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Rate of Convergence for Approximate Integration of the Wiener Process

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We consider approximate methods for calculation of integrals if the Wiener process and find the rate of convergence.

1. Introduction

The goal of this work is to find the rate of convergence for a sequence of approximators $\{A_n(x)\}$ to the integral $A(x) = \int_0^1 x(t)dt$ based on a fixed rule (e.g. trapezoidal rule, Simpson's rule) when x is a stochastic process which is continuous only. There are methods to estimate the rate of convergence, but usually they need additional conditions on the integrand x(e.g. x to be differentiabile, bounded etc.).

It is well-known that if $x \in \mathbf{C}^{(2)}[0,1]$, i.e. if x has two continuous derivatives and the sequence $\{A_n\}$ of approximators is based on the trapezoidal rule, then $A(x) - A_{2^n}(x) = o(1/\mu^n)$ for any $\mu \in [1,4)$. Note however that the standard methods say nothing about the rate of convergence when $x \in \mathbf{C}[0,1] \backslash \mathbf{C}^{(2)}[0,1]$, i.e. when x is continuous but not smooth. In this paper we offer a possible method to estimate the rate of convergence by considering a probabilistic model and arguments from theory of probability. In particular, we have established that the rate of convergence of $A(x) - A_{2^n}(x)$ to zero is $o(1/\mu^n)$ for any $\mu \in [1,2)$ and for almost all $x \in \mathbf{C}[0,1]$ and the sequence of approximators generated by the trapezoidal rule. As a consequence, this holds for any absolutely continuous function.

2. Preliminaries

In our recent paper (Kopanov (1994)) we have described methods to estimate the error of calculation when using a fixed sequence of approximators $\{A_n\}$ of the integral $A(x) = \int_0^1 x(t)dt$ especially in the case when the integrand x is a standard Wiener process.

Let us note that there exist standard methods to endow any separable Banach space B with a probability measure μ thus getting a probability space (X, F, μ)(see e.g. Kuo(1975)). Since $\mathbf{C}[0,1]$ is a separable Banach space , we can endow it with a probability measure μ. The construction of the measure μ can be done by different methods and it is possible to construct an infinite set of measures on $\mathbf{C}[0,1]$(or on B in the general situation). The concrete measure we have chosen in this study is the measure w generated by the standard Wiener process, and this measure is called a Wiener measure. Models involving other measures μ often lead to similar. Let us note that the Wiener measure w is the standard probability measure on $\mathbf{C}[0,1]$.

The construction of the Wiener measure can be extended on any separable Banach space $\mathbf{B}$. This approach is described e.g. in (Kuo (1975)). The Banach space $\mathbf{B}$ with that Gaussian probability measure μ is called abstract wiener space.

Generally we can describe the abstract Wiener space $\mathbf{B}$ as follows: Let $\mathbf{H}$ is a separable Hilbert space. For any $E = \{x \in \mathbf{H}, Px \exists F\}$ where P is a projection with a finite dimension in $\mathbf{H}$ and F is a Borel subset of PH, we define

$$\mu(E) = (2\pi)^{-n/2} \int_F \exp(-|x|^2 dx.$$

Here $n = \dim(PH)$ and dx is the Lebesgue measure in PH. The measure μ is not σ-additive in $\mathbf{H}$ but μ is σ-additive in some separable Banach space $\mathbf{B}, \mathbf{H} \subset \mathbf{B}$, and for any separable Banach space $\mathbf{B}$ there exists a Hilbert space $\mathbf{H}, \mathbf{H} \subset \mathbf{B}$, and a Gaussian measure μ above.

The triple $(i, \mathbf{H}, \mathbf{B})$ is called abstract Wiener space.

R e m a r k. If $\mathbf{B} = \mathbf{C}[0,1]$, then $\mathbf{H} = C'$ - the space of absolutely continuous functions in the unit interval (see Kuo (1975)).

3. Main results

For our further reasoning we need a suitable convergence type theorems and we shall refer to such results from the book by Shiryaev (1984):

Theorem. *Let $(\Omega, \mathbf{F}, P)$ be a probability space, ξ and $\{\xi_n\}$ be random variables of the space $L_2(P)$ and*

$$\sum_{n=1}^{\infty} E\{|\xi - \xi_n|^2\} < \infty.$$

Then $\xi_n \to \xi$ almost surely as $n \to \infty$.

Let $\mathbf{B}$ be an abstract Wiener space with invariant subspace $\mathbf{H}$ and let L be a linear functional on $\mathbf{B}$; $L : \mathbf{B} \to R$, and $\{L_n\}$ be a sequence of approximators for L. Then $D_n = L - L_n$ is the error and let δ_n^2=Var(D_n).

We use the following assumption:

(A) There exist constants $C > 0$ and $k \geq 1$ such that $\delta_n^2 \leq C/n^k$ for any natural n.

Further it will become clear that this assumption is reasonable. In particular, if $\mathbf{B} = \mathbf{C}[0,1]$ is endowed with the Wiener measure and we use the trapezoidal method, then $\delta_n^2 = 1/12n^2$, so in this case $C = 1/12$ and $k = 2$. For details see Kopanov (1994).

Let us formulate one of the results of the present paper.

Theorem 1. *There exists a linear subspace $Z \subset \mathbf{B}$ such that:*
1)Z is dense in $\mathbf{B}$ and has probability one;
2) $\lim_{n\to\infty} \rho^n(L(x) - L_{2^n}(x)) = \lim_{n\to\infty} \rho^n.D_{2^n}(x) = 0$ for all $x \in Z$ and $\rho \in [1, 2^{k/2})$.
3)$\mathbf{H} \subset Z$.

Proof. Since $D_n \sim N(0, \delta_n^2)$ we have $\rho^n.D_{2^n} \sim N(0, \rho^{2n}.\delta_{2^n}^2)$. Therefore

$$\sum_{n=1}^{\infty} Var(\rho^n.D_{2^n}(x)) = \sum_{n=1}^{\infty} \rho^{2n}.\delta_{2^n}^2 \leq \sum_{n=1}^{\infty} \rho^{2n}.C/2^{kn} = C.\sum_{n=1}^{\infty}(\rho^2/2^k)^n =$$

$$= C/(1 - (\rho^2/2^k)) < \infty.$$

By the convergence theorem given at the begining of this section we conclude that $\lim_{n\to\infty} \rho^n.D_{2^n}(x) = 0$ P-a.s. for each $\rho \in [1, 2^{k/2})$.

Now let $Z_\rho = \{x \in \mathbf{B} : \lim_{n\to\infty} \rho^n.D_{2^n}(x) = 0\}, \rho \in [1, 2^{k/2})$. It is easy to see that Z_ρ is a linear subspace of $\mathbf{B}$(directly from the definition) and it is a Borel subset, too. Furthermore, $P(Z_\rho) = 1$ for $\rho \in [1, 2^{k/2})$. It is obvious also by the definition that $1 \leq \rho_1 < \rho_2 < 2^{k/2}$ implies the inclusion $Z_{\rho_2} \subseteq Z_{\rho_1}$. Let us choose an increasing sequence $\{\rho_n\}$ such that $\rho_1 = 1$ and $\lim_{n\to\infty} \rho_n = 2^{k/2}$. Denote

$$Z = \bigcap_{n=1}^{\infty} Z_{\rho_n}.$$

Clearly Z is a linear subspace of $\mathbf{B}$ and $P(Z) = \lim_{n\to\infty} P(Z_{\rho_n}) = 1$. Moreover, Z is dense because $P(Z) = 1$ and the abstract Wiener measure is positive on all nonempty open sets. Thus we have proven statements 1) and 2).

It remains to show that $\mathbf{H} \subset$ Z. If $x \in \mathbf{B}\backslash Z$, we have to show that $x \in \mathbf{B}\backslash\mathbf{H}$. Consider the translated measure $P_x : B \to [0,1]$ defined by $P_x(B) = P(x+B)$ for any $B \in B$. It is well-known (see e.g. Kuo(1975)) that P_x and P are either equivalent or orthogonal. They are equivalent if and only if $x \in \mathbf{H}$. We need to show that P_x and P are orthogonal.

Since $x \in \mathbf{B}\backslash Z$, then there exists $\rho \in [1, 2^{k/2})$ such that $x \notin Z_\rho$. We know already that $P(Z_\rho) = 1$ and if we show that $P_x(Z_\rho) = 0$, this would imply the orthogonality of the measures P_x and P and the fact that $x \notin \mathbf{H}$.

Indeed, take $y \in Z_\rho$ and consider x+y. Assume $x + y \in$ Z. Then

$$\lim_{n\to\infty} \rho^n . D_{2^n}(x+y) = 0.$$

But

$$\lim_{n\to\infty} \rho^n . D_{2^n}(x) = \lim_{n\to\infty} \rho^n . D_{2^n}(x+y) - \lim_{n\to\infty} \rho^n . D_{2^n}(y) = 0 - 0 = 0.$$

Thus, if y and $x + y \in Z_\rho$, then $x \in Z_\rho$, which is a contradiction. Hence $x + Z_\rho \subseteq \mathbf{B}\backslash Z_\rho$. However $P(Z_\rho) = 1$ implies that $P(\mathbf{B}\backslash Z_\rho) = 0$ and we have

$$P_x(Z_\rho) = P(x + Z_\rho) \le P(\mathbf{B}\backslash Z_\rho) = 0.$$

4. Corollaries

Let **B** be the space $\mathbf{C}[0,1]$ of the continuous functions and $\mathbf{H} = \mathbf{C}'$ - the space of the absolutely continuous functions (the standard case of abstract Wiener space).

It is easy to see that Theorem 1 means in fact that $L(x) - L_{2^n}(x) = o(1/\rho^n)$ for any $\rho \in [1, 2^{k/2})$ and for almost all $x \in \mathbf{C}[0,1]$ and also for all $x \in \mathbf{C}'$(k is the constant in Assumption (A)). If now $L(x) = \int_0^1 x(t)dt$ and $\{L_n\}$ is the sequence of approximators constructed by trapezoidal rule, then $k = 2$(for details see e.g. Kopanov (1994)). This case is perhaps one of the most important since here the variance δ_n^2 of the error is the smallest possible among all the methods.

5. Counterexamples

The following question arises from Theorem 1: Is it possible to find $\rho \in [1, 2^{k/2})$ such that $\lim_{n\to\infty} \rho^n.D_{2^n}(x) = 0$ for all $x \in \mathbf{B}$. The answer to this question is negative as can be seen by the following counterexample given first in a general and then in a specific form.

General counterexample. Let $u(t) = |t|, t \in [-1,1]$, and let us extend the definition of u to the real line by requiring $u(t+2) = u(t)$ for all t. Let $\{a_n\}_{n=0}^{\infty}$ be a summable sequence of real numbers and define

$$u(t) = \sum_{n=0}^{\infty} a_n.u(4^n t), t \in [0,1].$$

Note that u is a generalization of the well-known counterexample of a continuous but nowhere differentiable function constructed by Van der Warden. Let now $L(x) = \int_0^1 x(t)dt$ and let $\{L_n\}$ be the sequence of approximators constructed by the trapezoidal rule. We now compute

$$L(u) = \sum_{n=0}^{\infty} a_n. \int_0^1 u(4^n t)dt = \sum_{n=0}^{\infty} a_n/2.$$

For $m \geq 1$,

$$L_{2^{2m}}(u) = \sum_{n=0}^{\infty} a_n.L_{2^{2m}}(u(4^n t)) = \sum_{n=0}^{m} a_n/2,$$

$$L_{2^{2m+1}}(u) = \sum_{n=0}^{\infty} a_n.L_{2^{2m+1}}(u(4^n t)) = \sum_{n=0}^{m} a_n/2.$$

Therefore

$$\rho^n.D_{2^n}(u) = \rho^n. \sum_{i=[n/2]+1}^{\infty} a_i/2.$$

Obviously, we can choose the sequence $\{a_n\}$, and then the function u such that the trapezoidal estimates converge as fast as desired. These estimates can even be divergent.

Specific couterexample.

Let $a_n = 1/n(n+1), n \geq 1$, and $a_0 = 0$. Thus we have

$$u(t) = \sum_{n=0}^{\infty} u(4^n t)/n.(n+1), \quad t \in [0,1],$$

and

$$\rho^n.D_{2^n}(u) = \rho^n. \sum_{i=[n/2]+1}^{\infty} a_i/2 = \rho^n/([n/2]+1).$$

Obviously,

$$\lim_{n\to\infty} \rho^n.D_{2^n} = 0 \text{ for } \rho \in [0,1], \text{ and}$$

$$\lim_{n\to\infty} \rho^n.D_{2^n} = +\infty \text{ for } \rho > 1.$$

This shows that $u \in C[0,1] \setminus \cup Z_\rho, \rho \in (1,2)$.

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