Provided for non-commercial research and educational use.
Not for reproduction, distribution or commercial use.

Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

The attached copy is furnished for non-commercial research and education use only. Authors are permitted to post this version of the article to their personal websites or institutional repositories and to share with other researchers in the form of electronic reprints.
Other uses, including reproduction and distribution, or selling or licensing copies, or posting to third party websites are prohibited.

For further information on Mathematica Balkanica visit the website of the journal
http://www.mathbalkanica.info
or contact:
Mathematica Balkanica - Editorial Office;
Acad. G. Bonchev str., Bl. 25A, 1113 Sofia, Bulgaria
Phone: +359-2-979-6311, Fax: +359-2-870-7273,
E-mail: balmat@bas.bg

Mathematica Balkanica

New Series Vol.9, 1995, Fasc.4

Uniform Asymptotic Stability for a Scalar Autonomous Differential Equation with "Maxima"

H.D. Voulov

Presented by V. Kiryakova

The scalar differential equation

$$\dot{x}(t) = f(x(t), \max_{s\in[t-h,t]} x(s)),$$

is considered under the assumptions $f \in C^1(\mathbb{R}^2)$ and $yf(y,y) < 0$ for $y \neq 0$. A sufficient condition for uniform asymptotic stability of the zero solution is obtained in an explicite form. In the case $f(x,y) = ax^\lambda + by^\lambda, \lambda = \frac{(2n-1)}{(2m-1)}$, $m, n \in \mathbb{N}$, a necessary and sufficient condition for uniform asymptotic stability of the zero solution is found.

1.Introduction

The mathematical simulation of the processes taking part in some systems with automatic regulation ($c.f$[3]) leads to differential equation of the form

$$(1) \qquad \dot{x}(t) = F(t, x(t), \max_{s\in[g(t),t]} x(s)),$$

where x is an unknown function, $\dot{x}$ is its right-hand derivaqtive and F and g are previously given continuous functions, $F(t,0,0) = 0$ and $g(t) \leq t \quad for \quad t > t_0$.

To eludicate the idea, let us consider the following situation .Suppose that we have a system with automatic regulation and the process going on is described by the scalar differential equation

$$\dot{x}(t) = ax(t) + u(x_t),$$

where $a \in R$ and the forcing term $u(x_t)$ corresponds the automatic control which is to keep x near zero and the function $x_t(s) = x(t+s), s \leq 0$ represents the history of the process described. If the regulator is constructed in such a way that

$$u(x_t) = -b \max \max_{s\in[t-h,t]} x(s), \qquad b > 0, \quad h > 0,$$

then it will help the quantity x to increase only when $x(s) < 0$ for all $s \in [t-h, t]$.Such kinds of regulators are suitable if, for some reason, it is undesirable to keep $x(t) > 0$.The problem arises how to choose b to get a stable system.

It is important to notice that in equation (1) the deviation of the argument depends on the unknown function x .That is why equation (1) is nonlinear even if the function $F(t,.,.)$ is linear.Some stability results about equation (1) are obtained in [4] – [10].

In the present paper the case $F(t,x,y) = f(x,y) \in R, g(t) = t-h, \quad h = const > 0$, is considered under the assumption $yf(y,y) < 0$ for $y \neq 0$.Extending some results from [9], a sufficient condition for uniform asymptotic stability of the zero solution is obtained in an explicite form.

In the case $f(x,y) = ax^{\lambda} + bx^{\lambda}, \quad ,\lambda = (2n-1)/(2m-1), \quad m,n \in N$, a nacessary and sufficient condition for uniform asymptotic stability of the zero solution is found.

2.Preliminaries

We consider the initial value problem

$$\dot{x}(t) = f(x(t)), \max_{s\in[t-h,t]} x(s)), \tag{2}$$

$x_\sigma = \varphi \in C[-h,0]$, where $t \geq \sigma \in I = (t_0, +\infty), h = const > 0$ and $x_\sigma(s) = x(\sigma+s)$ for $s \in [-h,0]$.

Definition 1. *The function $x : [\sigma-h,a) \to R$ is called a solution of the initial value problem (2) ,$x_\sigma = \varphi, (\sigma,\varphi) \in I \times C[-h,0]$, if $\sigma < a, x \in C[\sigma-h,a)$ and it satisfies equalities $x_\sigma = \varphi$ and (2) for $t \in [\sigma,a)$.*

Definition 2. *The solution $x : [\sigma-h,a) \to R$ of the initial value problem (2) ,$x_\sigma = \varphi, (\sigma,\varphi) \in I\times C[-h,0]$, is called a* **noncontinuable solution** *if there is no solution $x_1 : [\sigma-h,a_1) \to R$ of the initial value problem (2) , $x_\sigma = \varphi$ with $a < a_1, x(t) = x_1(t) \quad$ for $t \in [\sigma,a)$.*

Let introduce the following conditions:

H1. $f \in C(R^2), \quad yf(y) < 0 \quad$ for $y \neq 0$.

H2. The function x is the unique solution of the initial value problem (2), $x_\sigma = 0, \quad \sigma \in I$.

By virtue of $[1, p.41, Theorem 2.1]$ and Zorn's lemma, condition H1 quarantees that there exists a noncontinuable solution x of the initial value problem (2), $x_\sigma = \varphi, (\sigma, \varphi) \in I \times C[-h, 0]$. For any noncontinuable solution x of the initial value problem (2), $x_\sigma = \varphi, (\sigma, \varphi) \in I \times C[-h, 0]$, we shall denote by $d(x)$ the right-hand endpoint of its definition domain and

$$D(x) = \{t \in (\sigma, d(x)) | \max_{s \in [t-h,t]} x(s) \neq x(t)\},$$

$$L(x) = \{t \in D(x) | \max_{s \in [t-h,t]} x(s) \neq x(t-h)\}.$$

Theorem 1. *Let conditions $H1, H2$ hold and x be a noncontinuable solution of the initial value problem* (2) , $x_\sigma = \varphi, (\sigma, \varphi) \in I \times C[-h, 0]$. *Then* $D(x) = (\sigma, r), \quad \sigma \leq r$, *the function* $\max_{s \in [t-h,t]} x(s)$*is nonincreasing on* $[\sigma, r)$ *and the function x is nondecreasing and nonpositive on $[r, d(x))$.*

The proof folloms from [9,Theorems 1.4 and 1.7].

It is clear that $x = 0$ is a solution of the initial value problem (2), $x_\sigma = 0, \ \sigma \in I$ if condition H1 holds.

Definition 3. *The zero solution of equation (2) is said to be:*

(i) if there exists a positive function $\delta = \delta(\sigma, \varepsilon)$ defined for $\sigma \in I, \varepsilon > 0$ such that for $(\sigma, \varphi) \in I \times C[-h, 0]$, $\|\varphi\| < \delta(\sigma, \varepsilon)$ each noncontinuable solution x of the initial value problem (2), $x_\sigma = \varphi$ *satisfies the relations $d(x) = +\infty$ and $|x(t)| < \varepsilon$ for $t \geq \sigma$;*

(ii) **uniformly stable** *if it is stable and the function $\delta(\sigma, \epsilon)$ does not depend on $\sigma \in I$;*

(ii) **uniformly** *if for $\sigma \in I$ there exists $\delta_0(\sigma) > 0$ such that for $\varphi \in C[-h, 0]$, $\|\varphi\| < \delta_0(\sigma)$ each noncontinuable solution x of the initial value problem (2),$x_\sigma = \varphi$ satisfies the relations $d(x) = +\infty$ and $x(t) \to 0$ as $t \to = \infty$;*

(iv) **uniformly attractive** *if there exist a number $\delta_0 > 0$ and a positive function T defined for $\varepsilon > 0$ so that for $(\sigma, \varphi) \in I \times C[-h, 0]$, $\|\varphi\| < \delta_0$ each noncontinuable solution x of the initial value problem* (2) , $x_\sigma = \varphi$ *satisfies the relations $d(x) = +\infty$ and $|x(t)| < \varepsilon$ for $t \geq \sigma + T(\varepsilon)$;*

(v) **asymptotically stable** *if it is stable and attractive;*

(vi) **uniformly asymptotically stable** *if it is uniformly stable and uniformly attractive.*

Lemma 1. *Let $f \in C(R^2), \sigma \in R$ and x be a noncontinuable solution of the initial value problem (2) , $x_\sigma = \varphi$, $(\sigma, \varphi) \in I \times C[-h, 0]$, defined for $t < d(x)$.Let there exists a number $r > 0$ such that $\|x_t\| \leq r$ for $t \in (\sigma, d(x))$. Then $d(x) = +\infty$.*

The proof follows from [1,p.43,Theorem 3.2].

Introduce the following conditions H3-H5:

H3. There exists a number $q > 0$ such that $L(x) \subset (\sigma, \sigma + q)$ for any solution x of the initial value problem (2) ,$x_\sigma = \varphi$, $(\sigma, \varphi) \in I \times C[-h, 0]$.

H4. There exists a number $r > 0$ such that any solution x of the initial value problem $\dot{x}(t) = f(x(t), x(t)), x(\sigma) = x_0 \in (-r, 0)$ tends to zero as $t \to +\infty$.

H5. There exists an open set $V \subset I \times C[-h, 0]$ such that $I \times \{0\} \subset V$ and for $(\sigma, \varphi) \in V$, where the function φ is nonincreasing, if x is a nonincreasing solution of the initial value problem $\dot{x}(t) = f(x(t), x(t-h))$, $x_\sigma = \varphi$, which is defined for $t \geq \sigma$, then $x(t) \to 0$ as $t \to +\infty$.

Theorem 2. *Let conditions H1,H3 and H4 hold.*

Then the zero solution of equation (2) is asymptotically stable if and only if it is stable and condition H5 holds.

The proof is given in [9,Theorem 2.3].

The following theorem is used for verifying condition H3.

Theorem 3. *Let conditions H1 and H2 hold but condition H3 does not hold.*

Then there exist numbers $x_k, \overline{y_k}, y_k^*$ for $k = 1, 2$ such that $\overline{y_k} \geq y_k^*$ and

$$(-1)^k f(x_k, \overline{y_k}) \geq 0 \geq (-1)^k f(x_k, y_k^*)$$

for $k = 1, 2$.

The proof follows from [9,Theorems 2.6 and 2.8].

Theorem 4. *Let $f(x, y) = ax + by$, $a, b \in R$*

Then the zero solution of equation (2) is uniformly asymtotically stable if and only if $b + \psi(a) < 0$, where

$$\psi(a) = \left\{ \begin{array}{lll} a & \text{for} & ah \leq 1 \\ e^{ah}(eh)^{-1} & \text{for} & ah > 1 \end{array} \right\} \tag{3}$$

The proof is given in [9,Theorem 3.1].

3.Main results

Theorem 5. *Let the function f be continuous on a neighbourhood of $(0,0)$ and differentiable at $(0,0)$.Let $f(0,0) = 0$ and $f_y(0,0) + \psi(f_x(0,0)) < 0$, where ψ is defined by (3).*

Then the zero solution of equation (2) is uniformly asymptotically stable.

Proof. For $\varphi \in C[-h,0]$ we set

$$g(\varphi) = f(\varphi(0), \max_{s\in[-h,0]} \varphi(s))$$

and

$$g_0(\varphi) = a\varphi(0), +b \max_{s\in[-h,0]} \varphi(s),$$

where $a = f_x(0,0)$, $b = f_y(0,0)$.From Theorem 4 it follows that the zero solution of the equation $\dot{x}(t) = g_0(x_t)$ is uniformly asymptotically stable and, therefore, it is exponentially stable in view of [2,Theorem 5].In a neightbourhood U of $(0,0)$ we have that

$$f(x,y) = ax + by + r(x,y),$$

where $r(x,y)(|x| + |y|)^{-1} \to 0$ as $(x,y) \to 0$ because the function f is differentiable at $(0,0)$ and $f(0,0) = 0$. Since f is continuous on U and $|\max_{s\in[-h,0]} \varphi(s)| \leq \|\varphi\|$, $|\varphi(0) \leq \|\varphi\|$ for $\varphi \in C[-h,0]$, then the functional g is continuous, $|g_0(\varphi) - g_0(\psi)| \leq (|a|+|b|)\|\varphi - \psi\|$ for small $\|\varphi\|, \|\psi\|$ and $|g(\varphi) - g_0(\varphi)|(\|\varphi\|)^{-1} \to 0$ as $\|\varphi\| \to 0$.From [2,Theorems5,6 and 7] it follows that the zero solution of equation (2) is uniformly asymptotically stable. ■

Unfortunately, the theorem proved above does not say anything if either $f_x(0,0) = 0 = f_y(0,0)$ or f is not differentiable at $(0,0)$.Trying to find out some more information about these two casess let us consider equation (2) with

$$f(x,y) = ax^\lambda + by^\lambda,$$

where $a, b \in \mathbb{R}, \lambda = (2n-1)(2m-1)^{-1}$, $m, n \in \mathbb{N}$.

Theorem 6. *Let $f(x,y) = ax^\lambda + by^\lambda$, where $\lambda = (2n-1)(2m-1)^{-1}$, $m, n \in \mathbb{N}$ $a, b \in \mathbb{R}$. Then the zero solution of equation (2) is uniformly asymptotically stable if and only if $a + b < 0$ and either $n > m$, or $n = m$ and $b + \psi(a) < 0$, where $\psi(a)$ is given by (3), or $n < m$ and $a \leq 0$.*

Proof.

Necessity.

For $n = m$ the proof follows from Theorem4. For $a + b \geq 0$ the zero solution of equation (2) is not assymptotically stable becaause for each $\sigma \in I, \epsilon > 0$ the initial value problem (2), $x_\sigma(s) \equiv \epsilon$ has a nondecreasing positive solution.For $a + b < 0$, $a > 0, n < m$, the zero solution of equation (2) is not stable because the initial value problem (2),$x_\sigma = 0$ has a nonzero solution x given by $x(t) = -((1 - \lambda)(t - \sigma))^{\frac{1}{(1-\lambda)}}$ for $t \in (\sigma, \sigma + h)$.

Sufficiency.

In view of [1,p.104,Lemma1.1] it suffices to prove that the zero solution of equation (2) is asymptotically stable. For $n = m$, $b + \psi(a) < 0$, this follows from Theorem 4. Let $n \neq m$, $a + b < 0$ and $a \leq 0$. For $b \geq 0$ we have $a < 0$, $|b| < |a|$ and the assertion follows by Liapunov's direct method (see for instance [1,p.127,Theorem 4.2]).For $b < 0$ and $a \leq 0$ any noncontinuable solution of the initial value problem (2),$x_\sigma = \varphi$, $(\sigma, \varphi) \in I \times C[-h, 0]$ satisfies the inequalities

$$(4) \qquad (a + b)(max_{s \in [t-h,t]} x(s))^\lambda \leq \dot{x}(t) \leq (a + b)(x(t))^\lambda$$

for $t \in (\sigma, d(x))$ whence it follows that condition H2 holds and , by virtue of Theorem 1, $|x(t)| \leq ||\varphi|| + h||\varphi||^\lambda |a + b|$. Therefore, the zero solution of equation (2) is stable.From relations (4) and $\lambda > 1$ it follows that each nonincreasing solution of equation (2) is nonnegative and it tends to zero as $t \to +\infty$.Therefore, condition H5 holds.Moreover, condition H3 is met because if this is not the case,then by Theorem 3 there exist numbers $\overline{y_1}, \overline{y_2}, y_1^*, y_2^*$ such that $\overline{y_k} \geq y_k^*$ and

$$(-1)^k((\overline{y}_k)^\lambda - (y_k^*)^\lambda)b > 0$$

for $k = 1, 2$ and, hence, $0 < b < 0$ which is a contradiction. From Theorem 2 it follows that the zero solution of equation (2) is asymptotically stable for $b < 0, a \leq 0$

It remains to prove this for $b < 0 < a, a + b < 0, \lambda = (2n - 1)(2m - 1)^{-1}, n > m$.

From $a+b < 0$ and $\lambda > 1$ it follows that conditions H1, H2 and H4 hold.In the same way as above, it follows that condition H3 holds as well.Without loss of generality, we may assume $a = 1$, $b = -c$, $c > 1$ so that equation (2) takes the following form

$$(5) \qquad \dot{x}(t) = (x(t))^\lambda - c(\max_{s \in [t-h,t]} x(s))^\lambda$$

where $\lambda = (2n - 1)(2m - 1)^{-1}$, $n > m$, $c > 1$, $h > 0$.

Let $\tau = \lambda - 1$. Since $\tau > 0$, there exists a number $\epsilon_0 > 0$ such that

$$(c+1)h\tau\epsilon_0^{\tau} < \min\{1 - 2^{-\tau}, 1 - c^{\frac{-\tau}{\lambda}}\} \tag{6}$$

Let $\epsilon \in (0, \epsilon_0), \varphi \in C([-h, 0]), \|\varphi\| < \epsilon, \sigma \in I$, and x be a solution of the initial value problem (5) , $x_\sigma = \varphi$.We shall prove that

$$|x(t)| < 2\epsilon \qquad for \quad t > \sigma \tag{7}$$

Introduce the following sets

$$W_1(x) = \{t \in D(x) | x(t) \le -\epsilon\},$$

$$W_2(x) = \{t \in D(x) | \dot{x}(t) \le 0\}.$$

If $W_1(x) = 0$ then from the inequality $\|\varphi\| < \epsilon$ and from the continuity of the function x is follows by Theorem 1 that relation (7) is valid.

Let $W_1(x) \ne 0$ and $t_1 = \inf W_1(x)$. Since $\|x_\sigma\| < \epsilon$ we have $t_1 > \sigma$ and $x(t_1) = -\epsilon < x(t)$ for $t \in [\sigma, t_1)$.Hence, $t_1 \in W_1(x) \subset D(x) = (\sigma, r)$, $\sigma < r < +\infty$.

For $s \in (\sigma, r) \setminus W_2(x)$, in view of equality (5) and Theorem 1, we have $[s, r) \cap W_2(x) = 0$. Therefore, $W_2(x) = (\sigma, t_2], \sigma < t_1 \le t_2 \le r \le +\infty$. For $t \in [t_1, t_2] \cap [t_1, t_1 + h]$ we have

$$x(t) \le -\epsilon \le -|\max_{s \in [t-h, t]} x(s)|$$

which, combined with the relations (5) and $c > 0$ yields

$$(x(t))^{-\lambda}\dot{x}(t) \le c + 1.$$

Integrating this inequality and taking intoaccount that $x(t_1) = -\epsilon$, we obtain

$$|x(t)| \le \epsilon(1 - (c+1)h\tau\epsilon^{\tau})^{\frac{-1}{\tau}} \tag{8}$$

for $t \in [t_1, t_2] \cap [t_1, t_1 + h]$.

If $t_1 + h \le t_2$, and $\dot{x}(t_1 + h) \le 0$ and, by virtue of the relations $\max_{s \in [t_1, t_1 + h]} x(s) = -\epsilon, \epsilon \in (0, \epsilon_0)$, (5), (6) and (8) we have $\dot{x}(t_1 = h) > 0$, which is a contradiction. Hence, $t_2 < t_1 + h$. From the relations (6),(8) and $W_2(x) = (\sigma, t_2]$, in view of Theorem 1, it follows that $|x(t)| \le |x(t_2)| \le 2\epsilon$ for $t \in [\sigma, d(x))$. By Lemma 1 we have $d(x) = +\infty$ and, hence, relation (7) holds. Thus, the zero solution of equation (5) is stable.

Further on we shall prove that condition H5 holds. Let $(\sigma, \varphi) \in I \times C[-h, 0], ||\varphi|| < \epsilon_0$, where the function φ is nonincreasing and let x be a nonincreasing solution of the initial value problem

$$\dot{x}(t) = (x(t))^\lambda - (x(t-h))^\lambda, \qquad x_\sigma = \varphi,$$

defined for all $t \geq \sigma$,. For $n \in \mathbb{N}$ by induction on n the inequality

$$x(t)c^{\frac{n}{\lambda}} \geq x(t+nh) \tag{9}$$

is obtained. But x is also a solution of the initial value problem (5),$x_\sigma = \varphi$, and for $t > \sigma, \epsilon \in (||\varphi||, \epsilon_0)$, inequality (7) holds. Let $\epsilon \in (||\varphi||, \epsilon_0)$.

For $t > \sigma$, in view of inequalities (7) and (9), we have $x(t) \geq -2\epsilon c^{\frac{-n}{\lambda}}$. Passing to the limit as $n \to \infty$ we obtain that $x(t) \geq 0$ for $t > \sigma$. On the other hand, from equality (5) it follows that $\dot{x}(t) \leq (1-c)(x(t))^\lambda$ for $t > \sigma$. Hence, $x(t) \to 0$ as $t \to \infty$, i.e. condition H5 holds. Then, from Theorem 2 it follows that the zero solution of equation (5) is asymptotically stable.

References

[1] J. K. H a l e, Theory of Functional Differential Equations, *Springer-Verlag*, New York - Heidelberg - Berlin, 1977.

[2] J. K a t o, Stability problem in functional differential equations with infinite delay, *Funk. Ekvac.*, **21**, 1978, 63-80.

[3] A. R. M a g o m e d o v, On some problems of differential equations with "maxima", *Izv. Acad. Sci. Azerb. SSR, Ser. Phys.-Techn. and Math. Sci.*, **108**(1), 1977, 104-108 .

[4] A. R. M a g o m e d o v, G. M. N a b i e v, On some questions of the stability of solutions of linear differential equations with "maxima", *Dokl. Acad. Sci. Azerb. SSR*, **42**(2), 1986, 3-6.

[5] G. M. N a b i e v , Some problems of the stability theory of the solutions of differential equations with maxima 1, *Dokl.Acad.Sci.Azerb.SSR*, **40**(8), 1984, 14-19 .

[6] G. M. N a b i e v, Some problems of the stability theory of the solutions of differential equations with maxima 2, *Dokl.Acad.Sci.Azerb.SSR*, **40**(9), 1984, 16-20

[7] G. M. N a b i e v, Some problems of the stability theory of the solutions of differential equations with maxima 3, *Dokl.Acad.Sci.Azerb.SSR*, **41**(4), 1985, 11-14 .

[8] H. D. V o u l o v, Uniform asymptotic stability for a scalar differential equations with maxima, *Compt.Rend.Bulg.Acad.Sci.*,**44**(6), 1991, 5-7.

[9] H. D. V o u l o v, D. D. B a i n o v, Asymptotic stability for differential equations with maxima, *Rend.Circ.Mat.Palermo*, **II**, Tomo XL, 1991, 385-420.
[10] H. D. V o u l o v, D. D. B a i n o v, Uniform stability for nonautonomous differential equation with maxima, *Appl.Math.Comput.* (to appear)

Department of Mathematics
Technical University of Sofia
1156 Sofia
BULGARIA

Received 24.06.1993