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On Strongly Sequential Ordered Topological Algebras

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Let A be a real unital Hausdorff topological algebra which is also a partially ordered algebra such that every element $a \in A$ is a difference $a = a_1 - a_2$ of positive elements $a_1, a_2 \in A$ satisfying $a_1 a_2 = 0$. It is shown in this paper that if order intervals of A are topologically bounded then A is an Archimedean f-algebra with locally solid topology, and if in addition A is strongly sequential then the algebric unit e of A is a strong order unit and the topology of A is generated by the e-uniform norm.

1. I. Introduction and preliminaries

A partially ordered algebra is an associative real algebra which is simultaneously a partially ordered vector space such that the positive cone A^+ of A is closed for multiplication, i.e. $a, b \in A^+$ implies $ab \in A^+$.

Definition 1. *A partially ordered algebra A is said to be*

(1) p-decomposable if every element $a \in A$ is expressed as $a = a_1 - a_2$ with $a_1, a_2 \in A^+$ and $a_1 a_2 = 0$;

(2) an f-algebra if its underlying partially ordered vector space is a vector lattice such that $a, b \in A$ with $a \wedge b = 0$ implies $ac \wedge b = ca \wedge b = 0$ for all $c \in A^+$.

It is clear that every f-algebra is p-decomposable. In this paper we are concerned with unital p-decomposable partially ordered algebras which are also topological algebras, i.e. topological vector spaces with jointly continuous multiplication. We show that if such an algebra A is Hausdorff then the boundedness of its order intervals implies that A (under the original partial order) is an Archimedean f-algebra with locally solid topology. Recall now the notion of a strongly sequential topological algebra from [6].

Definition 2. *A topological algebra A is said to be strongly sequential, if there exist a neigborhood U of* 0 *in A such that* $\lim_{n\to\infty} a^n = 0$ *holds for all* $a \in U$.

Every normed (topological) algebra is strongly sequential. Let A be an Archimedean f-algebra with unit e, and A_e the principal order ideal of A generated by e. Then one can introduce in A_e the e-uniform norm.

$$\|a\|_e = inf\{\lambda > 0 : |a| \leq \lambda e\}, a \in A_e$$

, which makes A_e a lattice-normed f-algebra.

It is shown in this note that if A is a unital p-decomposable partially ordered algebra which is also a Hausdorff strongly sequential topological algebra, then the boundedness of its order intervals implies that the algebraic unit e of A is a strong order unit and that the e-uniform topology generates the original topology of A.

We give also a refinement of this result in a particular case when A is a normed algebra, and present a characterisation of algebras of type $C(X)$, where X is a compact Hausdorff space.

For the basic theory of partially ordered algebras and f-algebras we refer the reader to [4], [2] and [12], while for general theory of partially ordered topological vector spaces we refer to [10],[11] and [3].

2. Results

Recall that a partially ordered algebra A is said to be Archimedean if $a = 0$ whenever $ka \leq b$ for all integers k and some $b \in A$.

Proposition 1. *An Archimedean p-decomposable partially ordered algebra A with unit element is an f-algebra.*

Proof. Let a be a positive nilpotent element of A and $k \in \mathbb{Z}$. Set $b = ka - e$, where e denotes the unit element of A. Since A is p-decomposable, there exists positive elements $b_1, b_2 \in A$, such that $b = b_1 - b_2$ and $b_1 b_2 = 0$. Then

$$kb_1 a - b_1 = b_1(ka - e) = b_1 b = b_1^2 \geq 0,$$

hence $b_1 \leq kb_1 a$, and therefore $b_1 \leq k^n b_1 a^n$ for all $n \in \mathbb{N}$($k \leq 0$ implies that $b_1 = 0$). Since a is nilpotent, this yields $b_1 = 0$ and consequently $ka \leq e$. The Archimedean property of A implies that $a = 0$, so A is without non-zero positive nilpotent elements. Let

$$c = c_1 - c_2,\ c_1, c_2 \in A^+,\ c_1 c_2 = 0$$

be a p-decomposition of $c \in A$. Then $c_2c_1 \geq 0$ and $(c_2c_1)^2 = 0$, thus $c_2c_1 = 0$. This implies that if c is a nilpotent, then c_1 and c_2 are nilpotents, therefore $c_1 = c_2 = 0$ and $c = 0$. It follows by [5] that A is an f-algebra, thus the proof is complete. □

Remark 1. The p-decomposable partially ordered algebras are closely related to the so called almost f-algebras, i.e. lattice-ordered algebras in which $a \wedge b = 0$ implies $ab = 0$. Precisely, it can be proved that a p-decomposable partially ordered algebra is an almost f-algebra if an only if it is lattice-ordered. It is shown in [1, Theorem 1.11] that an Archimedean almost f-algebra with unit element is an f-algebra, thus Proposition 1 extends this result on Archimedean p-decomposable algebras.

Lemma 1. *Let A be an Archimedean f-algebra with unit e, and let $a, b, c \in A$. If $|b| \leq |a|$, then $c' = (c \wedge e) \vee (-e)$ satisfies $|b - ac'| \leq |b - ac|$.*

Proof. Let $u \in A$. Then

$$a(e-u)^- \leq a^+(e-u)^- + a^-(e+u)^- = (|a| - au)^- \leq (b - au)^-,$$

and therefore

$$b - a(u \wedge e) = b - au + a(e-u)^- \leq (b - au)^+.$$

Similarly we can obtain $b - a(u \wedge e) \geq -(b - au)^-$, thus

$$|b - a(u \wedge e)| \leq |b - au|.$$

If we replace in this inequality a by $-a$ and u by $-u$, we get

$$|b - a(u \vee (-e))| \leq |b - au|,$$

hence $|b - ac'| \leq |b - a(c \vee e)| \leq |b - ac|$ as claimed. □

We are prepared to prove the following result, which can be compared with [8, Theorem 2.2].

Theorem 1. *Let A be a Hausdorff topological algebra with unit e. Then the following statements are equivalent.*

(i) A is a p-decomposable partially ordered algebra with topologically bounded order intervals.

(ii) A is an Archimedean f-algebra with locally solid (or locally order-convex) topology.

Proof. The implication $(ii) \Longrightarrow (i)$ is clear. To prove the converse suppose that (i) is satisfied. Since the topology of A is Hausdorff, the boundedness of order intervals of A implies that A is Archimedean, therefore by proposition 1 A is an f-algebra. In order to show that the topology of A is locally solid suppose that U ia a given neighborhood of 0. Choose neighborhoods U_1 and V_1 of 0 satisfying

$$U_1 + U_1 \subset U, \; V_1 \subset U_1, \; (V_1)^2 \subset U_1,$$

take a real $r > 0$ such that $[-e, e] \subset rV_1$, and set $V = r^{-1}V_1$.

Let $a \in V, b \in A$ and $|b| \leq |a|$. The multiplication $x \longmapsto ax$ by $a \in A$ is an orthomorphism in A. Since by [9. Theorem 16.1] the range R of an orthomorphism in an Archimedean Riesz space is uniformnly dense in the o-ideal generated by R, there exist elements $d \in A^+, c_n \in A, n = 1, 2, ...,$ such that $|b - ac_n| \leq (1/n)d$ for n=1,2,... By lemma 1 the sequence $c_n' = (c_n \wedge e) \vee (-e) \in [-e, e]$ satisfies

$$|b - ac_n'| \leq |b - ac_n| \leq (1/n)d \,\text{for}\; n = 1, 2, ...$$

Chose $n \in \mathbb{N}$ such that $[-d, d] \subset nV_1$, and note that

$$b = ac_n' + (b - ac_n') \in V(rV_1) + V_1 = (V_1)^2 + V_1 \subset U.$$

It follows easily that the topology of A is locally solid as claimed. □

The implication $(ii) \implies (i)$ of theorem 1 remains valid for algebras without unit, while the reverse implication (and also proposition 1) does not hold in this case.

E x a m p l e 1. Let ℓ^1 be a partially ordered algebra of real absolutely summable sequences with componentwise defined operations, and normed by

$$||(x_1, x_2, ...)||_0 = |x_1| + \sum_{k=1}^{\infty} |x_{k+1} - x_k|,$$

as in example 3.1 of [8]. Observe that $||.||_0$ is an algebra norm on ℓ^1. Endow the coordinatewise ordered direct sum

$$A = \ell^1 \oplus \ell^1 = \{(x, y) : x, y \in \ell^1\}$$

with the multiplication

$$(x, y)(u, v) = (0, xu), \; x, y \in \ell^1$$

and the norm

$$||(x, y)|| = ||x||_0 + ||y||_0, \; x, y \in \ell^1.$$

Then A is an Archimedean p-decomposable normed partially ordered algebra. Since $0 \leq y \leq x \in \ell^1$ implies $||y||_0 \leq 2||x||_1$, it follows that the intervals of A are norm bounded, so A satisfies the condition (i) of theorem 1. It is easy to see that A is not an f-algebra. Moreover, the topology of A is not locally order-convex. Indeed, if $\{e_1, e_2, \ldots\}$ is the standard basis of ℓ^1 and

$$x = \sum_{k=1}^{n} e_{2k-1},\ y = \sum_{k=1}^{2n} e_k;\ n \in \mathbb{N},$$

then the elements $a = (x,0), b = (y,0) \in A$ satisfy $0 \leq a \leq b$ and $\|a\| = 2n, \|b\| = 2$, therefore by [10, Proposition II.1.7] A is not locally order-convex.

In further work we need the follwing

Lemma 2. *Let A be an f-algebra with unit element e and simultaneously a topological algebra with locally order-convex topology. Then* $\lim_{n\to\infty} a^n = 0(a \in A$ *implies that* $|a| \leq e$.

P r o o f. We claim that each $a \in A$ satisfies

$$(|a| - e)^+ \leq |a|^n$$

for all $n \in \mathbb{N}$. The inequality clearly holds for $n = 1$. Suppose by induction that

$$(|a| - e)^+ \leq |a|^k$$

for some $k \in \mathbb{N}$. Then $(|a| - e)(|a| - e)^+ \geq 0$ implies that

$$(|a| - e)^+ \leq |a|(|a| - e)^+ \leq |a|^{k+1},$$

so the claim follows. If now $a^n \to 0$, then $|a|^{2n} = a^{2n} \to 0$, so the local order-convexity of A implies that $(|a| - e)^+ = 0$, and therefore $|a| \leq e$. □

R e m a r k 2. If A (from the above lemma) is also infrasequential [6, Definition 3.6], then e is a strong unit of A. Indeed, in this case for each $a \in A$ there exists a real $\lambda = \lambda(a) > 0$ such that $(\lambda a)^n \to 0$, hence by lemma 2 we have $|a| \leq \lambda e$. If A is not infrasequential, e need not be a strong unit of A. By way of example, let A be the f-algebra $C(\mathbb{R})$ of real continuous functions equipped with the uniform topology on compact subset of $\mathbb{R}$. Then A is locally solid topological f-algebra with unit $1_{\mathbb{R}}$ which is not a strong order unit of A.

In order to prove the main result we need an algebraic.

Lemma 3 *Let A be an Arhimedean f-algebra with unit e, and let $a \in A^+, 0 < r \in \mathbb{R}$. If a does not belong to the principal order ideal A_e generated by e, then there exists an element $b \in [0, e]$ such that*

$$(ab - re)^+ > 0 \text{ and } ab \in A_e$$

Proof. For each $n \in \mathbb{N}$ set

$$a_n = (a - ne)^+, \; b_n = e - a_n + a_{n+1},$$

and note that $b_n \in [0, e]$. It follows from

$$(e + a)a_{n+1} = (a^2 - na - (n+1)e)^+ \leq (a^2 - na)^+ = aa_n$$

that

$$a - (n+1)e \leq a_{n+1} \leq a(a_n - a_{n+1}).$$

Consequently

$$0 \leq ab_n = a(e - a_n + a_{n+1}) \leq (n+1)e,$$

so $ab_n \in A_e$. We claim that if $a \notin A_e$, then there exists an $n \in \mathbb{N}$ such that $b = b_n$ satisfies the required conditions. Indeed, if $ab_n \leq re$ holds for all $n \in \mathbb{N}$, then the estimate $a_n - a_{n+1} \leq (1/n)a^2$ gives

$$a \leq re + a(a_n - a_{n+1}) \leq re + (1/n)a^3$$

for all $a \in \mathbb{N}$. This yields $a \leq re$, a contradiction with $a \notin A_e$, so the proof is complete.

We are now in a position to prove the main theorem.

Theorem 2. *Let A be a strongly sequential topological algebra with unit e. Then the following statements are equivalent.*

(i) A is a p-decomposable partially ordered with closed cone A^+ and with bounded interval $[0,e]$.

(ii) A is a p-decomposable partially ordered algebra with Hausdorff topology and bounded order intervals.

(iii) A is an Archimedean f-algebra with strong order unit e and with topology generated by the e-uniform norm.

Proof. $(i) \Longrightarrow (ii)$. Assume that A satisfies (i). The closedness of A^+ implies easily that A is Archimedean with Hausdorff topology. It follows by theorem 1 that A is an Archimedean f-algebra. To see that A satisfies (ii) it suffices to prove that e is a strong unit of A. By way of contradiction suppose that there exists an element $a \in A^+ \setminus A_e$. Since A is strongly sequential,

$$U = \{u \in A : \lim_{n \to \infty} u^n = 0\}$$

is a neighborhood of 0 in A. Choose a neighborhood V of 0 satisfying

$$V \subset U, \; V^2 \subset U,$$

and pick a real $r > 0$ such that

$$[0, e] \cup \{a\} \subset r^{1/2}V.$$

By lemma 3 there exists $b \in [0, e]$ satisfying

$$ab \in A_e, \ (ab - re)^+ > 0.$$

Theorem 1 tells us that the topology of A induces on A_e a locally solid topology, therefore by lemma 2

$$(1/r)ab \notin U.$$

This contradicts the relation

$$ab \in (r^{1/2}V)(r^{1/2}V) = rV^2 \subset rU,$$

thus e is a strong unit of A as claimed.

(ii) $\Longrightarrow$ *(iii)*. Suppose that A satisfies (ii). Then theorem 1 implies that A is an Archimedean f-algebra with locally solid topology. By lemma 2 we have

$$U = \{u \in A : \lim_{n\to\infty} u^n = 0\} \subset [-e, e],$$

hence $[-e, e]$ is a bounded convex neighborhood of 0. It follows that e is a strong order unit and that the functional $\mu_{[-e,e]}$ of Minkowski is a norm which generates on A its original topology. Since $\mu_{[-e,e]}$ coincides with the e-uniform norm (iii) follows.

(iii) $\Longrightarrow$ *(i)*. Obvious.

Let A be a topological algebra with unit element e. Then the condition (iii) of theorem 2 implies (ii), and (ii) implies (i). If e is a strong order unit of A, then (i) is equivalent to (ii), but even in this case the condition that A is strongly sequential cannot be dropped in theorem 2. This is shown by the following example.

Example 2. (see [6, 3.30]). Let A be the Archimedean f-algebra $C_b(\mathbb{R})$ of all bounded continuous real functions equipped with the locally convex topology defined by the family of seminorms

$$\{p_\varphi : \varphi \in C_b(\mathbb{R}), \ \varphi \text{vanishes at infinity}\},$$

where

$$p_\varphi(f) = \sup\{|f(x)\varphi(x)| : x \in \mathbb{R}\}.$$

Then A is an infrasequential locally solid topological f-algebra with unit $1_{\mathbb{R}}$ which is also a strong order unit of A. A satisfies the conditions (i) and (ii) of theorem 2. Since A is not strongly sequential, its topology is not generated by the e-uniform norm, hence A does not satisfy the condition (iii) of theorem 2.

If A is a normed algebra, theorem 2 can be sharpened as follows.

Corollary 1. *Let A be a normed algebra with unit e and simultaneously a p-decomposable partially ordered algebra such that the cone A^+ is closed and $a \in [-e, e]$ implies $\|a\| \leq 1$. Then A is an Archimedean f-algebra in which e is a strong order unit and the norm of A coincides with the e-uniform norm.*

Proof. Note that A is strongly sequential, use the implication $(i) \Longrightarrow (iii)$ of theorem 2, and observe that by lemma 2 each $a \in A$ with $\|a\| < 1$ satisfies $\|a\|_e \leq 1$. It follows easily that $\|a\|_e \leq \|a\|$ for all $a \in A$. Since each $a \in [-e, e]$ satisfies $\|a\| \leq 1$, we have $\|a\| \leq \|a\|_e$ for all $a \in A$, hence the norm of A coincides with the e-uniform norm.

A representation theorem of Kadison [7] together with theorem 2 shows that a p-decomposable unital Banach algebra with bounded intervals is algebraically, order and topologically isomorphic to $C(X)$ for some compact Hausdorff space X, and gives also the following characterisation of normed algebras of type $C(X)$. □

Corollary 2. *Let A be a partially ordered algebra which is also a Banach algebra with unit element e and with closed cone A^+. Then A is algebraically, order and isometrically isomorphic to $C(X)$ for some compact Hausdorff space X, if and only if A is p-decomposable and each $a \in [-e, e]$ satisfies $\|a\| \leq 1$.*

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