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On Minimal Spliting Subspaces and Markovian Extensions of a Stochastic Process [1]

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The problem of existence of a minimal Markovian process which contains a given stochastic process as a component is considered and partially solved in [6]. In this paper the problem (to be formulated precisely below) or finding all minimal Markovian extensions for a given second order stochastic process $\{Y(t), t \in T\}$ is solved.

1. Introduction and Notations.

The approach adopted in this paper is that of [3] and [6].

Let H be a given Hilbert space whose inner product is denoted by $(.,.)$. For subspaces H_1 and H_2 of H, $H_1 \perp H_2$ means that H_1 and H_2 are orthogonal; $H_1 \oplus H_2$ denotes direct summ; $H_1 \ominus H_2$ is the subspace of H_1 orthogonal to H_2; $H_1 \vee H_2$ is the closed linear hull of H_1 and H_2 and $H_1^{\perp}$ is the orthogonal complement of H_1 in H. The orthogonal projection of $h \in H$ onto H_1 is denoted by $P(h|H_1)$.

Definition 1.1. *(compare with [4], [7] and conditional independence from [5]) If H_1, H_2 and X are arbitrary subspaces of Hilbert space H, then it is said that H_1 and H_2 are conditionally ivrthogonal given X or that X is splitting for H_1 and H_2 if*

$$(1) \qquad (h_1, h_2) = (P(h_1|X), P(h_2|X)) \text{ for all } h_1 \in H_1, h_2 \in H_2.$$

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Then we write $H_1 \perp H_2 | X$.

When X is trivial, i.e. $X = \{0\}$, this reduces to the usual orthogonality $H_1 \perp H_2$.

The notion of splitting was first given in [4]. The splitting subspace X is minimal if there is no proper subspace of X satisfying (1).

It can be seen that, if $X \subset H_1$, (1) is equivalent to

$$H_1 \ominus X \perp H_2. \tag{2}$$

We need the following results later.

Lemma 1.2. *([3]) Let $S \subset H_2$. Then $P(H_1 \cap H_2^{\perp} | S) = P(H_1 | S) \cap H_2 \perp$.*

The next theorem gives all minimal splitting subspaces with respect to two given arbitrary subspaces.

Theorem 1.3. *([3]) The subspace X is a minimal splitting subspace from $H_1 \vee H_2$ if and only if $X = P(H_1|S)$ for some subspace S such that $H_2 \subseteq S \subseteq (H_2 \vee P(H_2|H_1))$.*

Proposition 1.4. *([3]) Let S be a subspace such that $H_2 \subseteq S \subseteq H_1 \vee H_2$ and define $X = P(H_1|S)$. Then $S = H_2 \vee X$.*

For the proofs of the previous results see [3].

Here, we first apply it to find all minimal Markovian extensions for a given stochastic process.

From Theorem 1.3 we have

$$X = P(H_1|S), H_2 \subseteq S \subseteq H_2 \vee P(H_2|H_1).$$

Let us suppose that S is the biggest permitted; i.e. $S = H_2 \vee P(H_2|H_1)$. Then we have

$$\begin{aligned} X &= P(H_1|H_2 \vee P(H_2|H_1)) = P(H_1|P(H_2|H_1)) \oplus [(H_2 \vee P(H_2|H_1)) \oplus P(H_2|H_1)]] \\ &= P(H_2|H_1) \oplus P(H_1|(H_2 \vee P(H_2|H_1)) \cap P(H_2|H_1)^{\perp} \\ &= P(H_2|H_1) \oplus P(H_1|H_2 \cap P(H_2|H_1)^{\perp}). \end{aligned}$$

It is easy to see that $P(H_1|H_2 \cap P(H_2|H_1)^{\perp}) = 0$, or equivalently that $H_1 \perp H_2 \cap P(H_2|H_1)^{\perp}$. Really let $a \in H_2 \cap P(H_2|H_1)^{\perp}$, which implies a $\perp P(a|H_1)$, that is $P(a|H_1) = 0$. Consequently, $H_1 \perp H_2 \cap P(H_2|H_1)^{\perp}$ and $X = P(H_2|H_1)$.

Thus, if S is the biggest permitted from Theorem 1.3, then $X = P(H_2|H_1)$.

If S is the smallest permitted from Theorem 1.3, then $X = P(H_1|H_2)$.

Let S be some spacevs between H_2 and $H_2 \vee P(H_2|H_1)$, i.e. $S = H_2 \vee S_1$, $S \subseteq P(H_2|H_1)$. Then

$$\begin{aligned} X &= P(H_1|S) = P(H_1|H_2 \vee S_1) = P(H_1|S_1) \oplus P(H_1|(H_2 \vee S_1) \ominus S_1)) = \\ &= S_1 \oplus P(H_1|(H_2 \vee S_1) \cap S_1^{\perp}) = S \oplus P(H_1|H_2 \cap S_1^{\perp}). \end{aligned}$$

Conclusion. Generally, S has to be of the form

$$S = H_2 \vee S_1, S_1 \subseteq P(H_2|H_1).$$

In this case, X that is splitting for H_1 and H_2 is given by

$$X = S_1 \oplus P(H_1|H_2 \cap S_1^{\perp}).$$

If $S_1 = \{o\}$, then $X = P(H_1|H_2)$.
If $S_1 = P(H_2|H_1)$, then $X = P(H_2|H_1)$.
In general case, that is when $H_2 \subseteq S_1 \subseteq P(H_2|H_1)$, we have

$$X = S_1 \oplus P(H_1|H_2 \cap S_1^{\perp}) \tag{3}$$

that is, X takes part from both $P(H_1|H_2)$ and $P(H_2|H_1)$. For different choices of S_1, different X (of the form (3)) will be obtained, no two of them are comparable.

R e m a r k. By using Lemma 1.2 it is easier to obtain the above results. Namely, since $S_1 \subseteq P(H_2|H_1) \subseteq H_1$, Lemma 1.1 gives

$$P(H_2 \cap S_1^{\perp}|H_1) = P(H_2|H_1) \cap S_1^{\perp}.$$

Now, it is clear that, if $S_1 = P(H_2|H_1)$, then $H_2 \cap S_1^{\perp} \perp H_1$, and thus $X = P(H_2|H_1)$. On the other hand, in order to be $H_2 \cap S_1^{\perp} \perp H_1$, it has to be $P(H_2|H_1) \cap S_1^{\perp} = 0$, that is $S_1 = P(H_2|H_1)$. That proves that the following is true:
i) It will be $X = P(H_2|H_1)$ if and only if $S_1 = P(H_2|H_1)$, i.e. $S = H_2 \vee P(H_2|H_1)$.

Straight forward reasoning proved that following:
ii) it will be $X = P(H_1|H_2)$ if and only if $S = \{o\}$, i.e. $S_1 = H_2$.

2. Main results

Let $\{Y(t), t \in T\}$ be a stochastic second order process such that the $Y(t)$ belongs to H for each $t \in T$; H_t^Y is a Hilbert space defined by $Y(t), t \in T$, and we shall write H^Y to denote $\bigvee_{t \in T} H_t^Y$. The main problem considered here is to determine Markovian processes $\{X(t), t \in T\}$ that contains a given process as a component, i.e. such that $H_t^Y \subseteq H_t^X, t \in T$. To exploit the results of minimal splitting subspaces from part 1, we need to define the past space $H_{\leq t}^Y = \overline{\mathcal{L}}\{Y(s), s \leq t\}$ and the future space $H_{\geq t}^Y = \overline{\mathcal{L}}\{Y(s), s \geq t\}$ for each $t \in T$; then $H^Y = H_{\leq t}^Y \vee H_{\geq t}^Y$.

In this paper the following definition of Markovian property will be used.

Definition 1.2. *See [6] The process $\{Y(t), t \in T\}$ is a Markovian one if*

$$H_{\leq t}^Y \bot H_{\leq t}^Y | H_t^Y \text{ for all } t \in T$$

It will be said that stochastic process $\{X(t), t \in T\}$ is an extension of a stochastic process $\{Y(t), t \in T\}$ if $H_t^Y \subseteq H_t^X$ for each $t \in T$ (see [6]).

The family $H = (H_t)_{t \in T}$ is the minimal with a certain property if any other family $H_\star = (H_t^\star)_{t \in T}$ with the same property is such that $H_t \subseteq H_t^\star, t \in T$. The next result gives all minimal Markovian extensions of a given stochastic process $\{Y(t), t \in T\}$.

Theorem 2.1 . *The process $\{X(t), t \in T\}$ is a minimal Markovian extension of a given process $\{Y(t), t \in T\}$ such that $H_t^Y \subseteq H_t^X \subseteq H^Y, t \in T$, if and only if*

1) $H_t^X = P(H_{\geq t}^Y | S)$ for some subspace S such that $H_{\leq t}^Y \subseteq S \subseteq H_{\leq t}^Y \vee P(H_{\leq t}^Y | H_{\geq t}^Y)$;

2) the family $\{S_t, t \in T\}$ where $S_t = H_{\leq t}^Y \vee H_t^X$ is nondecreasing, i.e. $S_u \subset S_t$, whenever $u \leq t$.

Proof. It is easy to see that $H_t^Y \subseteq H_t^X, t \in T$ i.e. that process $\{X(t), t \in T\}$ defined by 1) is an extension of the given process $\{Y(t), t \in T\}$. By condition 1) and Proposition 1.4, we have $H_t^X = P(H_{\geq t}^Y | H_{\leq t}^Y \vee H_t^X) = P(H_{\geq t}^Y | S_t)$. Using the orthogonal decomposition $A = P(B|A) \oplus (A \cap B^\bot)$ which holds for any subspaces A and B from H, we have

$$S_t \ominus H_t^X = S \cap H_{\geq t}^{Y,\bot}$$

which is, because of (2), nondecreasing in $t \in T$. Hence, since trivially, $S_t \ominus H_t^X \bot H_t^X$, $S_t \ominus H_t^X \bot H_{\geq t}^X$. By (2) we have $S_t \bot H_{\geq t}^X | H_t^X$ and due to condition 2) $H_{<t}^Y \vee H_{<t}^X \subset S_t$, so $H_{\leq t}^X \bot H_{\geq t}^X | H_t^X$ holds, i.e. $\{X(t), t \in T\}$ is Markovian process. To prove the minimality of H_t^X defined by 1), let us suppose that $H = (H_t)_{t \in T}$ is some other Markovian extension of $\{Y(t), t \in T\}$ such that $H_t^Y \subset H_t \subset H^Y$. Then we have $H_t = P(H_{\geq t} \bot H_{\leq t}) \supset P(H_{\geq t}^Y \bot H_{\leq t})$ which is equivalent to $H_{\geq t}^Y \bot H_{\leq t} | H_t$, from that it follows $H_{\geq t}^Y \bot H_{\leq t}^Y | H_t$. According to

Theorem 1.3. the minimal space splitting $H^Y_{\geq t}$ and $H^Y_{\leq t}$ is defined by condition 1) of this theorem, so $H^X_t \subset H_t, t \in T$, holds.

Conversely, let $H = (H_t)_{t \in T}$ be a minimal Markovian extension of a given process $\{Y(t), t \in T\}$ such that $H^Y_t \subset H_t \subset H^Y$. Then H_t is a minimal splitting for $H^Y_{\geq t}$ and $H^Y_{\leq t}$. Now, condition 1) follows from Theorem 1.3, i.e. H_t has to be equivalent $\bar{H}^X_t, t \in T$. It is easy to see that condition 2) holds. The proof is complete. ■

R e m a r k. The concept of conditional independence for a triple of σ-algebras is different from concept of conditional orthogonality (i.e. splitting) for Hilbert spaces. The extensions of the proofs from Hilbert space formulation is nontrivial mainly because one cannot take an orthogonal complement with respect to σ-algebras as one can with respect to a subspace in Hilbert space.

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