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Oscillations in Linear Differential-Difference Equations

D. P. Tsvetkov

Presented by P. Kenderov

In this paper it is shown that a differential-difference equation has a nonoscillatory solution if and only if its characteristic function has a real zero. In this way it is proposed an oscillation criterion for the equations discussed in [**1**].

Introduction

Consider a differential-difference equation of the following form

$$\sum_{i\in I}\left(\sum_{j\in J} c_{ij}x(t-\beta_{ij})\right)^{(i)}=0 \tag{1}$$

where $c_{ij},\beta_{ij}\in\mathbf{R}$ for $i\in I$, $j\in J$; I and J are finite sets of nonnegative whole numbers. Our purpose is to find conditions under which every real solution of such an equation, defined on certain interval $(\cdot,\infty)$, has oscillatory behavior.

In the "regular" cases, i.e. when Eq.(1) is neutral /or becomes neutral after a time-translation eventually that is essentially the same/ or Eq.(1) presents an entirely difference equation, O. A r i n o and I. G y ö r i [**2**] proved the following result.

(*) *Eq.*(1) *has a nonoscillatory solution if and only if its characteristic function*

$$\chi(z)\overset{\text{def}}{=}\sum_{i\in I}\sum_{j\in J}c_{ij}z^{i}exp(-z\beta_{ij})$$

has real zeros.

More precisely, Arino and Györy proved (*) for neutral equations but it is not difficult to see that the method, used in [**2**], can be applied in the same way to the difference equations. It is enough for a reader, who is not acquainted with the classification above, to think that Eq.(1) is "regular" when the zeros of the characteristic function χ belongs to certain half-plane $Re\, z \leq C$. When Eq.(1) is "regular", we have a well-posed initial-value problem; also solutions are not growing faster than exponentially.

Nonoscillatory solution /NS/ is said to be a solution which does not change its sign on certain interval $(\cdot, \infty)$. Hereafter we will precise this definition. The first part of (*) is evident. If the characteristic function has a real zero μ then $\exp(\mu t)$ is a NS of Eq.(1). The difficulties come in the another part.

In the "regular" cases, the solutions of Eq.(1) are assumed in the conventional sense /see, for instance, R. B e l l m a n and K. C o o k e [**3**:Ch.6] or A. D. M i s h k i s [**4**:Ch.1] or J. K. H a l e [**5**:Ch.7,Ch.12]/. Furthermore, we have a well-posed initial value problem. In particular, the solutions are continuable up to ∞ and continuous at least and, as we have already pointed out, the solutions *are not growing faster than exponentially.* The latter helps Arino and Györi to find a brief and clear proof by using the Laplace Transform /LT/.

Y. C h e n g [**1**], who refers his investigations to I. G y ö r y and G. L a d a s [**6**], studies the oscillation problem for Eq.(1) without a type restriction and states that he has found particular necessary and sufficient oscillation criteria. In this case, as it is announced by R. B e l l m a n and K. C o o k e in [**3**:Ch.5-6], LT is not á priori applicable. Therefore the approach of [**2**] is unusable without improving. We remove this trouble proving that an adaptation of a NS of Eq.(1) admits LT in the conventional mode /Theorem 1/. Combining this fact with the main idea of [**2**] we prove that *proposition* (*) *is valid without a type restriction about* Eq.(1) /Theorem 2/.

Thus we give a natural oscillation criterion for the first and the second kind of equations discussed in [**1**]. Unfortunately, in this abstract, Cheng does not formulate explicitly his results but it is clear that he does not hold ours.

Definitions

First of all we have to say what we call a solution of Eq.(1). It is known [**4,5**] that the forward initial value problem of Eq.(1) is ill-posed in the general case. Nevertheless, we may choose a convenient definition in terms of the distributions theory since Eq.(1) generates an operator in $C^\infty(\mathbf{R})$ which can be associated with a distribution of a compact support. Let

$$\mathcal{P}\psi \stackrel{\text{def}}{=} \sum_{i\in I}(-1)^i\Big(\sum_{j\in J} c_{ij}\psi(\cdot+\beta_{ij})\Big)^{(i)}$$

be the adjoint operator assigned to Eq.(1). We are convinced that there are no useful definitions outside the following one, since it replaces the classical solutions with weak ones; moreover it even does not include initial or any other conditions.

Definition 1. *We call a solution of Eq.*(1) *a real distribution* $x \in \mathcal{D}'(A_x, \infty)$ *such that*

$$\langle x, \mathcal{P}\psi\rangle = 0, \qquad \psi \in C_0^\infty(B_x,\infty), \tag{2}$$

where B_x *is chosen with the property*

$$\mathcal{P}\psi \in C_0^\infty(A_x,\infty) \qquad \text{when} \qquad \psi\in C_0^\infty(B_x,\infty)$$

and $\langle\cdot,\cdot\rangle$ *is the product of* $\mathcal{D}'\times C_0^\infty$. *Also we call a solution* x *of Eq.*(1) *a* C^∞*-solution when* $x \in C^\infty(E,\infty)$ *and we call a solution* x *of Eq.*(1) *locally integrable when* $x\in L^1_{loc}(E,\infty)$ *for some* $E\in\mathbf{R}$.

This definition has a translation-invariance property, i.e. $x(\cdot)$ is a solution if and only if $x(\cdot + Const)$ is a solution that is rather useful for us. An illustrative example of Definition 1 is given in the last section.

When we require x to be locally integrable, (2) must be replaced by

$$\int_R x(s)\mathcal{P}\psi(s)ds = 0, \qquad \psi\in C_0^\infty(B_x,\infty). \tag{2'}$$

It is not difficult to see the validity of

Proposition 1. *A solution* x *of Eq.*(1) *is a* C^∞*-solution if and only if there are constants* $\bar{E}\geq E$ *such that* $x\in C^\infty(E,\infty)$ *and*

$$\sum_{i\in I}\sum_{j\in J} c_{ij}x^{(i)}(t-\beta_{ij}) = 0$$

for $t\geq \bar{E}$.

Here we do not discuss the existence and the uniqueness of solutions with respect to the above definition. Also we do not assign solutions with any initial conditions. The fact of major importance for us is that we require a solution of Eq.(1) to be defined up to ∞. This corresponds to the nature of the problem considered.

When we say that a continuous function $g(t)$ does not change its sign for $t \geq E$, we mean that one of the sets $\{t \geq E : g(t) > 0\}$, $\{t \geq E : g(t) < 0\}$ is empty. Of course, a distribution $x \in \mathcal{D}'(A_x, \infty)$ has no sign but, for a given $\phi \in C_0^\infty(\mathbf{R})$ and large values of t, the convolution

$$\langle x, \phi(t - \cdot)\rangle \stackrel{\text{def}}{=} x_\phi(t)$$

is a C^∞-function and we can consider its sign.

In the sequel we assume expressions like x_ϕ as infinitely differentable functions defined on intervals $(\cdot, \infty)$.

The distribution calculus implies

Proposition 2. *Let x be a solution of Eq.(1). Then x_ϕ is a C^∞-solution.*

Proposition 3. *Let x be a* NS *of Eq.(1) and ϕ is as in Definition 2. Then x_ϕ is a C^∞-*NS.

We are prepared to propose

Definition 2. *A solution x of Eq.(1) we call* NS *if there exist constant E and $\phi \in C_0^\infty(\mathbf{R})$ such that the C^∞-solution $x_\phi(t)$ does not change its sign for $t \geq E$ and $x_\phi \not\equiv 0$ on every interval $(\bar{E}, \infty)$ with $\bar{E} \geq E$.*

R e m a r k. Our definitions are given for the sake of convenience with respect to the oscillatory problem. Perhaps they are not properly applicable for another problem. These definitions expand the conventional ones [**3,4,5**]. For instance, when Eq.(1) is retarded, a solution is assumed to be N-times continuously differentable where N is the maximal order of differentiation in Eq.(1). When Eq.(1) is neutral, a solution is assumed to be $(N-1)$-times continuously differentable with /at least/ piece-wise differentable $(N-1)$-th derivative. In both cases simple calculation shows that a solution of Eq.(1) satisfies (2′). Such a solution can be considered as a distribution, defined on certain interval $(\cdot, \infty)$, with locally integrable derivatives up to N-th order. Also Definition 1 comprises the case when one requires a solution of Eq.(1) to be a function x such that

$$\sum_{j\in J} x(\cdot - \beta_{ij}) \in C^i(E_{i,x}, \infty), \qquad i \in I,$$

and (1) holds for $t \in (E_x, \infty)$.

Using a translation with

$$\beta = \max_{i\in I,\, j\in J} -\beta_{ij}$$

we rewrite Eq.(1) in the form

$$\sum_{i\in I}\left(\sum_{j\in J} c_{ij}x(t-\beta_{ij}-\beta)\right)^{(i)}=0. \tag{3}$$

Obviously x is a /nonoscillatory/ solution of Eq.(1) if and only if x is a /nonoscillatory/ solution of Eq.(3) in which some of the numbers $\beta_{ij}+\beta$ equal zero. Thus, for sufficiently differentable x, the left-hand side of Eq.(3) becomes

$$\sum_{i=0}^{n} a_i x^{(i)}(t)+\sum_{i\in I^\circ}\sum_{j\in J^\circ} b_{ij}x^{(i)}(t-\alpha_{ij})$$

with $\alpha_{ij}=\beta_{ij}+\beta>0$ for $i\in I^\circ$, $j\in J^\circ$ and $a_n\neq 0$. Below we use only the latter form. We assume $I^\circ\neq\vee$ and $J^\circ\neq\vee$. Otherwise Eq.(1) reduces to an ordinary differential equation for which the results, we are going to prove, are trivial. Further, without loss of generality, we assume $a_n>0$. Eq.(3) has a characteristic function

$$\chi^\circ(z)\stackrel{\text{def}}{=}\sum_{i=0}^{n} a_i z^i+\sum_{i\in I^\circ}\sum_{j\in J^\circ} b_{ij}z^i exp(-z\alpha_{ij}).$$

Actually

$$\chi(z)=exp(\beta z)\chi^\circ(z).$$

Therefore χ has a real zero if and only if χ° has the same property.

Lemma 1. *Let x be a* NS *of Eq.*(1) *and ϕ is as in Definition* 2. *Then there is a function $\Phi(\cdot;x)\in C^\infty(\mathbf{R})$ with*

i. *$\Phi(t;x)=0$ for $t\le 0$ and $\Phi(\cdot;x)$ is nonnegative on the real axis. Also $\Phi(\cdot;x)\not\equiv 0$ on every interval $(\cdot,\infty)$.*

ii. *$\Phi(\cdot;x)$ satisfies the equation*

$$\sum_{i=0}^{n} a_i\Phi^{(i)}(t;x)+\sum_{i\in I^\circ}\sum_{j\in J^\circ} b_{ij}\Phi^{(i)}(t-\alpha_{ij};x)=h(t),\qquad t\in\mathbf{R}, \tag{4}$$

where $h\in C_0^\infty(\mathbf{R}_+)$.

Such a function $\Phi(\cdot;x)$ we call an adaptation of x.

Proof. According to Proposition 3, x_ϕ is a C^∞-NS. Therefore x_ϕ is defined as a C^∞-function for large values of the argument and satisfies Eq.(3)

on certain interval $(E_x, \infty) \subset \mathbf{R}_+$. In view of the fact that $\pm x_\phi$ is also a C^∞-NS, without loss of generality, we assume x_ϕ to be nonnegative at (E_x, ∞). We set

$$y(t) \stackrel{\text{def}}{=} \begin{cases} x_\phi(t) & \text{for} \quad t \geq E_x \\ 0 & \text{for} \quad t < E_x \end{cases}$$

and

$$\bar{y}(t) \stackrel{\text{def}}{=} \int_R y(s)\kappa(t-s)ds$$

where $\kappa \in C_0^\infty(\mathbf{R})$ with supp$\kappa \subset [-1,1]$ and $\kappa(s) \geq 0$ for $s \in \mathbf{R}$. The function, we are looking for, can be determined as

$$\Phi(\cdot; x) \stackrel{\text{def}}{=} \bar{y}(\cdot - C)$$

with large fixed $C \in \mathbf{R}_+$. It is sufficient to choose $C > E_x + 2$. One can complete the proof immediately. ■

The Main Result

Denote by $\mathcal{L}_a$ the set of the functions $g \in L_{1,loc}(a, \infty)$ for which there is $q \in \mathbf{R}$ such that

$$\int_a^\infty |g(t)|exp(-qt)dt < \infty.$$

Let

$$\mathcal{L} \stackrel{\text{def}}{=} \bigcup_{a \in \mathbf{R}} \mathcal{L}_a.$$

Our basic result is

Theorem 1. *Let x be a* NS *of Eq.*(1) *and let $\Phi(\cdot; x)$ be its adaptation. Then $\Phi(\cdot; x)$ belongs to $\mathcal{L}$. Moreover, for LT of $\Phi(\cdot; x)$, we have*

$$\chi^\circ(p) \int_0^\infty exp(-pt)\Phi(t;x)dt = \int_0^\infty exp(-pt)h(t)dt \tag{5}$$

for $p > p_0$.

The proof of Theorem 1 is in the next section. The integral on the left-hand side of (5) presents an analytic function in the half-plane $Re\, p > p_0$. Thus the validity of (5) for real $p > p_0$ implies its validity likewise for complex p with $Re\, p > p_0$.

R e m a r k. Further, when we use LT, we restrict ourselves to considering only real values of p since this is sufficient to perform our approach. Theorem

1 shows that the adaptation $\Phi(\cdot, x)$ admits LT as a solution of Eq.(4) *in the conventional mode.*

Corollary 1. *Let x be a* NS *of Eq.*(1) *and ϕ is as in Definition* 2. *Then x_ϕ belongs to $\mathcal{L}$.*

Proof. Let $\Phi(\cdot; x)$ be the adaptation of x and $p > p_0$. Remember that κ is chosen to be nonnegative. Then using the Fubini's theorem we get

$$\int_0^\infty \Phi(t; x)exp(-pt)dt = \int_{E_x}^\infty x_\phi(t)exp(-pt)dt \int_R \kappa(t)exp(-pt)dt$$

/see the proof of Lemma 1/ which completes the proof. ∎

Corollary 2. *Let x be a locally integrable solution of Eq.*(1) *which does not change its sign almost everywhere on certain interval* $(\cdot, \infty)$. *Then $x \in \mathcal{L}$.*

For a proof it is enough to retrace the proof of Corollary 1.

We have mentioned above that the absence of real zeros of the characteristic function is a necessary condition for the absence of a NS of Eq.(1). We will show that this condition is also sufficient. Our main result is

Theorem 2. *Eq.*(1) *has a* NS *if and only if its characteristic function has a real zero.*

Proof. The proof is based essentially on the proof of Theorem 2.1[**1**]. It remains to show that, in the case when the characteristic function has no real zero, Eq.(1) has no NS.

Assume that χ° has no real zero and, nevertheless, Eq.(1) has a NS x. Then the adaptation $\Phi(\cdot; x)$ is a nonnegative C^∞-function which, according to Theorem 1, satisfies (5). Also we have $\chi^\circ(p) \neq 0$ with large real values of p. These facts, in accordance with the the proof of Theorem 2.1[**1**], lead to the existence of a constant E_0 such that $\Phi(t; x) = 0$ for $t \geq E_0$. The latter contradicts point **i** of Lemma 1. ∎

As we have pointed out, Theorem 2 has already been proved when the expression of Eq.(1) corresponds to a neutral or difference equation. It shows that we may exclude out of consideration the equations of the mentioned types but the proofs we propose do not need this assumption.

Let us close this section with

Corollary 3. *Eq.*(1) *has a locally integrable solution, which does not change its sign and does not equal zero almost-everywhere on certain interval* $(\cdot, \infty)$, *if and only if its characteristic function has a real zero.*

For a proof it is enough to see that the adaptation is a C^∞-NS and then to repeat the proof of Theorem 2.

Proof of Theorem 1

Introduce the function

$$G_m(t\lambda) \stackrel{\text{def}}{=} \begin{cases} exp(t\lambda) - \sum_{i=0}^{m} \frac{(t\lambda)^i}{i!} & \text{for} \quad t \geq 0 \\ 0 & \text{for} \quad t < 0 \end{cases}$$

where $\lambda > 0$ is a real parameter.

It is easy to check the validity of

Lemma 2. *The function $G_m(\cdot\lambda)$ has the following properties.*

i. *$G_m(\cdot\lambda)$ is nonnegative on the real axis and*

$$\Big(G_m(t\lambda)\Big)^{(i)} = \lambda^i G_{m-i}(t\lambda), \qquad i = 0, \ldots, m, \quad t \in \mathbf{R}.$$

ii. *The following equalities hold*

$$\int_0^t g^{(i)}(t-\gamma-u)G_m(u\lambda)du = \int_0^t g(t-u)G_m^{(i)}((u-\gamma)\lambda)du, \quad i = 0, \ldots, m$$

for $t, \gamma \geq 0$ where $g \in C^m(\mathbf{R})$ is a function with $g(s) = 0$ for $s \leq 0$.

iii. *For LT of $G_m(\cdot\lambda)$ we have*

$$\int_0^\infty exp(-pt)G_m(t\lambda)dt = \frac{\lambda^{m+1}}{p^{m+1}}\frac{1}{p-\lambda}, \qquad p > \lambda. \bullet$$

Our preparation finishes with

Proposition 4. *Let $f, g : \mathbf{R} \to \mathbf{R}_+$ are locally integrable functions which equal to zero on $\mathbf{R}_-$ and $p \in \mathbf{R}$. Let also $f, g \not\equiv 0$ and*

$$\mathcal{J} \stackrel{\text{def}}{=} \int_0^\infty exp(-pt)\left(\int_0^t f(t-s)g(s)ds\right)dt < \infty.$$

Then f and g belongs to $\mathcal{L}$. Moreover

$$\mathcal{J} = \int_0^\infty exp(-pt)f(t)dt \int_0^\infty exp(-pt)g(t)dt. \bullet$$

The proof is a simple application of the Fubini's Theorem.

P r o o f of Theorem 1. Let x be a NS of Eq.(1) with an adaptation $\Phi(\cdot; x)$ and

$$N = \max_{i \in I} i \quad .$$

Then according to (4) we have

$$\sum_{i=0}^{n} a_i \Phi^{(i)}(t-u;x) + \sum_{i\in I^\circ}\sum_{j\in J^\circ} b_{ij}\Phi^{(i)}(t-u-\alpha_{ij};x) = h(t-u)$$

for $t,u \in \mathbf{R}$. Multiplying the latter with $G_N(\cdot\lambda)$, integrating at $[0,t]$ and using points **i-ii** of Lemma 2 we get

$$\int_0^t \Phi(t-u;x)G(u;\lambda)du = \int_0^t h(t-u)G_N(u\lambda)du, \quad t \geq 0, \tag{6}$$

where

$$G(\cdot;\lambda) = \sum_{i=0}^{n} a_i\lambda^i G_{N-i}(\cdot\lambda) + \sum_{i\in I^\circ}\sum_{j\in J^\circ} b_{ij}\lambda^i G_{N-i}((\cdot-\alpha_{ij})\lambda).$$

Now we are going to show that $G(\cdot;\lambda)$ has a constant positive sign at $(0,\infty)$ for every sufficiently large λ. We have

$$G(\cdot;\lambda) = \lambda^n G_{N-n}(\cdot\lambda)\Big(a_n + S_1(\cdot;\lambda) + S_2(\cdot;\lambda)\Big)$$

where

$$S_1(\cdot;\lambda) = \sum_{i=0}^{n-1} a_i\lambda^{i-n}\frac{G_{N-i}(\cdot\lambda)}{G_{N-n}(\cdot\lambda)}$$

and

$$S_2(\cdot;\lambda) = \sum_{i\in I^\circ}\sum_{j\in J^\circ} b_{ij}\frac{G_{N-i}((\cdot-\alpha_{ij})\lambda)}{exp((\cdot-\alpha_{ij})\lambda)}\,\frac{exp(\cdot\lambda)}{G_{N-n}(\cdot\lambda)}\,\frac{\lambda^{i-n}}{exp(\alpha_{ij}\lambda)}.$$

It is clear that the inequalities

$$0 \leq \frac{G_{N-i}(u\lambda)}{G_{N-n}(u\lambda)} \leq 1, \qquad 0 \leq i \leq n,$$

hold for $u \geq 0$ which implies that $S_1(u;\lambda)$ tends uniformly to zero with respect to $u \geq 0$ when $\lambda \to \infty$. Also

$$0 \leq \frac{G_{N-i}((u-\alpha_{ij})\lambda)}{exp((u-\alpha_{ij})\lambda)} \leq 1, \qquad u \geq 0, \quad i \in I^\circ, j \in J^\circ.$$

The function

$$\tau(u) = \frac{exp(u)}{G_{N-n}(u)}$$

is decreasing for $u \in (0, \infty)$. Then

$$|S_2(u;\lambda)| \leq \sum_{i\in I^\circ} \sum_{j\in J^\circ} |b_{ij}| \tau(\lambda\alpha_{ij}) \frac{\lambda^{i-n}}{exp(\alpha_{ij}\lambda)}, \qquad u \geq 0.$$

Here we use essentially the fact that every α_{ij} is a positive number. In this way

$$\lim_{\lambda\to\infty} \sup_{u\geq 0} \Big(|S_1(u;\lambda)| + |S_2(u;\lambda)| \Big) = 0$$

and this allows us to choose $\mu > 0$ such that

$$sign\Big(G(u;\lambda) \Big) = sign(a_n) = 1, \qquad \lambda \geq \mu, \quad u > 0.$$

We must keep in mind that $G(u;\lambda) = 0$ for $u \leq 0$. Further we assume $\lambda = \mu$.

Now using (6) we get

$$\int_0^\infty exp(-pt) \left(\int_0^t \Phi(t-u;x) G(u;\mu) du \right) dt < \infty$$

for real $p > p_0$ since h and $G_N(\cdot\lambda)$ are exponentially bounded. Then we are able to apply Proposition 4 which gives $\Phi(\cdot;x) \in \mathcal{L}$ and

$$\hat{G}(p;\mu)\hat{\Phi}(p;x) = \hat{G}_N(p\mu)\hat{h}(p) \tag{7}$$

where the hat denotes LT. We obtain (7) for real $p > p_0$ which provides its validity also for complex p with $Re\, p > p_0$.

Point iii of Lemma 2 yields

$$\hat{G}(p;\mu) = \frac{\mu^{N+1}}{p^{N+1}} \frac{1}{p-\mu} \chi^\circ(p)$$

as well

$$\hat{G}_N(p\mu) = \frac{\mu^{N+1}}{p^{N+1}} \frac{1}{p-\mu}.$$

Thus (7) becomes

$$\frac{\mu^{N+1}}{p^{N+1}} \frac{1}{p-\mu} \chi^\circ(p)\hat{\Phi}(p;x) = \frac{\mu^{N+1}}{p^{N+1}} \frac{1}{p-\mu} \hat{h}(p)$$

which implies (5).

Notes

In order to illustrate Definition 1, let us discuss an
E x a m p l e. Consider an equation of advanced type [**4**:Ch.1]

$$x(t)' - x(t+1) = 0 \tag{8}$$

which generates an adjoint operator of the form

$$\mathcal{P}\phi = -\phi'(\cdot) - \phi(\cdot - 1).$$

Let δ be the Dirac function. Then the distribution

$$x_*(t) = -\sum_{k=1}^{\infty} \delta^{(k-1)}(t-k) \ \in \ \mathcal{D}'(0,\infty)$$

is a solution of Eq.(8) in the sense of Definition 1. Actually

$$\langle x_*, \mathcal{P}\phi\rangle = \phi(0) = 0$$

for every $\phi \in C_0^\infty(0,\infty)$. Here we choose $A_{x_*} = 0$ and $B_{x_*} = 0$. In particular, x_* is not a NS since the characteristic function $z - exp(z)$ has no real zero.

Let us give an interpretation of Theorem 1. It is known /see for instance Leont'ev [**7**]/ that a solution x of Eq.(1) can be associated with a Dirichlet series

$$\sum_{\mu\in\Lambda} p_\mu(t)e^{\mu t} \tag{9}$$

where Λ is the set of the zeros of the characteristic function and p_μ are polynomials. It is reasonable to assume that $x \notin \mathcal{L}$ when the sum in (9) contains advanced subseries, i.e. there are $\{\mu_\nu\}$ with

$$\lim_{\nu\to\infty} Re\ \mu_\nu = \infty.$$

On the other hand, in this case it is quite possible that x oscillates.

We can extend the applicability of our approach. Consider an equation of the form

$$\sum_{i=0}^{n}\left(\int_0^{a_i} x(t-\theta)d\alpha_i(\theta)\right)^{(i)} + \sum_{i\in K}\left(\int_{b_i}^{c_i} x(t-\theta)d\beta_i(\theta)\right)^{(i)} = 0 \tag{10}$$

where α_i, $0 \le i \le n$, and β_i, $i \in K$, are real functions of bounded variation; K is a finite set of integers which is possible to be empty. We assume that every

α_i, $0 \le i \le n$, is not a constant in a neighborhood of zero as well $a_i > 0$ for $0 \le i \le n$ and $c_i > b_i > 0$ for $i \in K$. The definitions of the solution and NS of Eq.(10) remain essentially the same. We have to update only the definition of the adjoint operator. Unfortunately, here we need a regularity condition to use the idea of the proof of Theorem 1 without essential changes. Eq.(10) we call regular if α_n has a jump at zero, i.e. $\alpha_n(0+) - \alpha_n(0) \neq 0$. For such an equation one can receive complete analogs of Theorems 1 and 2. When K is empty, this result gives nothing new since our regularity condition makes Eq.(10) to be neutral or functional-difference one with a well-posed IVP.

Thus we describe an oscillation criterion for the last, third kind of equations discussed by Y. C h e n g in [**2**].

Note that an equation of a type (1) is regular in the sense aforementioned.

It is quite possible that the following improvement of Theorem 1 holds.

Conjecture. *Let x be a* NS *of Eq.*(1) *and ϕ is as in the Definition* 2. *Then x_ϕ is exponentially bounded, i.e. there are real constants C_x and σ_x such that*

$$|x_\phi(t)| \le C_x \, exp(\sigma_x t)$$

for t in the domain of x_ϕ.

Let us finish with the following remark.

Assume that Eq.(1) is of advanced type. For such un equation the characteristic function has an advanced subseries of zeros $\{z_i\}_{i=1}^{\infty}$. Let also Eq.(1) has a NS. Then (5) implies

$$Res\Big|_{z=z_i} \left(\frac{1}{\chi^\circ(z)} \int_0^\infty exp(-zt)h(t)dt \right) = 0$$

for $Re\, z_i > p_0$. The latter shows that, in general, the existence of a NS is an essential restrictive condition. This fact gives certain explanation of the exponential regularity of NS.

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Department of Mathematics and Informatics
University of Veliko Tarnovo
Veliko Tarnovo 5000
BULGARIA

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