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Some Non-Commutative Neutrix Convolution Products of Functions

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The non-commutative neutrix convolution product of the functions $\cos_{-}(\lambda x)$ and $\cos_{+}(\mu x)$ is evaluated. Further similar non-commutative neutrix convolution products are evaluated and deduced.

In the following we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. The convolution product $f * g$ of two distributions f and g in $\mathcal{D}'$ is then usually defined by the equation

$$\langle (f * g)(x), \phi \rangle = \langle f(y), \langle g(x), \phi(x+y) \rangle \rangle$$

for arbitrary ϕ in $\mathcal{D}$, provided f and g satisfy either of the conditions

(a) either f or g has bounded support,

(b) the supports of f and g are bounded on the same side,

see Gel'fand and Shilov [6].

Note that if f and g are locally summable functions satisfying either of the above conditions then

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt. \tag{1}$$

It follows that if the convolution product $f * g$ exists by this definition then

$$f * g = g * f, \tag{2}$$

$$(f * g)' = f * g' = f' * g. \tag{3}$$

This definition of the convolution product is rather restrictive and so a neutrix convolution product was introduced in [2]. In order to define the neutrix convolution product we first of all let τ be a function in $\mathcal{D}$ satisfying the following properties:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \leq \tau(x) \leq 1$,
(iii) $\tau(x) = 1$ for $|x| \leq \frac{1}{2}$,
(iv) $\tau(x) = 0$ for $|x| \geq 1$.

The function τ_ν is now defined by

$$\tau_\nu(x) = \begin{cases} 1, & |x| \leq \nu, \\ \tau(\nu^\nu x - \nu^{\nu+1}), & x > \nu, \\ \tau(\nu^\nu x + \nu^{\nu+1}), & x < -\nu, \end{cases}$$

for $\nu > 0$.

We now give a new neutrix convolution product which generalizes the one given in [2].

Definition 1. *Let f and g be distributions in $\mathcal{D}'$ and let $f_\nu = f\tau_\nu$ for $\nu > 0$. Then the neutrix convolution product $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_\nu * g\}$, provided that the limit h exists in the sense that*

$$\operatorname*{N-lim}_{\nu \to \infty} \langle f_\nu * g, \phi \rangle = \langle h, \phi \rangle,$$

for all ϕ in $\mathcal{D}$, where N is the neutrix, see van der Corput [1], having domain N' the positive reals and range N'' the real numbers, with negligible functions finite linear sums of the functions

$$\nu^\lambda \ln^{r-1} \nu,\ \ln^r \nu,\ \nu^\mu e^{\lambda\nu},\ \nu^\mu \cos \lambda\nu,\ \nu^\mu \sin \lambda\nu \quad (\lambda \neq 0,\ r = 1, 2, \ldots)$$

and all functions which converge to zero in the usual sense as ν tends to infinity.

Note that in this definition the convolution product $f_\nu * g$ is defined in Gel'fand and Shilov's sense, the distribution f_ν having bounded support.

In the original definition of the neutrix convoution product, the domain of the neutrix N was the set of positive integers $N' = \{1, 2, \ldots, n, \ldots\}$ and the negligible functions were finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\ \ln^r n \quad (\lambda > 0,\ r = 1, 2, \ldots)$$

and all funtions which converge to zero in the usual sense as n tends to infinity. In [5], the set of negligible functions was extended to include finite linear sums of the functions

$$n^\lambda e^{\mu n} \quad (\mu > 0).$$

It is easily seen that any results proved with the original definition hold with the new definition. The following theorems, proved in [2] therefore hold, the first showing that the neutrix convolution product is a generalization of the convolution product.

Theorem 1. *Let f and g be distributions in $\mathcal{D}'$ satisfying either condition (a) or condition (b) of Gel'fand and Shilov's definition. Then the neutrix convolution product $f \circledast g$ exists and*

$$f \circledast g = f * g.$$

Theorem 2. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix convolution product $f \circledast g$ exists. Then the neutrix convolution product $f \circledast g'$ exists and*

$$(f \circledast g)' = f \circledast g'.$$

Note however that equation (1) does not necessarily hold for the neutrix convolution product and that $(f \circledast g)'$ is not necessarily equal to $f' \circledast g$.

A number of neutrix convolution products have been evaluated. For example, $x_-^\lambda \circledast x_+^r$ see [2], $x_-^{-r} \circledast x_+^s$ see [3], $\ln x_- \circledast \ln x_+$ see [7] and $\ln x_- \circledast x_+^\mu$, $x_-^\mu \circledast \ln x_+$ see [4].

We now define the locally summable functions $e_+^{\lambda x}$, $e_-^{\lambda x}$, $\cos^{(}_{+}\lambda x)$, $\cos^{(}_{-}\lambda x)$, $\sin_+(\lambda x)$ and $\sin_-(\lambda x)$ by

$$e_+^{\lambda x} = \begin{cases} e^{\lambda x}, & x > 0, \\ 0, & x < 0, \end{cases} \qquad e_-^{\lambda x} = \begin{cases} 0, & x > 0, \\ e^{\lambda x}, & x < 0, \end{cases}$$

$$\cos_+(\lambda x) = \begin{cases} \cos(\lambda x), & x > 0, \\ 0, & x < 0, \end{cases} \qquad \cos_-(\lambda x) = \begin{cases} 0, & x > 0, \\ \cos(\lambda x), & x < 0, \end{cases}$$

$$\sin_+(\lambda x) = \begin{cases} \sin(\lambda x), & x > 0, \\ 0, & x < 0, \end{cases} \qquad \sin_-(\lambda x) = \begin{cases} 0, & x > 0, \\ \sin(\lambda x), & x < 0. \end{cases}$$

It follows that

$$\cos_-(\lambda x) + \cos_+(\lambda x) = \cos(\lambda x), \quad \sin_-(\lambda x) + \sin_+(\lambda x) = \sin(\lambda x).$$

The following theorem was proved in [5].

Theorem 3. *The neutrix convolution product $(x^r e_-^{\lambda x}) \circledast (x^s e_+^{\mu x})$ exists and*

$$(x^r e_-^{\lambda x}) \circledast (x^s e_+^{\mu x}) = D_\lambda^r D_\mu^s \frac{e_+^{\mu x} + e_-^{\lambda x}}{\lambda - \mu},$$

where $D_\lambda = \partial/\partial\lambda$ *and* $D_\mu = \partial/\partial\mu$, *for* $\lambda \neq \mu$ *and* $r, s = 0, 1, 2, \ldots$, *these neutrix convolution products existing as convolution products if* $\lambda > \mu$ *and*

$$(x^r e_-^{\lambda x}) \circledast (x^s e_+^{\lambda x}) = B(r+1, s+1) x^{r+s+1} e_-^{\lambda x},$$

where B *denotes the Beta function, for all* λ *and* $r, s = 0, 1, 2, \ldots$.

We now prove the following theorem.

Theorem 4. *The neutrix convolution products* $\cos_-(\lambda x) \circledast \cos_+(\mu x)$, $\cos_-(\lambda x) \circledast \sin_+(\mu x)$, $\sin_-(\lambda x) \circledast \cos_+(\mu x)$ *and* $\sin_-(\lambda x) \circledast \sin_+(\mu x)$ *exist and*

$$\cos_-(\lambda x) \circledast \cos_+(\mu x) = \frac{\lambda \sin_-(\lambda x) + \mu \sin_+(\mu x)}{\lambda^2 - \mu^2}, \tag{4}$$

$$\cos_-(\lambda x) \circledast \sin_+(\mu x) = -\frac{\mu \cos_-(\lambda t) + \mu \cos_+(\mu t)}{\lambda^2 - \mu^2}, \tag{5}$$

$$\sin_-(\lambda x) \circledast \cos_+(\mu x) = -\frac{\lambda \cos_-(\lambda x) + \lambda \cos_+(\mu x)}{\lambda^2 - \mu^2}, \tag{6}$$

$$\sin_-(\lambda x) \circledast \sin_+(\mu x) = -\frac{\mu \sin_-(\lambda x) + \lambda \sin_+(\mu x)}{\lambda^2 - \mu^2}, \tag{7}$$

for $\lambda \neq \pm\mu$.

P r o o f. We first of all note that since

$$\sin(\alpha x + \beta\nu) = \sin(\alpha x)\cos(\beta\nu) + \cos(\alpha x)\sin(\beta\nu),$$

it follows that

$$\mathrm{N}\!-\!\lim_{\nu\to\infty} \sin(\alpha x + \beta\nu) = \mathrm{N}\!-\!\lim_{\nu\to\infty} \nu \sin(\alpha x + \beta\nu) = 0 \tag{8}$$

for $\beta \neq 0$.

We now put $[\cos_-(\lambda x)]_\nu = \cos_-(\lambda x)\tau_\nu(x)$. Since $\cos_+(\mu x)$ and $[\cos_-(\lambda x)]_\nu$ are locally summable functions with $[\cos_-(\lambda x)]_\nu$ having compact support, the convolution product $[\cos_-(\lambda x)]_\nu * \cos_+(\mu x)$ is defined by equation (1) and so

$$[\cos_-(\lambda x)]_\nu * \cos_+(\mu x) = \int_{-\infty}^{\infty} [\cos_-(\lambda t)]_\nu \cos_+[\mu(x-t)]\, dt. \tag{9}$$

When $-\nu < x < 0$,

$$\begin{aligned}
&\int_{-\infty}^{\infty} [\cos_-(\lambda t)]_\nu \cos_+[\mu(x-t)]\, dt \\
&= \int_{-\nu}^{x} \cos(\lambda t)\cos[\mu(x-t)]\, dt + \int_{-\nu-\nu^{-\nu}}^{-\nu} \cos(\lambda t)\cos[\mu(x-t)]\tau_\nu(t)\, dt \\
&= \frac{\sin(\lambda x) - \sin[\mu x - (\lambda-\mu)\nu]}{2(\lambda-\mu)} + \frac{\sin(\lambda x) + \sin[\mu x + (\lambda+\mu)\nu]}{2(\lambda+\mu)} + O(\nu^{-\nu})
\end{aligned}$$

and it follows that

$$(10) \qquad \underset{\nu\to\infty}{\mathrm{N-lim}} \int_{-\infty}^{\infty} [\cos_-(\lambda t)]_\nu \cos_+[\mu(x-t)]\, dt = \frac{\lambda \sin(\lambda x)}{\lambda 2 - \mu 2},$$

on using equation (8).

When $x > 0$,

$$\int_{-\infty}^{\infty} [\cos_-(\lambda t)]_\nu \cos_+[\mu(x-t)]\, dt$$
$$= \int_{-\nu}^{0} \cos(\lambda t)\cos[\mu(x-t)]\, dt + \int_{-\nu-\nu^{-\nu}}^{-\nu} \cos(\lambda t)\cos[\mu(x-t)]\tau_\nu(t)\, dt$$
$$= \frac{\sin(\mu x) - \sin[\mu x - (\lambda - \mu)\nu]}{2(\lambda - \mu)} - \frac{\sin(\mu x) - \sin[\mu x + (\lambda + \mu)\nu]}{2(\lambda + \mu)} + O(\nu^{-\nu})$$

and it follows that

$$(11) \qquad \underset{\nu\to\infty}{\mathrm{N-lim}} \int_{-\infty}^{\infty} [\cos_-(\lambda t)]_\nu \cos_+[\mu(x-t)]\, dt = \frac{\mu \sin(\mu x)}{\lambda 2 - \mu 2},$$

on using equation (8).

It now follows from equations (9), (10) and (11) that for arbitrary ϕ in $\mathcal{D}$

$$\underset{\nu\to\infty}{\mathrm{N-lim}} \langle [\cos_-(\lambda x)]_\nu * \cos_+(\mu x), \phi(x)\rangle = \frac{\lambda}{\lambda^2 - \mu^2}\langle \sin_-(\lambda x), \phi(x)\rangle + $$
$$+ \frac{\mu}{\lambda^2 - \mu^2}\langle \sin_+(\mu x), \phi(x)\rangle$$

and equation (4) follows.

Differentiating equation (4), using Theorem 2, we have

$$\cos_-(\lambda x) \circledast [-\mu \sin_+(\mu x) + \delta(x)] = \frac{\lambda^2 \cos_-(\lambda x) + \mu^2 \cos_+(\mu x)}{\lambda^2 - \mu^2}$$

and so

$$\mu \cos_-(\lambda x) \circledast \sin_+(\mu x) = -\frac{\lambda^2 \cos_-(\lambda x) + \mu^2 \cos_+(\mu x)}{\lambda^2 - \mu^2} + \cos_-(\lambda x)$$
$$= -\frac{\mu^2 \cos_-(\lambda x) + \mu^2 \cos_+(\mu x)}{\lambda^2 - \mu^2}.$$

Equation (5) follows.

To prove equation (7) we put $[\sin_-(\lambda x)]_\nu = \sin_-(\lambda x)\tau_\nu(x)$. Then as above

$$[\sin_-(\lambda x)]_\nu * \sin_+(\mu x) = \int_{-\infty}^{\infty} [\sin_-(\lambda t)]_\nu \sin_+[\mu(x-t)]\,dt. \tag{12}$$

When $-\nu < x < 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} [\sin_-(\lambda t)]_\nu \sin_+[\mu(x-t)]\,dt &= \int_{-\nu}^{x} \sin(\lambda t)\sin[\mu(x-t)]\,dt + \\ &\quad + \int_{-\nu-\nu^{-\nu}}^{-\nu} \sin(\lambda t)\sin[\mu(x-t)]\tau_\nu(t)\,dt \\ &= \frac{\sin(\lambda x) + \sin[\mu x + (\lambda+\mu)\nu]}{2(\lambda+\mu)} + \\ &\quad - \frac{\sin(\lambda x) - \sin[\mu x - (\lambda-\mu)\nu]}{2(\lambda-\mu)} + O(\nu^{-\nu})\end{aligned}$$

and it follows that

$$\operatorname*{N-lim}_{\nu\to\infty} \int_{-\infty}^{\infty} [\sin_-(\lambda t)]_\nu \sin_+[\mu(x-t)]\,dt = -\frac{\mu\sin(\lambda x)}{\lambda^2-\mu^2}, \tag{13}$$

on using equation (8).

When $x > 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} [\sin_-(\lambda t)]_\nu \sin_+[\mu(x-t)]\,dt &= \int_{-\nu}^{0} \sin(\lambda t)\sin[\mu(x-t)]\,dt + \\ &\quad + \int_{-\nu-\nu^{-\nu}}^{-\nu} \sin(\lambda t)\sin[\mu(x-t)]\tau_\nu(t)\,dt \\ &= -\frac{\sin(\mu x) - \sin(\mu x + (\lambda+\mu)\nu]}{2(\lambda+\mu)} + \\ &\quad - \frac{\sin(\mu x) - \sin[\mu x - (\lambda-\mu)\nu]}{2(\lambda-\mu)} + O(\nu^{-\nu})\end{aligned}$$

and it follows that

$$\operatorname*{N-lim}_{\nu\to\infty} \int_{-\infty}^{\infty} [\sin_-(\lambda t)]_\nu \sin_+[(x-t)]\,dt = -\frac{\lambda\sin(\mu x)}{\lambda^2-\mu^2} \tag{14}$$

on using equation (8).

Equation (7) now follows as above on using equations (12), (13) and (14).

Differentiating equation (7), using Theorem 2, we get

$$\mu\sin_-(\lambda x) \circledast \cos_+(\mu x) = -\frac{\lambda\mu\cos_-(\lambda x) + \lambda\mu\cos_+(\mu x)}{\lambda^2-\mu^2}$$

and equation (6) follows.

Corollary. *The neutrix convolution products* $\cos_+(\lambda x) \circledast \cos_-(\mu x)$, $\cos_+(\lambda x) \circledast \sin_-(\mu x)$, $\sin_+(\lambda x) \circledast \cos_-(\mu x)$ *and* $\sin_+(\lambda x) \circledast \sin_-(\mu x)$ *exist and*

$$\begin{aligned} \cos_+(\lambda x) \circledast \cos_-(\mu x) &= -\frac{\lambda \sin_+(\lambda x) + \mu \sin_-(\mu x)}{\lambda^2 - \mu^2}, \\ \cos_+(\lambda x) \circledast \sin_-(\mu x) &= \frac{\mu \cos_+(\lambda x) + \mu \cos_-(\mu x)}{\lambda^2 - \mu^2}, \\ \sin_+(\lambda x) \circledast \cos_-(\mu x) &= \frac{\lambda \cos_+(\lambda x) + \lambda \cos_-(\mu x)}{\lambda^2 - \mu^2}, \\ \sin_+(\lambda x) \circledast \sin_-(\mu x) &= \frac{\mu \sin_+(\lambda x) + \lambda \sin_-(\mu x)}{\lambda^2 - \mu^2}, \end{aligned}$$

for $\lambda \neq \pm\mu$.

P r o o f. The results follow immediately on replacing x by $-x$ in equations (4), (5), (6) and (7).

Further results can be easily deduced. For example, it is easily proved that

$$\cos_+(\lambda x) * \cos_+(\mu x) = \frac{\lambda \sin_+(\lambda x) - \mu \sin_+(\mu x)}{\lambda^2 - \mu^2},$$

for $\lambda \neq \pm\mu$, and it follows that

$$\begin{aligned} \cos(\lambda x) \circledast \cos_+(\mu x) &= \cos_-(\lambda x) \circledast \cos_+(\mu x) + \cos_+(\lambda x) * \cos_+(\mu x) \\ &= \frac{\lambda \sin(\lambda x)}{\lambda^2 - \mu^2}. \end{aligned}$$

Replacing x by $-x$ in this equation we get

$$\cos(\lambda x) \circledast \cos_-(\mu x) = -\frac{\lambda \sin(\lambda x)}{\lambda^2 - \mu^2}$$

and so

$$\begin{aligned} \cos(\lambda x) \circledast \cos(\mu x) &= \cos(\lambda x) \circledast \cos_-(\mu x) + \cos(\lambda x) \circledast \cos_+(\mu x) \\ &= 0. \end{aligned}$$

Theorem 5. *The neutrix convolution products* $\cos_-(\lambda x) \circledast \cos_+(\lambda x)$, $\cos_-(\lambda x) \circledast \sin_+(\lambda x)$, $\sin_-(\lambda x) \circledast \cos_+(\lambda x)$ *and* $\sin_-(\lambda x) \circledast \sin_+(\lambda x)$ *exist and*

$$\cos_-(\lambda x) \circledast \cos_+(\lambda x) = \frac{2\lambda x \cos_-(\lambda x) + \sin_-(\lambda x) - \sin_+(\lambda x)}{4\lambda}, \tag{15}$$

$$(16)\qquad \cos_-(\lambda x) \circledast \sin_+(\lambda x) = \frac{2\lambda x \sin_-(\lambda x) + \cos_-(\lambda x) + \cos_+(\lambda x)}{4\lambda},$$

$$(17)\qquad \sin_-(\lambda x) \circledast \cos_+(\lambda x) = \frac{2\lambda x \sin_-(\lambda x) - \cos_-(\lambda x) + \cos_+(\lambda x)}{4\lambda},$$

$$(18)\qquad \sin_-(\lambda x) \circledast \sin_+(\lambda x) = \frac{\sin_-(\lambda x) - 2\lambda x \cos_-(\lambda x) - \sin_+(\lambda x)}{4\lambda},$$

for $\lambda \neq 0$.

P r o o f. We have

$$(19)\qquad [\cos_-(\lambda x)]_\nu * \cos_+(\lambda x) = \int_{-\infty}^{\infty} [\cos_-(\lambda t)]_\nu \cos_+[\lambda(x-t)]\,dt.$$

When $-\nu < x < 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} [\cos_-(\lambda t)]_\nu \cos_+[\lambda(x-t)]\,dt &= \int_{-\nu}^{x} \cos(\lambda t)\cos[\lambda(x-t)]\,dt + \\ &\quad + \int_{-\nu-\nu^{-\nu}}^{-\nu} \cos(\lambda t)\cos[\lambda(x-t)]\tau_\nu(t)\,dt \\ &= \frac{x\cos(\lambda x) + \nu\cos(\lambda x)}{2} + \\ &\quad + \frac{\sin(\lambda x) + \sin(\lambda x + 2\lambda\nu)}{4\lambda} + O(\nu^{-\nu})\end{aligned}$$

and it follows that

$$(20)\qquad \mathrm{N}\!-\!\lim_{n\to\infty} \int_{-\infty}^{\infty} [\cos_-(\lambda t)]_\nu \cos_+[\lambda(x-t)]\,dt = \frac{2\lambda x\cos(\lambda x) + \sin(\lambda x)}{4\lambda},$$

on using equation (8).

When $x > 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} [\cos_-(\lambda t)]_\nu \cos_+[\lambda(x-t)]\,dt &= \int_{-\nu}^{0} \cos(\lambda t)\cos[\lambda(x-t)]\,dt + \\ &\quad + \int_{-\nu-\nu^{-\nu}}^{-\nu} \cos(\lambda t)\cos[\lambda(x-t)]\tau_\nu(t)\,dt \\ &= \frac{\nu\cos(\lambda x)}{2} - \frac{\sin(\lambda x) - \sin(\lambda x + 2\lambda\nu)}{4\lambda} + \\ &\quad + O(\nu^{-\nu})\end{aligned}$$

and it follows that

$$(21)\qquad \mathrm{N}\!-\!\lim_{\nu\to\infty} \int_{-\infty}^{\infty} [\cos(\lambda t)]_\nu \cos_+[\lambda(x-t)]\,dt = -\frac{\sin(\lambda x)}{4\lambda},$$

on using equation (8).

Equation (15) now follows as above from equations (19), (20) and (21).

Differentiating equation (15), using Theorem 2, we get

$$\cos_-(\lambda x) \circledast [-\lambda \sin_+(\lambda x) + \delta(x)] = \frac{3\cos_-(\lambda x) - 2\lambda x \sin_-(\lambda x) - \cos_+(\lambda x)}{4}$$

and so

$$\begin{aligned} \lambda \cos_-(\lambda x) \circledast \sin_+(\lambda x) &= -\frac{3\cos_-(\lambda x) - 2\lambda x \sin_-(\lambda x) + \cos_+(\lambda x)}{4} - \cos_-(\lambda x) \\ &= \frac{2\lambda x \sin_-(\lambda x) + \cos_-(\lambda x) + \cos_+(\lambda x)}{4\lambda}. \end{aligned}$$

Equation (16) follows.

We now prove equation (18). We have

$$(22) \qquad [\sin_-(\lambda x)]_\nu * \sin_+(\lambda x) = \int_{-\infty}^{\infty} [\sin_-(\lambda t)]_\nu \sin_+[\lambda(x-t)]\,dt.$$

When $-\nu < x < 0$,

$$\begin{aligned} \int_{-\infty}^{\infty} [\sin_-(\lambda t)]_\nu \sin_+[\lambda(x-t)]\,dt &= \int_{-\nu}^{x} \sin(\lambda t)\sin[\lambda(x-t)]\,dt + \\ &\quad + \int_{-\nu-\nu^{-\nu}}^{-\nu} \sin(\lambda t)\sin[\lambda(x-t)]\tau_\nu(t)\,dt \\ &= \frac{\sin(\lambda x) + \sin(\lambda x + 2\nu x)}{4\lambda} + \\ &\quad - \frac{x\cos(\lambda x) - \nu\cos(\lambda x)}{2} + O(\nu^{-\nu}) \end{aligned}$$

and it follows that

$$(23) \qquad \mathrm{N}\!-\!\lim_{\nu\to\infty} \int_{-\infty}^{\infty} [\sin_-(\lambda t)]_\nu \sin_+[\lambda(x-t)]\,dt = \frac{\sin(\lambda x) - 2\lambda x\cos(\lambda x)}{4\lambda},$$

on using equation (8).

When $x > 0$,

$$\begin{aligned} \int_{-\infty}^{\infty} [\sin_-(\lambda t)]_\nu \sin_+[\lambda(x-t)]\,dt &= \int_{-\nu}^{0} \sin(\lambda t)\sin[\lambda(x-t)]\,dt + \\ &\quad + \int_{-\nu-\nu^{-\nu}}^{-\nu} \sin(\lambda t)\sin[\lambda(x-t)]\tau_\nu(t)\,dt \\ &= -\frac{\sin(\lambda x) - \sin(\lambda x + 2\lambda\nu)}{4\lambda} + \frac{\nu\cos(\lambda x)}{2} + \\ &\quad + O(\nu^{-\nu}) \end{aligned}$$

and it follows that

$$\text{(24)} \qquad \mathrm{N}-\lim_{\nu\to\infty} \int_{-\infty}^{\infty} [\sin_-(\lambda t)]_\nu \sin_+[\lambda(x-t)]\, dt = -\frac{\sin(\lambda x)}{4\lambda},$$

on using equation (8).

Equation (18) now follows as above on using equations (22), (23) and (24).

Differentiating equation (18), using Theorem 2, we get

$$\lambda \sin_-(\lambda x) \circledast \cos_+(\lambda x) = \frac{-\lambda \cos_-(\lambda x) + 2\lambda^2 \sin_-(\lambda x) - \lambda \cos_+(\lambda x)}{4\lambda},$$

and equation (17) follows.

Corollary. *The neutrix convolution products* $\cos_+(\lambda x) \circledast \cos_-(\lambda x)$, $\cos_+(\lambda x)$ $\circledast \sin_-(\lambda x)$, $\sin_+(\lambda x) \circledast \cos_-(\lambda x)$ *and* $\sin_+(\lambda x) \circledast \sin_-(\lambda x)$ *exist and*

$$\begin{aligned}
\cos_+(\lambda x) \circledast \cos_-(\lambda x) &= -\frac{2\lambda x \cos_+(\lambda x) + \sin_+(\lambda x) - \sin_-(\lambda x)}{4\lambda},\\
\cos_+(\lambda x) \circledast \sin_-(\lambda x) &= -\frac{2\lambda x \sin_+(\lambda x) + \cos_+(\lambda x) + \cos_-(\lambda x)}{4\lambda},\\
\sin_+(\lambda x) \circledast \cos_-(\lambda x) &= -\frac{-\cos_+(\lambda x) + 2\lambda x \sin_+(\lambda x) + \cos_-(\lambda x)}{4\lambda},\\
\sin_+(\lambda x) \circledast \sin_-(\lambda x) &= -\frac{\sin_+(\lambda x) - 2\lambda x \cos_+(\lambda x) - \sin_-(\lambda x)}{4\lambda},
\end{aligned}$$

for $\lambda \neq 0$.

P r o o f. The results follow immediately on replacing x by $-x$ in equations (15), (16), (17) and (18). Further results can again be easily deduced. Since,

$$\cos_+(\lambda x) * \cos_+(\lambda x) = \frac{\sin_+(\lambda x) + \lambda x \cos_+(\lambda x)}{2\lambda},$$

for $\lambda \neq 0$, it follows as above that

$$\begin{aligned}
\cos(\lambda x) \circledast \cos_+(\lambda x) &= \cos_-(\lambda x) \circledast \cos_+(\lambda x) + \cos_+(\lambda x) * \cos_+(\lambda x)\\
&= \frac{\sin(\lambda x) + 2\lambda x \cos(\lambda x)}{4\lambda},\\
\cos(\lambda x) \circledast \cos_-(\lambda x) &= -\frac{\sin(\lambda x) + 2\lambda x \cos(\lambda x)}{4\lambda},\\
\cos(\lambda x) \circledast \cos(\lambda x) &= 0,
\end{aligned}$$

for $\lambda \neq 0$.

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