Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica Balkanica

New Series Vol.9, 1995, Fasc.4

Conditions for Interpolation by Polynomials in Right Invertible Operators to be Admissible

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A characterization of initial operators for a right invertible operator with the finite dimesional kernel admitting the interpolation is given by means of determinants induced by basis coefficients of the kernel.

The following interpolation problem has been studied in PR[2] : Let D be a linear right invertible operator with the domain and range in a linear space X . Find a D-polynomial, i.e. an element of the kernel of D^N , which admits for given initial (or/and boundary) conditions given values (cf. also [K], PR[3], [N]).

In the present paper a characterization of initial operators for a right invertible operator D with the N-dimesional kernel admitting the interpolation is given by means of determinants induced by basis coefficients of the kernel of D . These determinants are polynomials of the degree N .

1. General interpolation problem.

We shall give here some definitions and main results of PR[2] (without proofs) which will be useful in our subsequent considerations.

Let X be a linear space over a field $\mathcal{F}$ of scalars. Here either $\mathcal{F} = \mathbf{R}$ or $\mathcal{F} = \mathbf{C}$. Let $L(X)$ be the set of all linear operators with domains and ranges in X , $L_0(X) = \{A \in L(X) : \text{dom } D = X\}$ and let $R(X)$ be the set of all right invertible operators belonging to $L(X)$. For a given $D \in R(X)$ write

$$\mathcal{R}_D = \{R \in L_0(X) : DR = I\}$$

$$\mathcal{F}_D = \{F \in L_0(X) : F^2 = F \ ; \ FX = \ker D \ and \ \exists_{R\in\mathcal{R}_D} \ FR = 0\} .$$

Any $F \in \mathcal{F}_D$ is an *initial operator* for D . We say that $F \in \mathcal{F}_D$ *is corresponding to an* $R \in \mathcal{R}_D$ if $FR = 0$.

We admit here and in the sequel that $\ker D \neq \{0\}$, i.e. operators D under consideration are right invertible but not invertible. Any element of $\ker D$ is said to be a *constant.* Right inverses and initial operators have the following properties (cf. PR[1]) :

An operator $F \in L_0(X)$ is an initial operator for a $D \in R(X)$ if and only if there is an $R \in \mathcal{R}_D$ such that $F = I - RD$ on dom D . Any projection F_1 **onto** $\ker D$ is an initial operator for D corresponding to a right inverse $R_1 = R - F_1 R$ and R_1 is independent of the choice of an $R \in \mathcal{R}_D$. If two right inverses (initial operators, respectively) are commutative each with another then they are equal. Initial operators preserve constants : $Fz = z$ for all $z \in \ker D$, $F \in \mathcal{F}_D$. Elements of the set $S = \bigcup_{n\in\mathbf{N}} \ker D^n$ are said to be *D-polynomials.* In particular, elements of the form : $R^k z$ for $z \in \ker D$, $R \in \mathcal{R}_D$, $k \in \mathbf{N}$) are said to be *D-monomials.* Elements $z_0, Rz_1, ..., R^n z_n$ are linearly independent for all $z_0, z_1, ..., z_n \in \ker D$, $R \in \mathcal{R}_D$ and $n \in \mathbf{N}$. It is easy to verify that $S = P(R)$, where

$$P(R) = \text{lin}\ \{R^k z : z \in \ker D \ ; \ k \in \mathbf{N}_0\} \ ; \quad \mathbf{N}_0 = \mathbf{N} \cup \{0\}$$

and that the set $P(R)$ is independent of the choice of an $R \in \mathcal{R}_D$.

Let $A(\mathbf{R}) = \mathbf{R}_+$ or $\mathbf{R}$. Let F be an initial operator for a $D \in R(X)$ corresponding to an $R \in \mathcal{R}_D$. A family $\{S_h\}_{h\in A(\mathbf{R})} \subset L_0(X)$ is said to be a family of *R-shifts* if $S_0 = I$ and

$$S_h R^k F = \sum_{j=0}^{k} \frac{h^{k-j}}{(k-j)!} R^j F \quad for \ all \ h \in A(\mathbf{R}) \ , \ k \in \mathbf{N}_0 \ .$$

R-shifts preserve constants : $S_h z = z$ for all $z \in \ker D$, $h \in A(\mathbf{R})$. The family of R-shifts is a semigroup if $A(\mathbf{R}) = \mathbf{R}_+$ and a group if $A(\mathbf{R}) = \mathbf{R}$ (with respect to the superposition as a structure operation). R-shifts are uniquely determined on the set S (cf. PR[1], also PR[5]).

Some more informations about right invertible operators and their applications can be found in PR[1], PR[3].

Definition 1.1. (cf. PR[2]) it An operator $F_0 \in \mathcal{F}_D$ has *the property* $c(R)$ for an $R \in \mathcal{R}_D$ if there exist scalars c_k such that

$$F_0 R^k z = \frac{c_k}{k!} z \quad for \ all \ z \in \ker D \ , \ k \in \mathbf{N}$$

and $c_k = 0$ for all $k \in \mathbf{N}$ if $F_0 = F$, where F is an initial operator for D corresponding to R. We shall write : $F_0 \in c(R)$. A set $\mathcal{F}_D^o \subset \mathcal{F}_D$ has *the property* (c) if for every $F_0 \in \mathcal{F}_D^o$ there exists an $R \in \mathcal{R}_D$ such that $F_0 \in c(R)$. We admit $c_0 = 1$, since $F_0 z = z$ for all $z \in \ker D$.

Theorem 1.1. (cf. PR[2]) *The set* $\mathcal{F}_D$ *of all initial operators for a* $D \in R(X)$ *has the property* $c(R)$ *if and only if dim* $\ker D = 1$.

Proposition 1.1. (cf. PR[2]) *Suppose that* $D \in R(X)$ *and dim* $\ker D = 1$, *i.e.* $\ker D = \{\mu e : \mu \in \mathcal{F}\}$ *for an* $e \neq 0$. *Denote by* X' *the space of all linear functionals over* X. *Then every initial operator is of the form* : $F_\varphi = \varphi(x)e$ *for* $x \in X$, *where* $\varphi \in X'$ *and* $\varphi(e) = 1$. *Moreover,* $c_k = c_k(\varphi) = k!\varphi(R^k e)$ *for* $k \in \mathbf{N}$, *where* $R \in \mathcal{R}_D$ *is this right inverse for which* $FR = 0$.

Proposition 1.2. (cf. PR[2]) *Suppose that* X *is a complete linear metric space,* $D \in R(X)$, F *is an initial operator for* D *corresponding to an* $R \in \mathcal{R}_D$, $\overline{P(R)} = X$ *and* $F_1 \neq F$ *is an arbitrarily fixed continuous initial operator having the property* $c(R)$. *Then not all numbers* c_k $(k \in \mathbf{N}_0)$ *do not vanish simultaneously.*

Clearly, if the system $\{F_0, ..., F_{N-1}\} \subset \mathcal{F}_D$ has the property $c(R)$ with constants d_{ik}, i.e. $F_i R^k z = \frac{d_{ik}}{k!} z$ for $i = 0, 1, ..., N-1$; $k \in \mathbf{N}_0$, and $F_0, ..., F_{N-1}$ are linearly independent then $V_N = \det(d_{ik})_{i,k=0,1,...,N-1} = 0$. The following question arises : is $V_N \neq 0$ for any system $\{F_0, ..., F_{N-1}\}$ of linearly independent initial operators having the property $c(R)$?

Proposition 1.3. (cf. PR[2]) *Suppose that* $D \in R(X)$, F *is an initial operator for* D *corresponding to an* $R \in \mathcal{R}_D$ *and* $\{S_h\}_{h \in A(\mathbf{R})}$ *is a family of R-shifts. Let* $F_h = FSh$ *and* $R_h = R - F_h R$ *for* $h \in A(\mathbf{R})$ ($F_0 = F$, , *hence we admit* $R_0 = R$). *Then* F_h *are initial operators for* D *corresponding to* R_h, *respectively,* $F_h \in c(R_{h_0})$ *for all* $h \in A(\mathbf{R})$ *and an arbitrarily fixed* $h_0 \in A(\mathbf{R})$ *and*

$$F_h R_{h_0}^k z = \frac{(h - h_0)^k}{k!} z \quad \text{for } z \in \ker D \ ; \ k \in \mathbf{N}_0 \ , \tag{1.1}$$

i.e. $c_k(h) = (h - h_0)^k$ $(k \in \mathbf{N}_0)$. *Moreover, if* $N \in \mathbf{N}$ *is arbitrarily fixed,* $h_j \in A(\mathbf{R})$ *and* $h_i \neq h_j \neq 0$ *for* $i \neq j$ $(i, j = 1, ..., N)$ *then the determinant* V_N *of the system* $\{F_{h_1}, ..., F_{h_N}\}$ *is the* V a n d e r m o n d e *determinant of numbers* $h_j - h_0$, *hence is different than zero.*

E x a m p l e 1.1. Suppose that all assumptions of Proposition 1.3 are satisfied. Then the following initial operators have the property $c(R)$ for all $h \in A(\mathbf{R})$:

(i) $F' = (1-d)F + dF_h$ (where $d \in \mathcal{F}$ is a parameter) with $c_k(h) = dh^k$;
(ii) $F'' = F + F_h D$ with $c_k(h) = kh^{k-1}$;
(iii) $F''' = F_h + FD$ with $c_k(h) = h^k + \delta_{ik}$ (where δ_{ik} is the K r o n e c k e r symbol) ;
(iv) $F^{(iv)} = \frac{1}{h} F_h R$ with $c_k(h) = \frac{h^k}{k+1}$;
(v) $F^{(v)} = (1-d)F + dF^{(iv)}$ with $c_k(h) = d\frac{h^k}{k+1}$;
(vi) $F^{(vi)} = \sum_{j=1}^{m} \alpha_j F_{h_j}$, where $\sum_{j=1}^{m} \alpha_j h_j^k = 1$, $\alpha_j h_j^k \in A(\mathbf{R})$, $h_j \neq h_k$ for $j \neq k$, $j = 1, ..., m$, with $c_k(h) = \sum_{j=1}^{k} \alpha_j h_j^k$.

Observe that for all listed here initial operators $c_k(h) \neq 0$ if $h \neq 0$.

E x a m p l e 1.2. Let $X = C(\mathbf{R})$, $D = \frac{\mathrm{d}}{\mathrm{d}t}$, $R = \int_0^t$ and $(Fx)(t) = x(0)$ for $x \in X$. Then R-shifts are defined as usual shifts operators : $(S_h x)(t) = x(t+h)$ for t , $h \in \mathbf{R}$, $x \in X$. Thus we have : $(F_h)(t) = x(h)$, $R_h = \int_0^t$ and $F_h R^k c = c\frac{h^k}{k!}$ for c , t , $h \in \mathbf{R}$, $x \in X$, $k \in \mathbf{N}_0$. We therefore conclude that $F_h \in c(R)$ and $c_k(h) = h^k$. Initial operators defined in Example 1.1 are of the form :

$$(F'x)(t) = (1-d)x(0) + dx(h) \, ; \; (F''x)(t) = x(0) + x'(h) \, ; \; (F'''x)(t) = x(h) + x'(0) \, ;$$

$$(F^{(iv)}x)(t) = \frac{1}{h}\int_0^h x(s)ds \; ; \; (F^{(v)}x)(t) = (1-d)x(0) + \frac{d}{h}\int_0^h x(s)ds \; ;$$

$$(F^{(vi)}x)(t) = \sum_{j=1}^{m} \alpha_j x(h_j)$$

for $x \in X$ and have also property $c(R)$. Recall that in this case we have $\dim \ker D = 1$ .Hence, by Theorem 1.1, all initial operators have the property $c(R)$.

E x a m p l e 1.3. Let $X = C[1, \infty)$, $D = t\frac{\mathrm{d}}{\mathrm{d}t}$, $(Rx)(t) = \int_0^t s^{-1}x(s)ds$ and $(Fx)(t) = x(1)$ for $x \in X$. Here we have : $(S_h x)(t) = x(e^h t)$; $(F_h x)(t) = x(e^h)$; $R^k c = c\frac{(\ln t)^k}{k!}$ for c , h , $t \in \mathbf{R}_+$, $x \in X$. Here $c_k = (\ln e^h)^k = h^k$ for $k \in \mathbf{N}_0$. Since $\dim \ker D = 1$, all initial operators have the property $c(R)$.

E x a m p l e 1.4. Suppose that $X = (s)$, i.e. X is the space of all sequences $x = \{x_n\}$, where $x_n \in \mathcal{F}$ for $n \in \mathbf{N}$. The operator of the *forward shift* : $D\{x_n\} = \{x_{n+1}\}$ is right invertible and $\dim \ker D = 1$. Write : $R\{x_n\} = \{x_{n-1}\}$, where we admit $x_{n-k} = 0$ if $n \leq k$ for all $x \in X$, $F\{x_n\} = x_1\{\delta_{n1}\}$. Then F is an initial operator for D corresponding to R . Since the kernel of D is one-dimensional, all initial operators have the property $c(R)$. Let $F_1 x = \sum_{j=1}^{m} \alpha_j x_j$ with $\sum_{j=1}^{m} \alpha_j = 1$. Clearly, F_1 is an initial operator for D and $F_1 = F$ if and only $a_j = \delta_{j1}$ $(j = 1, ..., m)$. Assume that this is not the case. It is not difficult to verify that

$$c_k = \begin{cases} \sum_{j=1}^{m} \alpha_j \delta_{j,k+1} , & \text{if } k \leq m-1; \\ 0 & \text{otherwise} \end{cases} \qquad (k \in \mathbf{N}_0)$$

and that not all c_k do not vanish simultaneously (cf. PR[2], Example 2.3). The list of examples can be essentially prolongated if we use *functional shifts* introduced by Z. Binderman (cf. for instance, B[1], B[2]) instead of R-shifts. Namely, R-shifts correspond are connected in a sense with the function e^h, functional shifts may be induced by an arbitrary function $f \in H(\Omega)$.

The general interpolation problem (shortly : **(GI)**, cf. PR[2]). Given n finite sets $I_j = \{n_{j,1}, ..., n_{j,p_j}\}$ of different nonnegative integers. Assume that these numbers are ordered, i.e. $n_{j,i} \leq n_{j,i+1} \leq N-1$, where $N = \sum_{j=1}^{n} p_j$, $p_j = \# I_j$ $(j = 1, ..., n)$. We are looking for a D-polynomial u of degree $N-1$ (i.e. for an element of $\ker D^N$) satisfying for n given different initial operators $F_0, ..., F_{n-1}$ the conditions

$$F_j D^k u = u_{jk} , \quad k \in I_j \; ; \; j = 1, ..., n , \tag{1.2}$$

where $u_{jk} \in \ker D$ are given, $u = \sum_{k=0}^{N-1} R^k z_k$ for an $R \in \mathcal{R}_D$ and $z_1, ..., z_{N-1} \in \ker D$ are to be determined.

Theorem 1.2. (cf. PR[2]) *Suppose that* $D \in R(X)$ *and* $F_0, ..., F_{N-1} \in \mathcal{F}_D$ *have the property* $c(R)$ *for an* $R \in \mathcal{R}_D$, *i.e.*

$$F_i R^k z = \frac{d_{ik}}{k!} z \quad \textit{for } z \in \ker D , \; i = 0, 1, ..., n-1 \; ; \; k \in \mathbf{N} \tag{1.3}$$

(recall that $d_{i0} = 1$ *for* $i = 0, 1, ..., n-1$ *since* $F_i z = z$ *for* $z \in \ker D$*). If*

$$\det V_N = \det \begin{pmatrix} V_{N,1} \\ ..... \\ V_{N,n} \end{pmatrix} \neq 0 , \qquad \textit{where} \tag{1.4}$$

$$V_{N,j} = \left(\frac{d_{j,m-k}}{(m-k)!} \right)_{k \in I_j \; ; \; m=k,...,N-1} , \qquad (j = 1, ..., n) \tag{1.5}$$

is the triangle matrix with all terms equal zero for $m < k$, *then the general interpolation problem* **(GI)** *has a unique solution for every* $u_{jk} \in \ker D$, $k \in$

I_j , $j = 1, ..., n$ *which is of the form :*

$$(1.6) \qquad u = W_N(u_0, ..., u_{N_1} \stackrel{\text{def}}{=} \sum_{j=0}^{N-1} V_{Nj}(R)u_j \ , \quad \textit{where}$$

$$(1.7) \qquad V_{Nj}(t) = \frac{1}{V_N} \sum_{k=0}^{N-1} (-1)^{k+1} k! V_{Njk} t^k \quad (j = 0, 1, ..., N-1) \ ,$$

V_{Njk} *is the minor determinant obtained by cancelling in* V_N *the* k*-th column and the* j*-th row* $(j, k = 0, 1, ..., N-1)$ *and the elements* $u_0, ..., u_{N-1}$ *are defined as follows :*

$$(1.8) \qquad \begin{array}{ll} u_\nu = u_{1\nu} & \textit{if } \nu = 0, 1, ..., p_1 - 1 \ ; \\ u_{p_1+\nu} = u_{2\nu} & \textit{if'} \ \nu = 0, 1, ..., p_2 - 1 \ ; \\ \cdots & \cdots \\ u_{p_1+...+p_{n-1}+\nu} = u_{n\nu} & \textit{if } \nu = 0, 1, ..., p_n - 1 \ . \end{array}$$

Observe that rank $V_{N,j} = p_j$ for $j = 1, ..., n$. Hence $\sum_{j=1}^n$ rank $V_{N,j} = N$ and N is the dimension of the square matrix V_N . However, we do not know, is rank $V_N = N$? On the other hand, we have

Corollary 1.1. (cf. PR[2]) *Suppose that all assumptions of Theorem 1.2 are satisfied and* $I_j = \{0\}$ $(j = 1, ..., n)$. *Then* $n = N$, $V_N = \det(d_{ik})_{i,k=0,1,...,N-1} \neq 0$ *and the* L a g r a n g e interpolation problem (shortly : **(L)**)

$$F_i u = u_i \ , \qquad u_i \in \ker D \qquad (i = 0, 1, ..., N-1)$$

has a unique solution for every $u_0, ..., u_{N-1} \in \ker D$ *which is of the form (1.6).*

In particular, if $F_0, ;..., F_{N-1}$ satisfy all assumptions of Proposition 1.3 (i.e. $F_j = F_{h_{j+1}}$) , then $V_N \neq 0$ and the problem **(L)** has a unique solution. Observe also that Corollary 1.1 gives us solutions of the *boundary value problem* for the operator D^N (cf. [K] and PR[3]).

Corollary 1.2. (cf. PR[2]) *Suppose that all assumptions of Theorem 1.2 are satisfied and* $I_j = \{j\}$ *for* $j = 1, ..., n$. *Then* $n = N$,

$$V_N = \det \left(\frac{d_{m,k-m}}{(k-m)!} \right)_{k=m,...,N-1 \ ; \ m=0,1,...,N-1} \neq 0$$

(as a triangle matrix with all terms equal zero for $k < m$*) and the* N e w t o n *interpolation problem* (shortly(**Nt**))

$$F_m D^m u = u_m \ , \qquad u_m \in \ker D \qquad (i = 0, 1, ..., N-1)$$

has a unique solution for every $u_0, ..., u_{N-1} \in \ker D$ *which is of the form (1.6).*

Note that Corollary 1.2 gives the solutions of them *mixed boundary value problem* for the operator D^N in a simpler form than that obtained by means of the T a y l o r - G o n t c h a r o v formula (cf. PR[1]).

Corollary 1.3. (cf. PR[2]) *Suppose that all assumptions of Theorem 1.2 are satisfied,* $I_j = \{0, 1, ..., p_j - 1\}$ *for* $j = 1, ..., n$, $\{S_h\}_{h \in A(\mathbf{R})}$ *is a family of R-shifts and* $F_j = F_{h_j} = FS_{h_j}$ *, where* $h_i \neq h_j$ *if* $i \neq j$ *. Then* $V_N \neq 0$ *and the H e r m i t e interpolation problem (shortly :* (**H**)*) has a unique solution for every* u_{jk} $(k = 0, 1, ..., p_j - 1 \ ; \ j = 1, ..., n)$ *which is of the form (1.6).*

We recall without the proof

Theorem 1.3. (cf. N g u y e n V a n M a u [N]) *Suppose that all assumptions of Theorem 2.1 are satisfied. Then* $V_N \neq 0$ *if and only if the operators* $F_j D^k$, $(k \in I_j \ ; \ j = 1, ..., n)$ *are linearly independent on the set*

$$P_N(R) = \text{lin}\ \{R^k z : z \in \ker D \ ; \ k = 0, 1, ..., N-1\}$$

(cf. Conditions (1.4)).

2. Finite dimensional kernels. Suppose that we are given an operator

$D \in R(X)$ such that $\dim \ker D = n \ (n \in \mathbf{N})$. It means that

$$\ker D = \text{lin}\ \{e_1, ..., e_n\} \ , \tag{2.1}$$

where $e_1, ..., e_n$ are linearly independent. Any initial operator F_1 for D is then of the form :

$$F_1 x = \sum_{j=0}^{n} a_j(x) e_j \ , \quad \textit{where } a_j \in X' \ , \ a_i(e_j) = \delta_{ij} \ (i, j = 1, ..., n) \ . \tag{2.2}$$

Proposition 2.1. *Suppose that* $D \in R(X)$, $\dim\ \ker D = n$, F *is an initial operator for* D *corresponding to an* $R \in \mathcal{R}_D$ *and* $F_1 \in \mathcal{F}_D$, $F_1 \neq F$ *. Then the operators* $F_1 R^k$ $(k \in \mathbf{N}_0)$ *preserve coordinates, i.e. there exist scalars* c_{jk} *such that*

$$F_1 R^k e_j = c_{jk} e_j \quad \textit{for } j = 1, ..., n \ ; \ k \in \mathbf{N}_0 \ , \ \textit{namely, } c_{jk} = a_j(R^k e_j) \ ,$$

provided that $\ker D$ *and* F_1 *are determined by Formulae (2.1), (2.2), respectively, if and only if*

$$a_{ijk} \overset{\text{def}}{=} a_i(R^k e_j) = \begin{cases} 0\,; & \text{if } i \neq j\,; \\ \neq 0 & \text{if } i = j\,. \end{cases} \tag{2.3}$$

If this condition is satisfied then $c_{jk} = a_j(R^k e_j)$ $(j = 1, ..., n\,;\ k \in \mathbf{N}_0)$.

Proof. Let $\ker D$ and F_1 be determined by Formulae (2.1), (2.2), respectively. By definitions,

$$F_1 R^k e_j = \sum_{i=0}^{n} a_i(R^k e_j)e_i \quad for\ every\ k \in \mathbf{N}_0\ ,\ j = 1, ..., n\ .$$

Since $e_1, ..., e_n$ are linearly independent, we get (2.3). ■

Corollary 2.1. *Suppose that all assumptions of Proposition 2.1 are satisfied and the initial operator* F_1 *preserve coordinates. Then* F_1 *has the property* $c(R)$.

The converse statement is not true if $n \neq 1$, as it is shown by Theorem 1.1.

For an initial operator F_1 determined by Formula (2.2) write :

$$\Delta_k(R;t) = \det\ (a_{ijk} - \delta_{ij}t)_{i,j=1,...,n}\ , \tag{2.4}$$

$R \in \mathcal{R}_D$, $a_ijk = a_j(R^k e_i)$ $(k \in \mathbf{N}_0)$. Clearly, $\Delta_k(R;t)$ are polynomials of degree n in the variable t with scalar coefficients.

Theorem 2.1. *Suppose that all assumptions of Proposition 2.1 are satisfied. Then the initial operator* F_1 *determined by Formula (2.2) has the property* $c(R)$ *if and only if all equations*

$$\Delta_k(R;t) = 0 \qquad (k \in \mathbf{N}) \tag{2.5}$$

have non-trivial solutions t_k . *If it is the case then* $F_1 R^k z = \frac{c_k}{k!} z$, *where* $c_k = k! t_k$ *for all* $z \in \ker D$ $(k \in \mathbf{N})$ *and, by definition,* $c_0 = 1$.

Proof. Let $z \in \ker D$ be arbitrarily fixed. Since $F_1 z = z$, we conclude that $z = \sum_{j=1}^{n} a_j(z)e_j$. Write : $b_j = a_j(z)$ for $j = 1, ..., n$. Hence for every fixed $k \in \mathbf{N}$ the equality $F_1 R^k z = tz$, where $t \in \mathcal{F}$, holds if and only if

$$\sum_{i=1}^{n} tb_i e_i = t\sum_{i=1}^{n} b_i e_= tz = F_1 R^k z = F_1 R^k \sum_{i=1}^{n} b_i e_i =$$

$$= \sum_{i=1}^{n} b_i F_1 R^k e_i = \sum_{i=1}^{n} b_i \sum_{j=1}^{n} a_j(R^k e_i) e_j = \sum_{j=1}^{n} (\sum_{i=1}^{n} \sum_{i=1}^{n} a_{ijk} b_i) e_j \ ,$$

i.e. if and only if

$$\sum_{j=1}^{n} (\sum_{i=1}^{n} a_{ijk} b_i - t b_j) e_i = 0 \quad (k \in \mathbf{N}) \ . \tag{2.6}$$

The linear independence of the basic elements $e_1, ..., e_n$ implies that (2.6) holds if and only if

$$\sum_{i=1}^{n} (a_{ijk} - t\delta_{ij}) b_i = 0 \quad \textit{for every } k \in \mathbf{N} \ . \tag{2.7}$$

For every fixed $k \in \mathbf{N}$ the system (2.7) has a non-trivial solution t_k if and only if its determinant is equal zero, i.e. if and only if the k-th equation (2.5) is satisfied. If this condition is satisfied, then by Definition 1.1, we get $c_k = k! t_k$ $(k \in \mathbf{N})$. Also by Definition 1.1, $c_0 = 1$. ∎

Theorems 1.1 and 2.1 together immediately imply

Corollary 2.4. *Suppose that all assumptions of Theorem 2.1 are satisfied and* $n = 1$ *. Then for every* $k \in \mathbf{N}$ *the equation* $\Delta_k(R; t) = 0$ *has a nontrivial solution, i.e.* $t_k = a_1(R^k e_1) \neq 0$ $(k \in \mathbf{N})$.

E x a m p l e 2.1. Suppose that all assumptions of Theorem 2.1 are satisfied and $n = 2$. Then $\Delta_k(R; t) = t^2 - (a_{11k} + a_{22k})t + a_{11k}a_{22k} - a_{12k}a_{21k}$, where $a_{ijk} = a_i(R^k e_j)$ $(i, j = 1, 2 \ ; \ k \in \mathbf{N})$.

E x a m p l e 2.2. Suppose that $D_o \in R(X)$, $\dim \ker D_o = 1$ and F_o is an initial operator for D_o corresponding to an $R_o \in \mathcal{R}_{D_o}$. Let $\ker D_o = \operatorname{lin} \{e\}$. By definition, $D_o e = 0$. Let $n \in \mathbf{N}$ be arbitrarily fixed. Clearly, $D = D_o^n \in R(X)$ and $R = R_o^n \in \mathcal{R}_D$. By the T a y l o r Formula (cf. PR[1]), the operator $F = \sum_{k=0}^{n-1} R_o^k F_o D_o^k$ is an initial operator for D corresponding to R . The elements $e_j = R_o^{j-1} e$ $(j = 1, ..., n)$ are linearly independent and $\ker D = \operatorname{lin} \{e_1, ..., e_n\}$. Write $r_j = R_o^j e$ $(j \in \mathbf{N}_0)$. Then $e_j = r_{j-1}$ for $j = 1, ..., n$ and $R^k e_j = (R_o^n)^k R_o^j e = r_{kn+j}$ $(j = 1, ..., n \ ; \ k \in \mathbf{N}_0)$. Let F_1 be defined by Formula (2.2). By Proposition 2.1, $F_1 R^k$ preserve coordinates if and only if

$$a_i(r_{kn+j}) = \begin{cases} 0 & \text{if } i \neq j \ ; \\ \neq 0 & \text{if } i = j \end{cases} \qquad (j = 1, ..., n \ ; \ k \in \mathbf{N}_0 \ .$$

If it is the case then $F_1 R^k ej = a_j(r_{kn+j})$ $(j = 1, ..., n\ ;\ k \in \mathbf{N}_0)$. By Theorem 2.1, F_1 has the property $c(R)$ if every equation

$$\det\left(a_i(r_{kn+j}) - t\delta_{ij} \right)_{i,j=1,...,n} = 0 \qquad (k \in \mathbf{N})$$

has a nontrivial solution t_k . If it so, then $c_k = k!t_k$ for $k \in \mathbf{N}$, $c_0 = 1$.

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Received 24.10.1994