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Several Theorems for Block Diagonally Dominant Matrices

M. Salah El-Sayed, M. G. Petkov

Presented by P. Kenderov

In this paper a block generalization of some theorems for scalar diagonally dominant matrices are given.

1.Block (composite) vector and matrix norms

Let $x \in R^n(C^n)$ be a n-dimensional real (complex) vector partitioned in p blocks X_i, i.e.

$$x = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix}$$

where $X_i \in R^{\alpha_i}(C^{\alpha_i}), \alpha_1 + \alpha_2 + \ldots + \alpha_p = n$.
If $\omega(x)$ is an arbitrary scalar norm of the vector $x = (x_1, x_2, \ldots, x_n)^T$, $x_i \in R^1(C^1)$, then it is easy to show that

$$v_{\omega,\infty}(x) = \max_i \omega(X_i), \tag{1}$$

$$v_{\omega,1}(x) = \sum_{i=1}^{p} \omega(X_i), \tag{2}$$

$$v_{\omega,2}(x) = (\sum_{i=1}^{p} \omega^2(X_i))^{\frac{1}{2}}, \tag{3}$$

are vector norms too. We call them block (composite) norms. For each of the norms (1) – (3) $\omega(x)$ can denote one of the common used norms :

$$\|x\|_\infty = \max_i |x_i|, \tag{4}$$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \tag{5}$$

$$\|x\|_2 = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}}. \tag{6}$$

In a similar way we can introduce block matrix norms too. Let the matrix $A = (a_{ij}) \in C^{mxn}$ be partitioned in pq blocks $A_{ij} \in C^{\alpha_i \times \beta_j}$ $(i = 1, 2, \ldots, p; j = 1, 2, \ldots q; \alpha_1 + \alpha_2 + \ldots + \alpha_p = m; \beta_1 + \beta_2 + \ldots + \beta_q = n)$, and let v be an arbitrary scalar matrix norm of $A = (a_{ij})$. Then the following functional defined for $A = (A_{ij})$ are block (composite) norms :

$$\mu_{v,\infty}(A) = \max_i \sum_{j=1}^{q} v(A_{ij}), \tag{7}$$

$$\mu_{v,1}(A) = \max_j \sum_{i=1}^{p} v(A_{ij}), \tag{8}$$

$$\mu_{v,2}(A) = (\sum_{i,j} v^2(A_{ij}))^{\frac{1}{2}}, \tag{9}$$

$$\mu_{v,\phi}(A) = \phi(p,q) \max_{i,j} v(A_{ij}), \tag{10}$$

where the function ϕ of two variables is supposed that it is satisfies the functional inequality

$$r\phi(p,q) \leq \phi(p,r)\phi(r,q),$$

$$\mu_{v,0}(A) = \sum_{i,j} v(A_{ij}). \tag{11}$$

Let us mention explicitly that $v(A)$ where $A = (a_{ij}) \in C^{mxn}$ can be also one of the following often used matrix norms

$$\|A\|_i, i = \infty, 1, 2, \tag{12}$$

(13) $$M_\phi(A) = \phi(m,n)\max_{i,j}|a_{ij}|,$$

(14) $$N(A) = (\sum_{i,j}|a_{ij}|^2)^{1/2}$$

2. Theorems for block diagonally dominant matrices

Let $I = (\delta_{ij}) \in R^{nxn}$ be the unit nxn-matrix and $I = (\Delta_{ij}) = (E_1, E_2, \ldots, E_p)$ be a block representation of it, and E_s be the s-th block (quasi-column) of I. Then the matrix

(15) $$(E_{i_1}, E_{i_2}, \ldots, E_{i_p})$$

where $(i_1, i_2, \ldots, i_p)$ is a rearrangement of the first p integers of $N_p = (1, 2, \ldots, p)$ is said to be a block permutation matrix.

Now, let

(16) $$A = (A_{ij}) \in C^{nxn}$$

be a block matrix with $A_{ij} \in C^{\alpha_i x \alpha_j}; i,j = 1,2,\ldots,p; \alpha_1 + \alpha_2 + \ldots + \alpha_p = n$.

Definition 1. *A block matrix of the form (16) is said to be reducible iff there is a block permutation matrix of the form (15) such that*

(17) $$PAP^T = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}$$

where M_{ii} are square matrices.

When a matrix of the form (16) is not reducible it is said to be irreducible matrix.

It is easy to show that the following definition 2 for reducibility is equivalent to definition 1.

Definition 2. *The matrix (16) is reducible iff there is a non-empty subset J of N_p, such that $A_{ij} = 0$ for every $i \in J$ and every $j \notin J$.*

Definition 3. *It is said that for a fixed pair $i,j \in N_p$ there exists a connecting chain for the blocks of (16) if there exist a sequence $i_1, i_2, \ldots, i_m \in N_p$ such that each block of the sequence $A_{ii_1}, A_{i_1i_2} \ldots A_{i_mj}$ of the matrix (16) is nonzero.*

Theorem 1. *A matrix $A = (A_{ij})$ of (16) is irreducible iff for each pair $i, j \in N_p$ there exists a connecting chain .*

We will not give a proof, since it is not essentially different from the corresponding proof for scalar matrices.

For the sake of completeness we give two definitions also.

Definition 4. *A matrix $A = (A_{ij})$ of (16) is said to be diagonally (strongly diagonally) dominant by rows with respect to a matrix norm μ if the numbers*

$$R_i = \mu^{-1}(A_{ii}^{-1}) - \sum_{j\neq i} \mu(A_{ij}), i = 1, 2, \ldots, p \tag{18}$$

are a nonnegative (positive).

Definition 5. *A matrix (16) is said to be irreducibly diagonally dominant, if it is irreducible, all $R_i \geq 0$ and $R_1 + R_2 + \ldots + R_p > 0$.*

Theorem 2. *(A block generalization of the theorem of O.Taussky-[3,8]). If a matrix $A = (A_{ij})$ of (16) is irreducibly diagonally dominant matrix with respect to a matrix norm μ subjected to a vector norm v, then a matrix A is nonsingular.*

Theorem 3. *Let the matrix $A = (A_{ij})$ of (16) is irreducibly diagonally dominant matrix with respect to a ordinated norm v and such that*

$$\sum_{j=1}^{p} v(A_{ij}) \leq 1, i = 1, 2, \ldots, p \tag{19}$$

where for at least one i we have a strong inequality. Then the spectral radius $\rho(A)$ of a matrix A is less then 1.

P r o o f. Since

$$\mu(A) = \max_i \sum_{j=1}^{p} v(A_{ij})$$

is a matrix norm, then $\rho(A) \leq 1$. Let us assume that A has an eigen- value λ with $|\lambda| = 1$. From the irreducibility of A, it follows that the matrix $A - \lambda I = (A_{ij} - \lambda\Delta_{ij})$ is irreducible too, and $v(A_{ii}) < 1$ for every i.
But from the last inequality and from $|\lambda| = 1$ it follows that $(\lambda I - A_{ii})^{-1}$ exists and

$$v[(\lambda I - A_{ii})^{-1}] \leq \frac{1}{1 - v(A_{ii})}$$

$$v^{-1}[(I - \lambda^{-1}A_{ii})^{-1}] \geq 1 - v(A_{ii}) \geq \sum_{j \neq i} v(A_{ij})$$

$$v[(\lambda I - A_{ii})^{-1}] \sum_{j \neq i} v(A_{ij}) \leq 1.$$

The last inequality must be strong for at least one i. Hence $\det(A - \lambda I) \neq 0$ which is a contradiction.

Further by $A \geq B$ we denote the nonnegativity of all elements of A-B. ■

Let us remind the definition of M-matrix.

Definition 6. *A real matrix $A = (a_{ij}) \in R^{nxn}$ is said to be M-matrix if $a_{ij} \leq 0$ for $i \neq j$ and $A^{-1} \geq 0$.*

We will need the following lemma, due to J. Von Neumann.

Lemma. *Let for a nxn-matrix B we have $B \geq 0$. Then $(I - B)^{-1} \geq 0$ if and only if $\rho(B) < 1$.*

Well known is the following result.

Theorem 4. *Let the matrix $A = (a_{ij}) = B - C, a_{ij} \leq 0$ for $i \neq j, B^{-1} \geq 0$ and $B^{-1}C \geq 0$. The matrix A is M-matrix iff spectral radius $\rho(I - B^{-1}A) < 1$.*

Theorem 5. *If $A_1 = B_1 - C_1$ is M-matrix, $A_2 = (a_{ij}) = B_2 - C_2, B_1^{-1} \geq B_2^{-1} \geq 0$, $C_1 \geq C_2 \geq 0$ and $a_{ij} \leq 0$ for $i \neq j$, then A_2 is M-matrix too.*

Proof. It is easy to verify, that

$$0 \leq I - B_2^{-1}A_2 \leq I - B_1^{-1}A_1.$$

Hence

$$\rho(I - B_2^{-1}A_2) \leq \rho(I - B_1^{-1}A_1) < 1.$$

From $\rho(I - B_2^{-1}A_2) < 1$ and theorem 4 we receive immediately, that A_2 is M-matrix too. ■

Theorem 6. *A matrix $A = (A_{ij})$ of (16) is M-matrix if $A_{ij} \leq 0$ for $i \neq j$, if all diagonal blocks A_{ii} are M-matrix and if $\rho(I - D^{-1}A) < 1$, where $D = diag(A_{11}, A_{22}, \ldots, A_{pp})$.*

Proof. From the hypothesis it follows easily that all non-diagonal scalar elements $a_{\alpha\beta}$ of A satisfy the inequality $a_{\alpha\beta} \leq 0$. It remains to show that $A^{-1} \geq 0$. Indeed, from $B = I - D^{-1}A \geq 0, \rho(B) < 1$ and from J. Von Neumann's lemma it follows that $(I - B)^{-1} \geq 0$, i.e. $A^{-1}D \geq 0$.Then, by right-hand multiplication by $D^{-1} \geq 0$, we get $A^{-1} \geq 0$. ■

Theorem 7. *A matrix $A = (A_{ij})$ of (16) is aM-matrix if it has a strongly, or irreducibly diagonally dominant, if $A_{ij} \leq 0$ for $i \neq j$ and if all A_{ii} are M-matrices.*

Proof. Let us we have the matrix of the form $B = I - D^{-1}A$, where $D = diag(A_{11}, A_{22}, \ldots, A_{pp})$. First, let suppose that A has a strongly dia- gonally dominant with respect to a norm v. Then the composite norm of $\mu_{v,\infty}(B)$ is less than 1. Hence, according to theorem 6 the matrix A is M-matrix. If A has an irreducibly diagonally dominant then the matrix B is irreducible and

$$\sum_{j=1}^{p} v(B_{ij}) \leq 1, i = 1, 2, \ldots, p$$

with a strong inequality for at least one i. Therefore, $\rho(B) < 1$ and we conclude again that A is a M-matrix.

Estimates for $\|A^{-1}\|_\infty$ and $\|A^{-1}B\|_\infty$ where A is a matrix with a strongly diagonally dominant are obtained in [1,4,5].

At last, we will obtain an estimate for $\mu_{v,\phi}(A^{-1})$, where $A = (A_{ij})$ is a matrix of (16) supposing that has a strongly diagonally domin- ant with respect to v,

$$\mu_{v,\phi}(A) = \phi(p,p) \max_{i,j} v(A_{ij})$$

and that the function $\phi(p,q)$ of two variables has the property

$$r\phi(p,q) \leq \phi(p,r)\phi(r,q)$$

In this case, we have

Theorem 8. *If the matrix (16) has a strongly dominant diagonal, then*

$$\mu_{v,\phi}(A^{-1}) \leq \phi(p,p)v(I) \max_{i} \frac{1}{R_i}$$

where

$$R_i = v^{-1}(A_{ii}^{-1}) - \sum_{j \neq i} v(A_{ij}), i = 1, 2, \ldots, p.$$

Proof. Let $A^{-1} = (L_{ij})$, and

$$\max_{i,j} v(L_{ij}) = v(L_{\alpha\beta}),$$

Then we have

$$\sum_{s=1}^{p} A_{\alpha s} L_{s\beta} = \Delta_{\alpha\beta}$$

where $\Delta_{\alpha\beta}$ is the generalized Kronecker's symbol. Further, we get

$$A_{\alpha\alpha} L_{\alpha\beta} + \sum_{s\neq\alpha} A_{\alpha s} L_{s\beta} = \Delta_{\alpha\beta}$$

$$L_{\alpha\beta} + \sum_{s\neq\alpha} A_{\alpha\alpha}^{-1} A_{\alpha s} L_{s\beta} = A_{\alpha\alpha}^{-1}\Delta_{\alpha\beta}$$

$$v(L_{\alpha\beta}) - \sum_{s\neq\alpha} v(A_{\alpha\alpha}^{-1}) v(A_{\alpha s}) v(L_{s\beta}) \leq v(A_{\alpha\alpha}^{-1}) v(\Delta_{\alpha\beta})$$

$$v(L_{\alpha\beta}) R_\alpha \leq v(I)$$

$$\mu_{v,\phi}(A^{-1}) \leq \frac{\phi(p,p) v(I)}{R_\alpha}$$

$$\mu_{v,\phi}(A^{-1}) \leq \phi(p,p) v(I) \max_i \frac{1}{R_i}.$$

The last inequality as other inequalities of similar sort can used for the study of the convergence of iterative methods for solving linear systems and for the study of condition numbers of matrices. ■

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Institute of Mathematics
Bulgarian Academy of Sciences
Acad. G. Bonchev st. bl.8
1113 Sofia
BULGARIA

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