescribed by the very conditions of the particular dynamical problem under consideration, whereas the expression "the common point A ... being in the general case variable" of Liaisons Nos 6–10 means that no information about the point of contact A is given in those same conditions, so that x_A, y_A, z_A are unknown quantities of the dynamical problem.

Similarly, the expression "point A fixed in the rigid body" of Liaisons Nos 1,3 and 5 means that ξ_A, η_A, ζ_A are completely determined constants prescribed by the very conditions of the particular dynamical problem under consideration, whereas the expression "the common point A ... being in the general case variable" of Liaisons Nos 6–10 means that no information about the point of contact A is given in those same conditions, so that

$$\xi_A = \xi_A(t),\ \eta_A = \eta_A(t),\ \zeta_A = \zeta_A(t) \tag{96}$$

are wholly unknown functions of the time t.

If P denotes any point in space and by definition $r = OP$ and

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \tag{97}$$

then the expression "surface given in space (possibly changeable in the course of time)" of Liaisons Nos 1, 6 and 8 means that a completely determined function $F(x, y, z, t)$ with

$$\mathrm{grad}F \neq \mathbf{0} \tag{98}$$

provided by definition

$$\mathrm{grad}F = \frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k} \tag{99}$$

is prescribed by the very conditions of the particular dynamical problem under consideration, the surface itself being defined by the equation

$$F(x, y, z, t) = 0. \tag{100}$$

Similarly, the expression "surface fixed in the rigid body" of Liaisons Nos 2, 6 and 7 means that a completely determined function $f(\xi, \eta, \zeta)$ with

$$\mathrm{grad}f \neq 0 \tag{101}$$

provided

$$\text{grad} f = \frac{\partial f}{\partial \xi}\bar{\xi}^0 + \frac{\partial f}{\partial \eta}\bar{\eta}^0 + \frac{\partial F}{\partial \zeta}\bar{\zeta}^0 \tag{102}$$

is prescribed by the very conditions of the particular dynamical problem under consideration, so that, if (24), then the surface itself is defined by the equation

$$f(\xi, \eta, \zeta) = 0. \tag{103}$$

At last, the expression "curve line given in space (possibly changeable in the course of time)" of Liaisons Nos 3, 7, 9 and 10 means that completely determined functions $F_\nu(x, y, z, t)$ $(\nu = 1, 2)$ with

$$\text{grad} F_1 \times \text{grad} F_2 \neq 0 \tag{104}$$

are prescribed by the very conditions of the particular dynamical problem under consideration, the curve line itself being defined by the system of equations

$$F_\nu(x, y, z, t) = 0 \quad (\nu = 1, 2). \tag{105}$$

Similarly, the expression "curve line fixed in the rigid body" of Liaisons Nos 4 and 8–10 means that completely determined functions $f_\nu(\xi, \eta, \zeta)$ $(\nu = 1, 2)$ with

$$\text{grad} f_1 \times \text{grad} f_2 \neq 0 \tag{106}$$

are prescribed by the very conditions of the particular dynamical problem under consideration, the curve line itself being defined by the system of equations

$$f_\nu(\xi, \eta, \zeta) = 0 \quad (\nu = 1, 2). \tag{107}$$

Under these notations the conditions of tangentiality of Liaisons Nos 6–8 and 10 are expressed by the relations

$$\text{grad}_A F \times \text{grad}_A f = \mathbf{0}, \tag{108}$$

$$\text{grad}_A F_1 \times \text{grad}_A F_2 \cdot \text{grad}_A f = 0, \tag{109}$$

$$\text{grad}_A F \cdot \text{grad}_A f_1 \times \text{grad}_A f_2 = 0, \tag{110}$$

$$(\text{grad}_A F_1 \times \text{grad}_A F_2) \times (\text{grad}_A f_1 \times \text{grad}_A f_2) = \mathbf{0}, \tag{111}$$

respectively, the index A indicating that the values of the corresponding quantities are taken in the common point A of the respective surface and curve lines.

Finally, the condition of intersection of the curve lines of Liaison No 9 is expressed by the relation

$$(\mathrm{grad}_A F_1 \times \mathrm{grad}_A F_2) \times (\mathrm{grad}_A f_1 \times \mathrm{grad}_A f_2) \neq \mathbf{0}. \tag{112}$$

Let us note *entre parenthéses* that equations (108) and (111) are of the kind

$$\mathbf{a} \times \mathbf{b} = \mathbf{0} \quad (\mathbf{a} \neq \mathbf{0},\ \mathbf{b} \neq \mathbf{0}) \tag{113}$$

in view of

$$\mathrm{grad}_A F \neq \mathbf{0}, \tag{114}$$

$$\mathrm{grad}_A f \neq \mathbf{0} \tag{115}$$

as well as

$$\mathrm{grad}_A F_1 \times \mathrm{grad}_A F_2 \neq \mathbf{0}, \tag{116}$$

$$\mathrm{grad}_A f_1 \times \mathrm{grad}_A f_2 \neq \mathbf{0} \tag{117}$$

in accordance with (98), (101), (104), (106) respectively. Now (113) implies

$$\mathbf{a} \times \mathbf{b} \cdot \mathbf{a} = 0, \quad \mathbf{a} \times \mathbf{b} \cdot \mathbf{b} = 0. \tag{118}$$

The corollaries (118) of (113) are not quite independent by virtue of (113) itself. In such a way the vector equation (113) is equivalent with 2 rather than 3 scalar equations, as a result of the first relation (118), say. Hence the same holds for (108) and (111), too.

In order to wind up the mathematical formalization of Liaisons Nos 1–10, let us recall to mind that, along with the relations exposed above, the following conditions must be also satisfied as the case may be:

$$F(x_A, y_A, z_A, t) = 0, \tag{119}$$

or

$$f(\xi_A, \eta_A, \zeta_A) = 0, \tag{120}$$

or

$$F_\nu(x_A, y_A, z_A, t) = 0 \quad (\nu = 1, 2) \tag{121}$$

or

$$f_\nu(\xi_A, \eta_A, \zeta_A) = 0 \quad (\nu = 1, 2). \tag{122}$$

It stands to reason that a rigid body may be submitted to more than one constraints, in other words that two or even more of Liaisons Nos 1–10 may be at hand simulataneously.

One of the aims of the present paper is to adapt Liaisons Nos 1–10 to that special kind of solids rigid rods are of. Before doing so, nowever, we shall first and foremost direct the reader's attention to a most important point: *is all this possible?*

As well as in any other case of a mathematical definition, any of the constraints listed above as Liaisons 1–10 being once proclaimed, the question of the *existence* of such a something arises quite naturally and completely unavoidably. One may define a lot of things calling them by certain names, but there is no guarantee beforehand, or apriorily, that such things exist. Existence and Possibility are the two faces of the same mathematical coin, and its name is *Consistency* of the respective conditions.

Are Liaisons Nos 1–10 possible or not: that is the question. Besides, it is not quite indifferent who is answering this question.

A geometer, for one, will certainly be of the opinion that his answer should be in the affirmative. For the sake of definiteness let us consider, for instance, Liaison No 10. Now the geometer is meditating in the following manner. For the mathematically adequate formulation of that constraint 12 auxiliary quantities have been called into play, namely (13) and

$$x_A,\ y_A,\ z_A,\ \xi_A,\ \eta_A,\ \zeta_A\,, \tag{123}$$

connected by means of 9 mutually independent relations, namely (94), (121), (122) and (111) (the latter being, as underlined, equivalent with 2 scalar equations). In such a way, only 3 of the unknowns (13) and (123) are mutually independent. Let us suppose that the functions in the left-hand sides of (105), (107) are mathematically propitious: their analytic nature is such that the system of 9 equations (94), (121), (122), (111) defines 9 implicit functions from among the variables (13) and (123) of 3 of them; or, for the sake of greater generality, let there exist 3 independent parameters

$$q_\lambda \quad (\lambda = 1, 2, 3) \tag{124}$$

so that the equations (94), (121), (122) and (111) determine the quantities (13) and (123) as such functions of (124) and possibly of the time t that they satisfy those equations identically for any triplet $(q_1, q_2, q_3) \in D \subset R^3$ and for any t, the domain D being appropriately chosen. Now, if (124) are any functions

$$q_\lambda = q_\lambda(t) \quad (\lambda = 1, 2, 3) \tag{125}$$

of the time t, the unknown variables (13) and (123) become completely determined functions (14), (15), (95), (96) of t, describing such a motion of the rigid body S that satisfies the conditions of Liaison No 10. *Quod erat demonstrandum* for a geometer.

We heard the geometer say — *audiatur et altera pars.* What does the mechanician say?

Before we answer that question, let us make a small crystallographical allegory. As it is well known, certain substances crystallize in the shape of various polyhedrons. For a geometer any polyhedron is acceptable in the capacity of a crystalline form. Nature, however, withstands such a standpoint. According to Haüy's fundamental law of crystallography (1784), if three non-parallel edges of a crystal are chosen as co-ordinate axes, then the position of any of its faces may be determined by means of small whole numbers. This law is a phenomenological reverberation of power interactions between the particles generating the crystal. Geometry is one thing — physics something quite different.

After this lyrical digression, let us emphasize that the geometer's affirmative answer is a *conditio sine qua non* for the mechanician: if a hypothetical liaison is *geometrically* impossible, then a *fortiori* it is impossible *mechanically* too. (*Exampli gratia*, imagine a dynamical problem concerning a mass-point, compelled to move simultaneously along two parallel straight lines.) There is, however, an essential difference between the attitudes of a geometer and a mechanician towards the constraints imposed on solids.

For a geometer the question what makes a rigid body S abide by the constraints it is submitted to in a dynamical problem is flat and plain poinless — as void of sense as, for instance, the question of the temperature of a triangle, the colour of a tetrahedron, or the odour of a sphere. For a mechanician this is not so. A geometer may imagine a pyramid built on sand out of spheres; for a mechanician, however, such a construction is condemned to collapse. Whereas a geometer professes monotheism, the scientific religion of a mechanician is ditheism. The geometer's God is Euclid; the mechanician's Gods are Euclid and Newton. A geometer's Commandments, though more than Ten, are Euclid's axioms; in addition, a mechanician lives lives in obedience to Newton's axiomata sive leges motus. The geometer's world is shaped by points, lines and planes; besides, the mechanician's world is hounted by forces. A geometer's space may be 4-dimensional, n-dimensional, and even trans-dimensional; the mechanician's space is 3-dimensional at all costs. The monopoly of Three Dimensions is an order in Councel of Her Majesty the Force — its proscriptions a mechanician is subordinated to are most severe ones: *dura lex, sed lex.*

The Holy Scripture an orthodoxal mechanician is oblidged to respect to the letter in solid dynamics has been taken down around 1750 by Euler. Gone by the appelation Discovery of a New Principle of Mechanics [1], it containes Solid Dynamics Six Commandments (54), (55) governing, as already repeatedly stressed, the motion of any rigid body, subjected to any mechanical constraints and submitted to the action of any forces.

The equations (54), (55) have just now been qualified as a Dynamical Testament; anyone can see that this is done in a polemical soliloquy. Mathematically spoken, the simple truth is that these relations should be formulated in a conditional clause. Namely, the dynamical demeanour of a solid S would be governed by Euler's equations (54), (55) if and only if these equations were ... true.

The meaning of this – at first sight seemingly joky, in point of fact, however, as solemn as grave — statement may become transparent taking cognizance of the following two curcumstances.

First, Euler's dynamical *equations* (54), (55) are mathematical formalizations of Euler's dynamical *axioms*, usually called laws, or principles, or postulates, or hypotheses, etc., of *momentum* (Ax 1 E) and of *moment of momentum* (Ax 2 E) of a rigid body. Now Ax 1 E states that there *exists* at least one Cartesian system of reference (called inertial according to Euler or simply inertial) for which (54) hold; and Ax 2 E states that (55) certainly hold provided $Oxyz$ is inertial. As a matter of fact, there exist infinitely many inertial systems of reference, if any: according to a dynamical criterion, the system Σ_1 being inertial, the system Σ_2 is also inertial if and only if the motion of Σ_2 with respect to Σ_1 is a rectalinear uniform translation. *A fortiori*, there exist infinitely many *non-inertial* systems of reference as well — such ones, with respect to which Euler's equations (54), (55) are utterly *wrong* — *wrong* in the utmost degree. Second, all those explanations are ***ad nec plus ultra*** provisional. Euler's dynamical principles Ax 1 E and Ax 2 E being *par excellence* mathematical *Axiome in Hilbertschen Sinne* of the word — inasmuch unprovable, as indisprovable, putting it another way — from a purely logical point of view one is faced with a situation of choice comparable to that of *asinus Buridani inter duo prata*: one is (from a purely logical point of view, we repeat) inasmuch in his own rights to take them at their face value, as to disdain them as cheap falsifications. In the first case one accepts equations (54), (55) as divine revelations; in the second case one repulses them as satanic seduction. In the first case one becomes adept of Eulerian dynamical tradition; in the second case one is facing one's own dynamical difficulties. The situation is to a heir's breadth the same as that of the Fifth Postulate in Euclidean geometry: take it, or be off. More explanations would seem like wish-wash.

All this once grasped, let us come back to the mechanical aspects of our curve-touch-curve liaison. As the reader might have already forgotten it, the question has been raised about the dynamical admissibility of such a constraint — about the possibility to materialize it mathematically, if one can put it this way. Now for a Newton-Eulerian mechanician anything concerning rigid bodies is possible, provided Euler's equations (54), (55) permit it. Let us therefore consult them apropos of this occasion.

We shall at first play the fool, and we shall afterwards try alter this impression. We shall reason in the following manner.

Let L_x, L_y, L_z and H_ξ, H_η, H_ζ denote the left-hand sides of the first, second and third equation (54) and (55) respectively, and let by definition

$$\begin{aligned} (126) \qquad \mathbf{L} &= L_x \mathbf{i} + L_y \mathbf{j} + L_z \mathbf{k}, \\ (127) \qquad \mathbf{H} &= H_\xi \bar{\xi}^0 + H_\eta \bar{\eta}^0 + H_\zeta \bar{\zeta}^0. \end{aligned}$$

Now (126), (127) and (48), (49), (51), (52) imply that the equations (54), (55) may be written in the form

$$\begin{aligned} (128) \qquad \mathbf{L} &= \mathbf{F} + \mathbf{R}, \\ (129) \qquad \mathbf{H} &= \mathbf{M}_G + \mathbf{N}_G \end{aligned}$$

respectively.

On the other hand, the definitions (12), (53), (32), the identity (47) and the relations (18) imply

$$(130) \qquad \begin{cases} x_G = x_\Omega + a_{11}\xi_G + a_{12}\eta_G + a_{13}\zeta_G, \\ y_G = y_\Omega + a_{21}\xi_G + a_{22}\eta_G + a_{23}\zeta_G, \\ z_G = z_\Omega + a_{31}\xi_G + a_{32}\eta_G + a_{33}\zeta_G. \end{cases}$$

Now ξ_G, η_G, ζ_G are, for any particular solid S, completely determined characteristic constants; hence (130) and the well known relations

$$(131) \qquad a_{\mu\nu} = a_{\mu\nu}(\psi, \phi, \theta) \quad (\mu, \nu = 1,2,3)$$

between the cosine-directors $a_{\mu\nu}$ $(\mu, \nu = 1,2,3)$ of $Oxyz$ snd $\Omega\xi\eta\zeta$ on the one hand, and Euler's angles ψ, ϕ, θ, on the other hand., in conjunction with the inference made above that

$$(132) \qquad x_\Omega = x_\Omega(q,t), \; y_\Omega = y_\Omega(q,t), \; z_\Omega = z_\Omega(q,t),$$

$$(133) \qquad \psi = \psi(q,t), \; \phi = \phi(q,t), \; \theta = \theta(q,t)$$

are completely determined functions of the parameters

$$(134) \qquad q = (q_1, q_2, q_3)$$

of the rigid body S and possibly of the time t, imply that the same holds also for

(135) $$x_G = x_G(q,t),\ y_G = y_G(q,t),\ z_G = z_G(q,t).$$

Let now

(136) $$\vec{R} = (\mathbf{R}_A, \mathbf{N}_A),$$

the moment N_A being taken with respect to O, be the reaction of Liaison No 10 on the rigid body S through the point of contact A, existing in accordance with Ax R. The latter implies

(137) $$\mathbf{N}_A = \mathbf{r}_A \times \mathbf{R}_A .$$

On the other hand, the relations

(138) $$\mathbf{M}_G = \mathbf{M} + \mathbf{F} \times \mathbf{r}_G ,\quad \mathbf{N}_G = \mathbf{N}_A + \mathbf{R}_A \times \mathbf{r}_G$$

hold. Now (137), (138) imply that the equations (128), (129) may be written in the form

(139) $$\mathbf{F} = \mathbf{L} - \mathbf{R}_A ,$$

(140) $$\mathbf{M} = \mathbf{H} + \mathbf{r}_G \times \mathbf{L} + \mathbf{R}_A \times \mathbf{r}_A$$

respectively.

Let it be noted that Ax R declines any information apropos of the reaction (136) of the constraint save (137). In other words, no restriction is imposed on $\mathbf{R}_A$ save that it must satisfy the equations (54) and (55) or, just the same, (139) and (140).

Let now (125) be a set of arbitrarily fixed functions of the time t and let the same hold for

(141) $$\mathbf{R}_A = \mathbf{R}_A(t).$$

In the light of the relations established above between the quantities (13), (123), on the one hand, and the parameters (124), on the other hand, as well as in view of (141), the right-hand sides of (139), (140) prove to be completely determined functions of t. In such a manner they define an infinite variaty of systems of active forces with bases F and moments M with respect to O, and all of them are satisfying Euler's dynamical equations (54) and (55). In such a way, Liaison No 10 turns out to be mechanically as well feasible in a countless many case.

This conclusion is, however, premature, precipitate and thoughtless.

Indeed, the affirmative answer of the possibility problem concerning Liaison No 10 given above depends essentially on the arbitrariness in the choice of $\mathbf{F}$ and $\mathbf{M}$ by means of the relations (139) and (140), whereas in any particular dynamical problem involving a solid S the active forces applied on S

are given beforehand by the very conditions of the problem, so **F** and **M** are predeterminated. In other words, **F** and **M** being given in advance (or, just the same, the quantities F_x, F_y, F_z and M_x, M_y, M_z in Euler's equations (54) and (55) being known), the possibility problem consists in the existence of such a reaction (141) and such functions (125) that satisfy identically (54) and (55) for any t.

Let us try another dynamical approach that enjoys a good reputation among, if not even a professional adoration of, Twentieth Century's mechanicians. We are speaking of Lagrange's dynamical equations (68), applicable in more restricted curcumstances, that is to say when the constraints of the type of Liaison No 10 are s m o o t h, i. e. provided

$$\mathbf{R}_A \cdot \operatorname{grad}_A F_1 \times \operatorname{grad}_A F_2 = 0 \tag{142}$$

in view of (104).

Also sprach Lagrange:

"Les méthodes que j'y expose ne demandent ni constructions, ni raisonnements géométriques ou méchaniques, mais seulement des opérations algébriques assujetties a une marche réguliére et uniforme ... formules générales, dont le simple developpment donne tous les équations nécessaires pour la solution de chaque probléme" (*Méchanique Analitique* (sic), 1788).

Now if all that were true, then the consistency of Lagrange's dynamical equations of motion of a solid submitted to a constraint of the kind of Liaison No 10 should be a steadfast guarantee for the possibility of that constraint in any one particular dynamical problem. Such a conclusion is, however, utterly wrong, since there exist counter-examples. In other words, there exist certain dynamical problems concerning rigid bodies submitted to a Liaison No 10, for which Lagrange's equations (68) displays consistency, whereas Euler's equations (54) and (55) establish clear-cut inconsistency.

The next point of our agenda is to adapt Liaisons Nos 1–10, or at least some of them, to the special case of rigid rods. The proviso "at least some of them" is due to the obvious curcumstance that for rigid rods some of these constraints are manifestly preposterous. Such ones, for instance, are Liaisons Nos 2, 6 and 7, where a "surface fixed in the rigid body" is supposed — a hypothesis point-blank unfeasible in the rigid rod case. As regards Liaisons Nos 4 and 8–10, where "a curve line fixed in the rigid body" is presumed, it is quite clear that such a "curve" line may be only the directrix of the rod. At last, as regards Liaisons Nos 1, 3 and 5, where "a point A fixed in the rigid body" is assumed, it is clear, by virtue of both Ax R and Ax L, that A must unconditionally lie in the directrix of the rod.

In such a manner, we may restate Liaisons Nos 1, 3–5, 8–10, giving them formulations accomodated to rigid rods: at that, for the sake of brevity, the clumsy expression "directrix of a rod" is substitute by "rod" only.

The revised in such a manner definitions read as follows:

Liaison No 1. A point A fixed on the rod is constrained to remain on a surface given in space (possibly changeable in the course of time).

Liaison No 3. A point A fixed on the rod is constrained to remain on a curve line given in space (possibly changeable in the course of time).

Liaison No 4. The rod is constrained to pass through a point A given in space (possibly changeable in the course of time).

Liaison No 5. A point A fixed on the rod is constrained to coincide with a point given in space (possibly changeable in the course of time).

Liaison No 8. The rod is constrained to touch a surface given in space (possibly changeable in the course of time), the common point A of the rod and the surface being in the general case variable on both of them.

Liaison No 9. The rod is constrained to intersect a curve line given in space (possibly changeable in the course of time), the common point A of the rod and the curve line being in the general case variable on both of them.

Liaison No 10. The rod is constrained to touch a curve line given in space (possibly changeable in the course of time), the common point A of the rod and the curve line being in the general case variable on both of them.

Analysis. Let the system of reference $Oxyz$ be inertial and the system of reference $\Omega\xi\eta\zeta$ be invariably connected with the rod L as indicated above: G being the mass-center of L, $G\xi$ the directrix of L, and $G\zeta$ satisfying (16). If (93) holds, then

$$\eta_A = 0, \quad \zeta_A = 0, \tag{143}$$

whence

$$\bar{\rho}_A = \xi_A \bar{\xi}^0. \tag{144}$$

Now (94) with $\Omega = G$ and (143) imply

$$x_A = x_G + a_{11}\xi_A, \quad y_A = y_G + a_{21}\xi_A, \quad z_A = z_G + a_{31}\xi_A. \tag{145}$$

Let (136) be the reaction in A generated by the constraint according to Ax R and let by definition

$$\mathbf{R}_A = R_{Ax}\mathbf{i} + R_{Ay}\mathbf{j} + R_{Az}\mathbf{k}. \tag{146}$$

If $\mathbf{N}_G$ denotes the moment of (136) with respect to G, then Ax R implies

$$\mathbf{N}_G = \bar{\rho}_A \times \mathbf{R}_A\,. \tag{147}$$

Now (147), (144) and

$$(148) \qquad \mathbf{R}_A = R_{A\xi}\bar{\xi}^0 + R_{A\eta}\bar{\eta}^0 + R_{A\zeta}\bar{\zeta}^0$$

imply

$$(149) \qquad \mathbf{N}_G = \xi_A(-R_{A\zeta}\bar{\eta}^0 + R_{A\eta}\bar{\zeta}^0).$$

Besides, the relations (18), in view of (16), imply

$$(150) \qquad R_{A\eta} = a_{12}R_{Ax} + a_{22}R_{Ay} + a_{32}R_{Az},$$

$$(151) \qquad R_{A\zeta} = a_{13}R_{Ax} + a_{23}R_{Ay},$$

and (149)–(151) imply

$$(152) \; \mathbf{N}_G = \xi_A(-(a_{13}R_{Ax} + a_{23}R_{Ay})\bar{\eta}^0 + (a_{12}R_{Ax} + a_{22}R_{Ay}) + a_{32}R_{Az}\bar{\zeta}^0).$$

In view of (146) and (152) Euler's dynamical equations (54) and (82) take the form

$$(153) \qquad m\ddot{x} = F_x + R_{Ax}, \; m\ddot{y} = F_y + R_{Ay}, \; m\ddot{z} = F_z + R_{Az}$$

and

$$(154) \quad \begin{cases} I(\dot{\omega}_\eta - \omega_\zeta\omega_\xi) = M_{G\eta} - \xi_A(a_{13}R_{Ax} + a_{23}R_{Ay}), \\ I(\dot{\omega}_\zeta + \omega_\xi\omega_\eta) = M_{G\zeta} + \xi_A(a_{12}R_{Ax} + a_{22}R_{Ay} + a_{32}R_{Az}). \end{cases}$$

In such a manner, for any of Liaisons Nos 1, 3–5, 8–10 three unknown quantities

$$(155) \qquad R_{Ax}, \; R_{Ay}, \; R_{Az}$$

are called into play. This general conclusion, supplemented by the admonition that the relations (19) and (22) must also be taken into account, is individualized as follows.

Liaison No 1. The point A being fixed on $G\xi$, the quantity ξ_A is a known constant. The equation of the surface being (100), the relation (119) holds. In such a way, Liaison No 1 brings into play 11 variables

$$(156) \qquad x_G, \; y_G, \; z_G, \; \psi, \; \phi, \; x_A, \; y_A, \; z_A, \; R_{Ax}, \; R_{Ay}, \; R_{Az}$$

with 9 equations at one's disposal for their determination, namely (145), (153), (154) and (119). If δ denotes the difference of the numbers of the unknown quantities and of the available equations, then $\delta = 2$.

Liaison No 3. The situation is similar to that of Liaison No 1, with the difference that now two equations (105), instead of (100) are at hand, hence

(121) instead of (119). In such a way, 10 equations are available for 11 unknowns (156). Hence $\delta = 1$.

Liaison No 4. In this case (95) are completely determined functions and

(157) $$\xi_A = \xi_A(t)$$

is unknown. In such a way, 9 variables

(158) $$x_G,\ y_G,\ z_G,\ \psi,\ \phi,\ \xi_A,\ R_{Ax},\ R_{Ay},\ R_{Az}$$

participate, for which 8 equations are available, namely (145), (153), (154). Hence $\delta = 1$.

Liaison No 5. The situation is similar to that of Liaison No 4, with the difference that now ξ_A is a known constant, the point A being fixed on $G\xi$. In such a way, 8 variables

(159) $$x_G,\ y_G,\ z_G,\ \psi,\ \phi,\ R_{Ax},\ R_{Ay},\ R_{Az}$$

take part, for which 8 equations are available, namely (145), (153), (154). Hence $\delta = 0$.

Liaison No 8. The unknown quantities are 12, namely

(160) $$x_G,\ y_G,\ z_G,\ \psi,\ \phi,\ x_A,\ y_A,\ z_A,\ \xi_A,\ R_{Ax},\ R_{Ay},\ R_{Az}$$

since the point of contact A is indeterminate on both the rod and the surface (100). The tangentiality condition (110) now becomes

(161) $$\bar{\xi}^0 \cdot \mathrm{grad}_A F = 0,$$

i. e.

(162) $$a_{11}\frac{\partial F}{\partial x_A} + a_{21}\frac{\partial F}{\partial y_A} + a_{31}\frac{\partial F}{\partial z_A} = 0$$

by virtue of (18) and (99). In such a manner, the variables (160) are connected by means of 10 equations, namely (145), (153), (154), (119) and (162). Hence $\delta = 2$.

Liaison No 9. The situation is similar to that of Liaison No 8, as regards the unknown quantities (160). The equations of the curve line being (105), the relations (121) now hold. In such a manner, the variables (160) are connected by means of 10 equations, namely (145), (153), (154), (121). Hence $\delta = 2$.

Liaison No 10. The unknown quantities are again (160), and the relations (121) hold. In addition, the tangentiality condition (111) now becomes

(163) $$\bar{\xi}^0 \times (\mathrm{grad}_A F_1 \times \mathrm{grad}_A F_2) = \mathbf{0}.$$

The equation (163) is equivalent with

$$(164) \qquad (\bar{\xi}^0 \cdot \operatorname{grad}_A F_2)\operatorname{grad}_A F_1 \;-\; (\bar{\xi}^0 \cdot \operatorname{grad}_A F_1)\operatorname{grad}_A F_2 \;=\; \mathbf{0}.$$

Similar to (162), it is proved that

$$(165) \qquad \bar{\xi}^0 \operatorname{grad}_A F_\nu \;=\; a_{11}\frac{\partial F_\nu}{\partial x_A} + a_{21}\frac{\partial F_\nu}{\partial y_A} + a_{31}\frac{\partial F_\nu}{\partial z_A} \quad (\nu = 1, 2).$$

As it has been already underlined, (111) is equivalent with two scalar equations; the same holds, consequently, for (164) with (165). In such a manner, the 12 unknown quantities (160) are connected by means of also 12 equations, namely (145), (153), (154), (121) and (164). Hence $\delta = 0$.

Let δ be the defect of the corresponding constraint. Now the analysis exposed above displays that the defect of any of the Liaisons Nos 1, 3–5, 8–10 is a non-negative number; at that, it is zero exactly in the cases of Liaisons Nos 5 and 10.

Since δ is by definition the surplus of the unknown quantities over the equations connecting them, the conclusion $\delta \geq 0$ displays that there may be mathematical hopes that Liaisons Nos 1, 3–5, 8–10 will be, in the general case, consistent. At that, in the cases Nos 5 and 10 one may expect that the dynamical problem will posseses one and only solution, while in the remaining cases the dynamical problems are, in the general case, mathematically imperfect, that is to say, admitting more than one solutions.

Let us note that conclusions almost identical to those made here concerning rigid rods may be derived in the general cases of Liaisons Nos 1–10 concerning arbitrary solids. This is, however, another topic.

The next point we shall discuss here is concerned with the so-called smooth constraints. As already mentioned, a constraint is qualified as smooth if the reaction R_A it generates is normal to that constraint at the point of contact A. We shall discuss the smoothness conditions in the cases of Liaisons Nos 1, 3–5 and 8–10.

Liaison No 1. Since the normal to the surface (100) at the point A is defined by $\operatorname{grad}_A F$, this surface will be smooth iff

$$(166) \qquad \mathbf{R}_A \;=\; \lambda \operatorname{grad}_A F,$$

λ being an appropriate indeterminate multiplier. The relation (166), together with (146) and (99), imply

$$(167) \qquad R_{Ax} \;=\; \lambda\frac{\partial F}{\partial x_A}, \quad R_{Ay} \;=\; \lambda\frac{\partial F}{\partial y_A}, \quad R_{Az} \;=\; \lambda\frac{\partial F}{\partial z_A}.$$

Since the components of $\text{grad}_A F$ are known, (167) implies that in the case of a smooth Liaison No 1 the unknown quantities are

$$x_G,\ y_G,\ z_G,\ \psi,\ \phi,\ x_A,\ y_A,\ z_A,\ \lambda \tag{168}$$

instead of (156), i. e. 9 altogether. Since they are connected by means of 9 equations, $\delta = 0$.

Liaison No 3. Since the normal plane to the curve line (105) at the point A is parallel to $\text{grad}_A F_\nu$, $(\nu = 1, 2)$, this curve will be smooth iff

$$R_A = \lambda \,\text{grad}_A F_1 + \mu \,\text{grad}_A F_2 \,, \tag{169}$$

λ and μ being appropriate indeterminate multipliers. The relation (169), together with (146) and (99), imply

$$R_{Ax} = \lambda \frac{\partial F_1}{\partial x_A} + \mu \frac{\partial F_2}{\partial x_A}\,,\ R_{Ay} = \lambda \frac{\partial F_1}{\partial y_A} + \mu \frac{\partial F_2}{\partial y_A}\,,\ R_{Az} = \lambda \frac{\partial F_1}{\partial z_A} + \mu \frac{\partial F_2}{\partial z_A}\,, \tag{170}$$

In the same way as above it is seen that in the case of a smooth Liaison No 3 the unknown quantities are

$$x_G,\ y_G,\ z_G,\ \psi,\ \phi,\ x_A,\ y_A,\ z_A,\ \lambda,\ \mu \tag{171}$$

instead of (156), i. e. 10 altogether. Since they are connected by means of 10 equations, $\delta = 0$.

Liaison No 4. An interesting mechanical trick must be applied in this case. Indeed, the constraint being a point A, the notion of perpendicularity becomes meaningless. On the other hand, the motion concept being relative, the rod itself may be regarded as a constraint for the point A. From this point of view, the smooth rod is acting on with a reaction $-\vec{R}_A$, directly contrary to $\vec{R}_A$ by virtue of Newton's *Lex III*, being at that normal to the rod. That is why $\vec{R}_A$ is also normal to the latter, whence

$$\mathbf{R}_A = \lambda \bar{\eta}^0 + \mu \bar{\zeta}^0 \,. \tag{172}$$

Now (172), (146) and (18), (16) imply

$$R_{Ax} = \lambda a_{12} + \mu a_{13}, \quad R_{Ay} = \lambda a_{22} + \mu a_{23}, \quad R_{Az} = \lambda a_{32} + \mu a_{33}, \tag{173}$$

and (173) imply that in the case of a smooth Liaison No 4 the unknown quantities are

$$x_G,\ y_G,\ z_G,\ \psi,\ \phi,\ \xi_A,\ \lambda,\ \mu \tag{174}$$

instead of (158), i. e. 8 altogether. Since they are connected by means of 8 equations, $\delta = 0$.

Liaison No 5. The meaning of smoothness is pointless.

Liaison No 8. The surface being defined by (100), it will be smooth iff (166) or, just the same, (167) hold. Ergo, in the case of a smooth Liaison No 8 the unknown quantities are

$$x_G,\ y_G,\ z_G,\ \psi,\ \phi,\ x_A,\ y_A,\ z_A,\ \xi_A,\ \lambda \tag{175}$$

instead of (160), i. e. 10 altogether. Since they are connected by means of 10 equations, $\delta = 0$.

Liaison No 9. Reasoning as in the case of Liaison No 4, one concludes that in the case of a smooth Liaison No 9 the condition

$$R_A = \lambda(\operatorname{grad}_A F_1 \times \operatorname{grad}_A F_2) \times (\operatorname{grad}_A f_1 \times \operatorname{grad}_A f_2) \tag{176}$$

must hold. Hence the unknown quantities are (175) instead of (160), i. e. 10 altogether. Since they are connected by means of 10 equations, $\delta = 0$.

Liaison No 10. This constraint supposed smooth, the relation (172) holds, whence (173). Therefore, the unknown quantities are

$$x_G,\ y_G,\ z_G,\ \psi,\ \phi,\ x_A,\ y_A,\ z_A,\ \xi_A,\ \lambda,\ \mu \tag{177}$$

instead of (160), i. e. 11 altogether. Since they are connected by means of 12 equations, $\delta = -1$.

Summing up, we conclude that in all cases of Liaisons Nos 1, 3, 4, 6 and 9 the smoothness hypothesis leads to perfect mathematical problems, under "perfect" being understood that the number of the unknown quantities of the dynamical problem under consideration equals the number of the equations available for their determination. (Let us note emphatically that the "perfectness" does in no wise necessarily mean that the particular dynamical problem has a solution indeed.) Since the smoothness of the constraints is a necessary condition for the very applicability of Lagrange's dynamical equations (68), in all those cases there may be no formal objection against their application.

Let us remind at that that in the case of a rigid rod the kinetic energy T takes the most simplified form (87) facilitating the application of those equations. Let it be also reminded that all the above conclusions concerning the "perfectness" of Liaisons Nos 1, 3, 4, 8 and 9 are derived on the basis of Eulerian approach towards the dynamical problem in question rather than on Lagrangean one. Indeed, in those implications the equations (153), (154) have played a fundamental role, while Lagrange's equations (68) themselves, where the reactions of the constraints have been expelled in the most approbrious way are as blind as *Alpa europaea* as regards the passive forces.

The case of Liaison No 10 is somewhat more peculiar — extravagant, one is inclined to say. In it the application of Lagrange's dynamical equations is somewhat in no way self-understood. A sagacious sailor in dynamical waters navigating his vessel by the aid of Euler's dynamical compass (153), (154) is prone to dread sunken rocks that may be deathly dangerous for a happy-go-lucky adept of Lagrangean dynamical tradition. This is, however, a topic we shall treat *alibi*, having the intention of squeezing some more drops of juice out of this green fruit. By that time, *lectori benevolo salutem.*

References

[1] L. Euler, Découverte d'un nouveau principe de mécanique, *Hist. Acad. Sci. Berlin*, **6**, 1750, 185–217 = *Opera omnia* **II 5**, 81–108.

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Received 28.04.1993