Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol.10, 1996, Fasc.1

On Large Additive Functions over Primes of Positive Density

Aleksandar Ivić

Presented by P. Kenderov

Several asymptotic formulas are obtained which involve certain large additive functions, defined on a set of primes having density δ $(0 < \delta < 1)$ in the set of all primes.

1. Introduction

Let Q be a set of primes such that there exists some constant δ satisfying $(0 < \delta < 1)$ and

$$\pi(x, Q) := \sum_{p \le x, p \in Q} 1 = \delta \mathrm{Li}\, x + O(x \log^{-B} x), \tag{1.1}$$

where B is a constant satisfying $B > 2$. Here and later p denotes primes, $\mathrm{Li}\, x = \int_2^x dt/\log t$, $f(x) = O(g(x))$ (same as $f(x) \ll g(x)$) means that $|f(x)| \le Cg(x)$ for some constant $C > 0$, $g(x) > 0$ and $x \ge x_0$. Such a situation naturally arises if Q consists of primes belonging to a finite union of arithmetic progressions, or if Q consists of the set of primes that divide some value of a fixed polynomial with integer coefficients. A stronger error term in (1.1) naturally leads to sharper final results in many instances, but for most applications (1.1) is a very reasonable assumption. Since $\mathrm{Li}\, x \sim x/\log x$ as $x \to \infty$, it makes sense also to assume that $B > 1$ in (1.1). The condition $B > 2$ was imposed by D. A. Goldston and K. S. McCurley [2], who obtained an asymptotic formula for the function

$$\Psi(x, y, Q) := \sum_{n \le x, P(n,Q) \le y} 1 \qquad (1 < y \le x), \tag{1.2}$$

where $P(n,Q)$ denotes the largest prime factor of n which belongs to Q. If n has no prime factors from Q, then it seems appropriate to set $P(n,Q)=0$, and such n are then not counted by the right–hand side of (1.2). The condition $B>2$ in (1.1) is not essential, as was shown by R. R. Warlimont [15], who evaluated (1.2) in the case when only $B>1$ is assumed in (1.1). Nevertheless, to avoid somewhat more complicated error term in the formula for $\Psi(x,y,Q)$ that Warlimont obtained, and which affects e.g. the right–hand side of (1.4), henceforth we shall suppose that $B>2$ holds in (1.1). As companion functions to $P(n,Q)$ we define large additive functions

$$\beta(n,Q)=\sum_{p|n,p\in Q} p,\qquad B(n,Q)=\sum_{p^\alpha||n,p\in Q}\alpha p, \tag{1.3}$$

where as usual $p^\alpha||n$ means that p^α divides n, but $p^{\alpha+1}$ does not. It also seems appropriate to define $\beta(n,Q)=B(n,Q)=0$ if n has no prime factors from Q. The function $P(n,Q)$, $\beta(n,Q)$ and $B(n,Q)$ are then analogues of the functions

$$P(n)=\max\{p:p|n\},\qquad \beta(n)=\sum_{p|n}p,\qquad B(n)=\sum_{p^\alpha||n}\alpha p.$$

The function $\beta(n)$ is additive (meaning $\beta(mn)=\beta(m)+\beta(n)$ for all m, n such that $(m,n)=1$), while $B(n)$ is totally additive (meaning $B(mn)=B(m)+B(n)$ for all m, n). From the above definitions it easily follows that $\beta(n,Q)$ is additive and $B(n,Q)$ is totally additive. Thus these functions may be thought of as large additive functions ("large", since e.g. $\beta(n,Q)=B(n,Q)=n$ if $n=p\in Q$) over a set of primes Q having positive density δ in the set of all primes, as indicated by (1.1).

An extensive literature exists on various asymptotic formulas involving the functions $P(n)$, $\beta(n)$, and $B(n)$. Here we shall mention the monograph [4] and the papers [1], [5], [7], [8], [9], [11], [13], [16], where references to other works can be found. No results seem to have appeared so far on the functions $\beta(n,Q)$ and $B(n,Q)$, and only very recently J. -M. De Koninck [3] investigated the sum of reciprocals of $P(n,Q)$. He proved that

$$\sum_{n\le x}{}' \frac{1}{P(n,Q)}=\left(1+O\left(\frac{1}{\log\log x}\right)\right)\eta(Q)\frac{x}{(\log x)^\delta}, \tag{1.4}$$

where $\eta(Q)$ is a positive constant which may be written down in closed form, while in general $\sum_{n\le x}{}' 1/f(n)$ denotes the sum other those n not exceeding x for which $f(n)\ne 0$. (J.- M. De Koninck [3] defines $P(n,Q)=+\infty$ if n has no prime factors from Q, but this is only a technical distinction from the definition made here and bears no consequences on any of the results). It may be noted that (1.4) is in sharp distinction from the asymptotic formula

$$\sum_{2\le n\le x} \frac{1}{P(n)} = x\exp\{-(2\log x\log\log x)^{1/2} + O(\log x\log\log\log x)^{1/2})\}. \tag{1.5}$$

This result was proved in [9], and then sharpened in [13] and [8]. Hence the sums in (1.4) and (1.5) are of a completely different order of magnitude. One expects that this effect will carry over to some problems involving the functions $\beta(n,Q)$ and $B(n,Q)$, namely in some problems these functions will behave differently from $\beta(n)$ and $B(n)$. On the other hand we may expect that in some problems $\beta(n,Q)$ and $B(n,Q)$ will behave similarly as $\beta(n)$ and $B(n)$. The aim of this paper is to investigate this topic by proving some asymptotic formulas involving the functions $P(n,Q)$, $\beta(n,Q)$ and $B(n,Q)$. In particular we shall evaluate the summatory functions of these functions and investigate the local densities of $B(n,Q) - \beta(n,Q)$. We also note that the sum of reciprocals of $\beta(n,Q)$, $B(n,Q)$ and some related problems are treated by J. -M. De Koninck and the author [6].

2. The summatory functions

In this section the asymptotic formula for the summatory function of $P(n,Q)$, $\beta(n,Q)$ and $B(n,Q)$ is given. The result is contained in

Theorem 1. *If*

$$A_j = (-1)^{j-1}\frac{d^{j-1}\{\zeta(s)s^{-1}\}}{ds^{j-1}}\Big|_{s=2}$$

for $j = 1, 2, \ldots$, *then we have*

$$\sum_{n\le x}\beta(n,Q) = \sum_{j\le B}\frac{\delta A_j x^2}{\log^j x} + O\left(\frac{x^2}{\log^B x}\right). \tag{2.1}$$

Moreover (2.1) remains valid if $\beta(n,Q)$ *is replaced by* $P(n,Q)$ *or* $B(n,Q)$.

Proof. The above result is the analogue of the asymptotic formula

$$\sum_{n\le x}\beta(n) = \sum_{j\le M}\frac{\delta A_j x^2}{\log^j x} + O\left(\frac{x^2}{\log^{M+1} x}\right), \tag{2.2}$$

proved by J.-M. De Koninck and A. Ivić [5], where $M \geq 1$ is an arbitrary, but fixed integer (the A_j's were not evaluated explicitly, except for $A_1 = \pi^2/12$). Naturally in (2.1) the sum over J stops at B, since the error term $O(x^2/\log^B x)$ is inherent in the condition (1.1) and cannot be avoided, while such restrictions are absent in (2.2). If p denotes primes and m, n denote natural numbers, then we have first

$$\sum_{n\leq x}\beta(n,Q) = \sum_{n\leq x}\sum_{n=pm,p\in Q} p = \sum_{m\leq x/2}\sum_{p\leq x/m,p\in Q} p.$$

To estimate the last sum over primes write (1.1) as

$$\pi(x,Q) = \delta\int_2^x \frac{dt}{\log t} + R(x,Q), \qquad R(x,Q) = O\left(\frac{x}{\log^B x}\right). \tag{2.3}$$

Writing the sum as a Stieltjes integral we have
(2.4)

$$\begin{aligned}\sum_{n\leq x}\beta(n,Q) &= \sum_{m\leq x/2}\int_{2-0}^{x/m} t\,d\pi(t,Q) = \sum_{m\leq x/2}\delta\int_2^{x/m}\frac{tdt}{\log t} + \sum_{m\leq x/2}\int_{2-0}^{x/m} t\,dR(t,Q)\\ &= \delta\int_2^x\Big(\sum_{m\leq x/t}1\Big)\frac{tdt}{\log t} + O\left(\sum_{m\leq x/2}\frac{x^2}{m^2\log^B(x/m)}\right) + O\left(\sum_{m\leq x/2}\int_2^x\frac{tdt}{\log^B t}\right)\\ &= \delta\int_2^x\Big[\frac{x}{t}\Big]\frac{tdt}{\log t} + O\left(\frac{x^2}{\log^B x}\right),\end{aligned}$$

where $[y]$ denotes the greatest integer not exceeding y. The same argument gives

$$\sum_{n\leq x}\beta(n) = \int_2^x\Big[\frac{x}{t}\Big]\frac{tdt}{\log t} + O\left(\frac{x^2}{\log^{M+1} x}\right) \tag{2.5}$$

for any integer $M \geq 1$, if we use the prime number theorem in the form

$$\pi(x) = \sum_{p\leq x}1 = \int_2^x\frac{dt}{\log t} + O\left(\frac{x}{\log^{M+1} x}\right);$$

sharper forms of the prime number theorem (see Ch. 12 of [5]) would not give any improvements. Hence (2.1) follows on comparing (2.2), (2.4) and (2.5). Of course, it follows directly from (2.4) on noting that by a change of variable

$t = x/u$ we obtain

$$\begin{aligned}\int_2^x [\frac{x}{t}]\frac{tdt}{\log t} &= x^2\int_1^{x/2}\frac{[u]du}{u^3\log(\frac{x}{u})}\\ &= \frac{x^2}{\log x}\int_1^{x/2}\frac{[u]}{u^3}\left\{\sum_{k=0}^{M-1}\left(\frac{\log u}{\log x}\right)^k + O\left(\left(\frac{\log u}{\log x}\right)^M\right)\right\}du\\ &= x^2\left(\sum_{j=1}^{M}\frac{A_j}{\log^j x} + O\left(\frac{1}{\log^{M+1} x}\right)\right)\end{aligned}$$

with

$$A_j = \int_1^\infty \frac{[u]}{u^3}\log^{j-1} udu = (-1)^{j-1}\frac{d^{j-1}\{\zeta(s)s^{-1}\}}{ds^{j-1}}\,|_{s=2}, \tag{2.6}$$

so that in particularar $A_1 = \frac{1}{2}\zeta(2) = \pi^2/12$. To see that the second equality in (2.6) holds, set $K(s) = \int_1^\infty [u]u^{-s-1}du$, which is a regular function of s for Re $s > 1$. Then we have

$$A_j = (-1)^{j-1}\frac{d^{j-1}K(s)}{ds^{j-1}}\,|_{s=2}. \tag{2.7}$$

But integration by parts gives

(2.8+ $$K(s) = \frac{1}{s}\int_{1-0}^\infty \frac{d[u]}{u^s} = \frac{1}{s}\sum_{n=1}^\infty n^{-s} = \frac{\zeta(s)}{s} \qquad (\text{Re } s > 1),$$

so that (2.6) follows from (2.7) and (2.8). Note that by Leibniz's rule one may alternatively write A_j as

$$A_j = \frac{(j-1)!}{2^j}\sum_{k=0}^{j-1}\frac{(-2)^k}{k!}\zeta^{(k)}(2) \qquad (j = 1, 2, \ldots).$$

To see that (2.1) remains valid if $\beta(n,Q)$ is replaced by $P(n,Q)$, let $P_k(n)$ denote the k-th largest prime factor of n if n has at least k prime factors (some of which may be equal), and otherwise let $P_k(n) = 0$. K. Alladi and P. Erdős [1] proved that

$$\sum_{n\le x} P_2(n) = \frac{B_2 x^{3/2}}{\log^2 x} + O\left(\frac{x^{3/2}\log\log x}{\log^3 x}\right)$$

with suitable $B_2 > 0$, and A. Ivić [12] sharpened this formula. Since

$$0 \leq \sum_{n \leq x} (\beta(n,Q) - P(n,Q)) \leq \sum_{n \leq x} (\beta(n) - P(n)) \leq \frac{\log x}{\log 2} \sum_{n \leq x} P_2(n) \ll \frac{x^{3/2}}{\log x},$$

it follows that (2.1) remains valid if $\beta(n,Q)$ is replaced by $P(n,Q)$. By Theorem 2, whose proof is independent of Theorem 1, we have

$$\sum_{n \leq x} B(n,Q) = \sum_{n \leq x} \beta(n,Q) + O(x \log\log x),$$

so that (2.1) also remains valid if $\beta(n,Q)$ is replaced by $B(n,Q)$.

3. The summatory function of $B(n,Q) - \beta(n,Q)$

K. Alladi and P. Erdős [1] proved that

$$\sum_{n \leq x} (B(n) - \beta(n)) = x \log\log x + O(x),$$

and the authors [11] sharpened this formula to

$$\sum_{n \leq x} (B(n) - \beta(n)) = x \log\log x + Cx + O\left(\frac{x}{\log x}\right), \tag{3.1}$$

where C is a suitable constant. The method of proof of (3.1) can be elaborated to yield

$$\sum_{n \leq x} (B(n) - \beta(n)) = x \log\log x + Cx + x \sum_{j=1}^{M} \frac{C_j}{\log^j x} + O\left(\frac{x}{\log^{M+1} x}\right), \tag{3.2}$$

where $M \geq 1$ is an arbitrary, but fixed integer and C_j's are the suitable constants. Moreover the method in question works also for the summatory function of $B(n,Q) - \beta(n,Q)$, in which case it gives

Theorem 2. *There exist effectively computable constants $C(Q)$, $C_1(Q)$, $C_2(Q)$,... such that*

$$\sum_{n \leq x} (B(n,Q) - \beta(n,Q)) = \delta x \log\log x + C(Q)x + x \sum_{j \leq B} \frac{C_j(Q)}{\log^j x} + O\left(\frac{x}{\log^B x}\right). \tag{3.3}$$

Proof. We shall give the proof of (3.3), noting that the case $\delta = 1$, $B = M$ leads essentially to (3.2). We have

$$\sum_{n\le x}(B(n,Q)-\beta(n,Q)) = \sum_{n\le x}\sum_{p^a\|n,p\in Q}(a-1)p = \sum_{p^am\le x,(p,m)=1,a\ge 2,p\in Q}(a-1)p$$

$$= \sum_{p^am\le x,a\ge 2,p\in Q}(a-1)p - \sum_{p^{a+1}m\le x,a\ge 2,p\in Q}(a-1)p = \sum_{p^am\le x,a\ge 2,p\in Q} p.$$

By the standart splitting-up argument of Dirichlet we have , if $a \ge 2$ is a fixed integer,

$$\begin{aligned}\sum_{p^am\le x,p\in Q} p &= \sum_{p\le x^{1/(2a)},p\in Q} p[\frac{x}{p^a}] + \sum_{m\le x^{1/2}}\sum_{p\le (x/m)^{1/a},p\in Q} p \\ &- \sum_{m\le x^{1/2}} 1 \sum_{p\le x^{1/(2a)},p\in Q} p = \textstyle\sum_1 + \sum_2 - \sum_3,\end{aligned}$$

say. Now by using (2.3) it follows that

$$\begin{aligned}\sum_{p\le x,p\in Q} p &= \int_{2-0}^{x} td\pi(t,Q) = \delta\int_2^x \frac{tdt}{\log t} + O\left(\frac{x^2}{\log^B x}\right) \\ &= \delta\sum_{j\le B}\frac{C_j}{\log^j x} + O\left(\frac{x^2}{\log^B x}\right)\end{aligned} \tag{3.4}$$

with suitable constants C_j, and in particular $C_1 = 1/2$. Thus by using (3.4) it is seen that the total contribution of $\sum_3$ for $a \ge 2$ is

$$\begin{aligned}&[x^{1/2}]\left(\delta x^{1/2}\sum_{j\le B}\frac{C_j}{\log^j x^{1/4}} + O\left(\frac{x^{1/2}}{\log^B x}\right)\right) + O(x^{2/3}) \\ &= \delta x\sum_{j\le B}\frac{4^jC_j}{\log^j x} + O\left(\frac{x}{\log^B x}\right).\end{aligned}$$

The contribution of $\sum_1$ and $\sum_2$ for $a \geq 3$ is

$$\begin{aligned}
&\sum_{p^a \le x^{1/2}, a\ge 3, p\in Q} (\frac{x}{p^{a-1}} + O(p)) + O\left(\sum_{m\le x^{1/2}} \sum_{3\le a \ll \log x} \frac{(x/m)^{2/a}}{\log x}\right) \\
= \; & x \sum_{a\ge 3, p\in Q} \frac{1}{p^{a-1}} + O\left(x \sum_{p^a \ge x^{1/2}, a\ge 3} \frac{1}{p^{a-1}}\right) + O(x^{1/3}) + O\left(\sum_{3\le a\ll \log x} \frac{x^{1/a+1/2}}{\log x}\right) \\
= \; & x \sum_{p\in Q} \frac{1}{p^2 - p} + O\left(x \sum_{3\le a\ll \log x} \frac{x^{1/a-1/2}}{\log x}\right) + O(x^{5/6}) \\
= \; & x \sum_{p\in Q} \frac{1}{p^2-p} + O(x^{5/6}).
\end{aligned}$$

For $a = 2$ we have

$$\begin{aligned}
\sum\nolimits_1 &= \sum_{p\le x^{1/4}, p\in Q} p[\frac{x}{p^2}] = x \sum_{p \le x^{1/4}, p\in Q} \frac{1}{p} + O(x^{1/2}) \\
&= x \int_{2-0}^{x^{1/4}} \frac{d\pi(t,Q)}{t} + O(x^{1/2}).
\end{aligned}$$

But by (2.3) it follows that

(3.5)

$$\begin{aligned}
\int_{2-0}^{y} \frac{d\pi(t,Q)}{t} &= \delta \int_2^y \frac{dt}{t\log t} + \int_{2-0}^{y} \frac{dR(t,Q)}{t} \\
&= \delta \log\log y - \delta \log\log 2 + \frac{R(y,Q)}{y} + \int_2^y R(t,Q)\frac{dt}{t^2} \\
&= \delta \log\log y - \delta\log\log 2 + O\left(\frac{1}{\log^B y}\right) + \int_2^\infty \frac{R(t,Q)dt}{t^2} + O\left(\int_y^\infty \frac{dt}{t\log^B t}\right) \\
&= \delta \log\log y + D_1 + O\left(\frac{1}{\log^B y}\right),
\end{aligned}$$

where

$$D_1 = \int_2^\infty \frac{R(t,Q)dt}{t^2} - \delta\log\log 2.$$

If we take $y = x^{1/4}$ in (3.5) and insert the resulting formula in the expression for $\sum_1$ we shall obtain

$$\sum\nolimits_1 = \delta \log\log x + D(Q)x + O\left(\frac{x}{\log^B x}\right),$$

where

$$D(Q) = \int_2^\infty \frac{R(t,Q)dt}{t^2} - \delta(\log 4 + \log\log 2).$$

For $a = 2$ we obtain, using (3.4),

$$\begin{aligned}\sum\nolimits_2 &= \sum_{m \le x^{1/2}} \sum_{p \le (x/m)^{1/2}, p \in Q} p \\ &= \delta x \sum_{m \le x^{1/2}} \frac{1}{m} \left(\sum_{j \le B} \frac{2^j C_j}{\log^j(x/m)} + O\left(\frac{1}{\log^B(x/m)}\right)\right).\end{aligned}$$

We recall (see e.g. (A. 23) of [10]) if $f(t) \in C^1[X,Y]$, $X < Y$, $\psi(t) = t - [t] - 1/2$, then the classical Euler–Maclaurin summation formula asserts that

$$\sum_{X < n \le Y} f(n) = \int_X^Y f(t)dt - \psi(Y)f(Y) + \psi(X)f(X) + \int_X^Y \psi(t)f'(t)dt.$$

We apply this with $X = 1/2$, $Y = x^{1/2}$, $f(t) = t^{-1}\log^{-j}(x/t)$ to obtain, if M is any fixed integer satisfying $M \ge j+1$,

(3.6)

$$\begin{aligned}&\sum_{m \le x^{1/2}} \frac{1}{m \log^j(x/m)} \\ &= \int_{1/2}^{x^{1/2}} \frac{dt}{t\log^j(x/t)} + O(x^{-1/2}) + \int_{1/2}^{x^{1/2}} \psi(t)\left(\frac{j}{\log(x/t)} - 1\right)\frac{dt}{t^2\log^j(x/t)} \\ &= \int_{x^{1/2}}^{2x} \frac{du}{u\log^j u} + \sum_{k=j}^{M} \frac{C_{j,k}}{\log^k x} + O\left(\frac{1}{\log^{M+1} x}\right).\end{aligned}$$

Noting that

$$\int_{x^{1/2}}^{2x} \frac{du}{u\log u} = \log(\log 2x) - \log(\log x^{1/2}) = \log 2 + \sum_{k=1}^{M} \frac{D_{1,k}}{\log^k x} + O\left(\frac{1}{\log^{M+1} x}\right),$$

where $D_{1,k} = (-1)^{k-1}\log^k 2/k$, and that for $j \ge 2$

$$\int_{x^{1/2}}^{2x} \frac{du}{u\log^j u} = \frac{1}{1-j}((\log 2x)^{1-j} - (\log x^{1/2})^{1-j}) = \sum_{k=j-1}^{M} \frac{E_{j,k}}{\log^k x} + O\left(\frac{1}{\log^{M+1} x}\right)$$

with suitable constants $E_{j,k}$, it follows when we substitute (3.6) in the expression for $\sum_2$ that with suitable constants F_j

$$\sum\nolimits_2 = \delta x \left(\log 2 + \sum_{j \le B} \frac{F_j}{\log^j x} + O\left(\frac{1}{\log^B x}\right)\right).$$

From the above expressions for $\sum_1$, $\sum_2$ and $\sum_3$ we obtain the assertion of Theorem 2.

4. Other problems involving $B(n,Q) - \beta(n,Q)$

In this section we shall investigate the sum of reciprocals of $B(n,Q) - \beta(n,Q)$, and the corresponding local densities. We shall prove first

Theorem 3. *There is a constant $A(Q) > 0$ such that*

(4.1)
$$\sum_{n\le x}{}' \frac{1}{B(n,Q)-\beta(n,Q)} = A(Q)x + O(x^{1/2}\log x),$$

where $\sum'$ denotes summation over n such that $B(n,Q) \neq \beta(n,Q)$.

Proof. The proof uses the analytic method given in the proof of Theorem 6.5 of De Koninck–Ivić [4]. For $0 \le t \le 1$ and $\mathrm{Re}\, s > 1$ we have

(4.2)
$$\sum_{n=1}^{\infty} t^{B(n,Q)-\beta(n,Q)} n^{-s} = \prod_{p\in Q}(1 + p^{-s} + t^p p^{-2s} + t^{2p}p^{-3s} + \ldots)\prod_{p\notin Q}(1-p^{-s})^{-1}$$
$$= \prod_p (1-p^{-s})^{-1}\prod_{p\in Q}(1 + (t^p - 1)p^{-2s} + (t^{2p} - t^p)p^{-3s} + \ldots) = \zeta(s)G_Q(s,t),$$

where for $\mathrm{Re}\, s > 1/2$ the function $G_Q(s,t)$ has the Dirichlet series representation

$$G_Q(s,t) = \sum_{n=1}^{\infty} g_Q(n,t)n^{-s},$$

since uniformly in $0 \le t \le 1$

(4.3)
$$\sum_{n\le x} |g_Q(n,t)| \ll x^{1/2}.$$

From the product representation (4.2) it follows that $g_Q(n,t)$ is multiplicative function of n satisfying $g_Q(n,t) = 0$ and

$$-1 \le g_Q(p^\alpha, t) \le 0 \qquad (\alpha \ge 2),$$

which easily implies (4.3). From (4.2) and (4.3) we obtain

(4.4)
$$\sum_{n\le x} t^{B(n,Q)-\beta(n,Q)} = \sum_{n\le x} g_Q(n,t)[\frac{x}{n}] = x\sum_{n\le x} g_Q(n,t)/n + O(\sum_{n\le x}|g_Q(n,t)|)$$
$$= F_Q(t)x + O(x^{1/2}),$$

uniformly for $0 \le t \le 1$, where

$$F_Q(t) = G_Q(1,t) = \prod_{p\in Q}\left(1 + \sum_{k=2}^{\infty}(t^{p(k-1)} - t^{p(k-2)})p^{-k}\right)$$

and therefore

$$F_Q(0) = \prod_{p\in Q}(1 - p^{-2}). \tag{4.5}$$

Let T be the set of numbers n such that $B(n,Q) = \beta(n,Q)$, and let $\chi_T(n)$ be the characteristic function of T. Then T consists of numbers n which have no prime factors from Q (in which case $\beta(n,Q) = B(n,Q) = 0$), or of numbers n all of whose prime factors from Q exactly divide n. Thus for Re $s > 1$

$$\sum_{n=1}^{\infty}\chi_T(n)n^{-s} = \prod_{p\in Q}(1+p^{-s})\prod_{p\notin Q}(1-p^{-s})^{-1} = \zeta(s)\prod_{p\in Q}(1-p^{-2s}).$$

A simple convolution argument gives then, in view of (4.5),

$$\sum_{n\le x, B(n,Q)=\beta(n,Q)} 1 = \sum_{n\le x}\chi_T(n) = F_Q(0)x + O(x^{1/2}). \tag{4.6}$$

From (4.4) and (4.6) it follows that, uniformly for $0 \le t \le 1$, we have

$$\begin{aligned}\sum_{n\le x}{}' t^{B(n,Q)-\beta(n,Q)-1} &= \sum_{n\le x, B(n,Q)\ne\beta(n,Q)} t^{B(n,Q)-\beta(n,Q)-1}\\ &= x(F_Q(t) - F_Q(0))t^{-1} + O(x^{1/2}t^{-1}).\end{aligned} \tag{4.7}$$

Now we integrate (4.7) over t from $\eta = \eta(x) = x^{-2/3}$ to 1. This gives

$$\sum_{n\le x}{}'\left(\frac{1}{B(n,Q)-\beta(n,Q)} - \frac{\eta(x)}{B(n,Q)-\beta(n,Q)}\right) = x\int_\eta^1 (F_Q(t)-F_Q(0))\frac{dt}{t} + O(x^{1/2}\log x),$$

and on simplifying (4.1) follows with

$$A(Q) = \int_0^1 (F_Q(t) - F_Q(0))\frac{dt}{t}.$$

We turn now to the problem of local densities of $B(n,Q) - \beta(n,Q)$. A nonnegative, integer valued arithmetic function $f(n)$ possesses local density d_k if, for a given integer $k \ge 0$,

$$d_k = \lim_{x\to\infty} x^{-1} \sum_{n\le x, f(n)=k} 1$$

exists. In the case of the function $f(n) = B(n,Q) - \beta(n,Q)$ local densities exist for any k, although it is easy to show that $d_k = 0$ for almost all k. More precisely we shall prove

Theorem 4. *For every integer $k \geq 0$ there exists a constant $d_k = d_k(Q) \geq 0$ such that uniformly in k*

$$\sum_{n\leq x, B(n,Q)-\beta(n,Q)=k} 1 = d_k x + O(x^{1/2}\log x). \tag{4.8}$$

P r o o f. The function $f(n) = B(n,Q) - \beta(n,Q)$ is an s–function in the terminology of A. I v i ć and G. T e n e n b a u m [14]. Namely every n can be written uniquely as $n = qs$, $(q,s) = 1$, where $q = q(n)$ is squarefree (i.e. $\mu^2(q) = 1$) and $s = s(n)$ is squarefull ($s = 1$ or $p^2|s$ whenever $p|s$). The function $f(n)$ is then an s–function if $f(n) = f(s(n))$ for every n. Thus from (1.6) of [14] it follows that (4.8) holds uniformly in k with the error term $O(x^{1/2}\log^2 x)$. The simplest way to improve this to $O(x^{1/2}\log x)$ is to follow the proof of Theorem 3, supposing this time that $t = e^{i\theta}$ with θ real. Then it is seen that (4.3) will be replaced by

$$\sum_{n\leq x} |g_Q(n,t)| \ll x^{1/2}\log x, \tag{4.9}$$

since now we have $|t^p - 1| = |e^{i\theta p} - 1| \leq 2$, while for $0 \leq t \leq 1$ we had $-1 \leq t^p - 1 \leq 0$. Consequently we now have $g_Q(p, e^{i\theta}) = 0$ and $|g_Q(p^\alpha, e^{i\theta})| \leq 2$ for $\alpha \geq 2$, hence

$$\sum_{n\leq x} |g_Q(n, e^{i\theta})| \leq \sum_{n\leq x} g(n),$$

where for Re $s > \frac{1}{2}$

$$\sum_{n=1}^{\infty} g(n)n^{-s} = \prod_p (1 + 2p^{-2s} + 2p^{-3s} + \ldots) = \zeta^2(2s)H(s), \tag{4.10}$$

and $H(s)$ is a Dirichlet series which is absolutely convergent for Re $s > \frac{1}{3}$. Hence a convolution argument based on (4.10) easily yields (4.9). Therefore we shall obtain

$$\sum_{n\leq x} e^{i\theta(B(n,Q)-\beta(n,Q))} = F_Q(e^{i\theta})x + O(x^{1/2}\log x) \tag{4.11}$$

uniformly in θ. If one integrates (4.11) over θ from 0 to 2π then (4.8) follows immediately with

$$d_k = \frac{1}{2\pi}\int_0^{2\pi} F_Q(e^{i\theta})e^{-i\theta k}\,d\theta,$$

since for any integer m one has

$$\int_0^{2\pi} e^{im\theta}\,d\theta = \begin{cases} 2\pi & \text{if } m = 0, \\ 0 & \text{if } m \neq 0. \end{cases}$$

Note that the constant $A(Q)$ in (4.1) may be written as

$$A(Q) = \sum_{k=1}^{\infty} \frac{d_k}{k}.$$

5. Sums of reciprocals

It is known (see [8], [9], [13], [16]) that sums of reciprocals of $\beta(n)$ and $B(n)$ behave similarly as the sum of reciprocals of $P(n)$ (see (1.5)). In particular T. Xuan [16] proved that

$$\sum_{2\le n\le x} \frac{1}{\beta(n)} = \left(D + O\left(\frac{(\log\log\log x)^2}{\log\log x}\right)\right) \sum_{2\le n\le x} \frac{1}{P(n)} \tag{5.1}$$

for some constant D which satisfies $\frac{1}{2} < D < 1$, and (5.1) remains valid if $\beta(n)$ is replaced by $B(n)$. Hence in view of (1.4) it is reasonable to expect that

$$\sum_{n\le x}{}' \frac{1}{\beta(n,Q)} = \left(D_1(Q) + O\left(\frac{1}{\log\log x}\right)\right) \sum_{n\le x}{}' \frac{1}{P(n,Q)} \tag{5.2}$$

and

$$\sum_{n\le x}{}' \frac{1}{B(n,Q)} = \left(D_2(Q) + O\left(\frac{1}{\log\log x}\right)\right) \sum_{n\le x}{}' \frac{1}{P(n,Q)} \tag{5.3}$$

hold with $0 < D_2(Q) \le D_1(Q) < 1$. The asymptotic formulas (5.2) and (5.3) appear to be hard, and at present I am unable to prove them . They may be put into the equivalent forms

$$\sum_{n\le x}{}' \frac{1}{\beta(n,Q)} = \left(\eta_1(Q) + O\left(\frac{1}{\log\log x}\right)\right) \frac{x}{(\log x)^{\delta}} \tag{5.4}$$

and

$$\sum_{n\le x}{}' \frac{1}{B(n,Q)} = \left(\eta_2(Q) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^\delta}, \tag{5.5}$$

respectively. In (5.4) and (5.5) we have $\eta_1(Q) = D_1(Q)\eta(Q)$ and $\eta_2(Q) = D_2(Q)\eta(Q)$, where $\eta(Q)$ is the constant appearing in (1.4), so that $0 < \eta_2(Q) \le \eta_1(Q) \le \eta(Q)$.

The result that will be proved involves reciprocals of $P(n,Q)$ and is the following generalization of (1.4).

Theorem 5. *Let $a > 0$ be real and b a fixed natural number, and let*

$$F_Q(x;a,b) = \sum_{n\le x,P^b(n,Q)|n}{}' \frac{1}{P^a(n,Q)}, \tag{5.6}$$

where $\sum'$ denotes summation over those n for which $P(n,Q) \ne 0$. Then there is a constant $C(Q) > 0$ such that

(5.7)

$$F_Q(x;a,b) = C(Q)\sum_{m=1,P^b(m)|m,P(m)\in Q}^{\infty} \frac{l_Q(P(m))}{mP^a(m)}\left(1 + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^\delta},$$

where

$$l_Q(y) = \prod_{p\le y,p\in Q}(1-\frac{1}{p}).$$

P r o o f. Follows on the lines of the proof of (1.4) in [3]. Note that the series in (5.7) is convergent, since

$$\begin{aligned} 0 &< \sum_{P(m)>x,P^b(m)|m,P(m)\in Q} \frac{l_Q(P(m))}{mP^a(m)} \\ &\le \sum_{P(m)>x}\frac{1}{mP^a(m)} = \sum_{p>x}\frac{1}{p^{a+1}}\sum_{P(n)\le p}\frac{1}{n} \\ &= \sum_{p>x}\frac{1}{p^{a+1}}\prod_{q\le p}(1-\frac{1}{q})^{-1} \ll \sum_{p>x}\frac{\log p}{p^{a+1}} \ll \frac{1}{x^a}, \end{aligned}$$

and $a > 0$ by hypothesis. The result of Theorem 5 is to be contrasted with the corresponding result for sums of $1/P^a(n)$. A. I v i ć and C. P o m e r a n c e [13] proved a result which implies, for $r > 0$ fixed,

(5.8)

$$\sum_{2\le n\le x}\frac{1}{P^r(n)} = x\exp\left\{-(2r\log x\log\log x)^{1/2}\left(1 + O\left(\frac{\log\log\log x}{\log\log x}\right)\right)\right\}.$$

The method of proof can be easily adapted to yield an asymptotic formula for the sum

$$F(x;a,b) = \sum_{2\le n\le x, P^b(n)|n} \frac{1}{P^a(n)}, \tag{5.9}$$

which will be of the same type as the formula in (5.8). The sum in (5.9) is the analogue of the sum in (5.6). But it should be observed that changing a and b in (5.6) only affects the constant in (5.7), whereas such a change in (5.9) changes the shape of the exponential factor (of the same type as in (5.8)), hence it changes the order of magnitude of the sum in question. A true asymptotic formula for the sums in (5.8) and (5.9) ((5.8) gives an asymptotic formula only for the logarithm of the sum) can be obtained by the method of [8], where the case $r = 1$ of (5.8) is worked out in detail.

Returning to the proof of (5.7), let $z = (\log x)^{2/a}$. Then we have

(5.10)

$$\begin{aligned} F_Q(x;a,b) &= \sum'_{n\le x, P^b(n,Q)|n, P(n,Q)\le z, P(n,Q)<P(n)} \frac{1}{P^a(n,Q)} + O\left(x\sum_{p>z}\frac{1}{p^{a+1}}\right) \\ &= H_Q(x;a,b) + O\left(\frac{x}{(\log x)^{\delta}\log\log x}\right), \end{aligned}$$

say, where

$$H_Q(x;a,b) = \sum'_{n\le x, P^b(n,Q)|n, P(n,Q)\le z, P(n,Q)<P(n)} \frac{1}{P^a(n,Q)}.$$

Here we used (5.8) (with $r = a$) to estimate the contribution of those n for which $P(n) = P(n,Q)$, and this contribution is trivially absorbed by the first O–term in (5.10). If $P(n,Q) < P(n)$, then $n = mr$ with $p(r) > P(m) = P(n,Q)$, wher $p(r)$ is the smallest prime factor of r, and r has no prime factors from Q, which is denoted by (r,Q)=1. This representation is unique, so that

$$H_Q(x;a,b) = \sum_{2\le m\le x, P(m)\in Q, P^b(m)|m, P(m)\le z} \frac{1}{P^a(m)} \sum_{r\le x/m, (r,Q)=1, p(r)>P(m)} 1.$$

Now the inner sum in this expression is evaluated just as in the proof of the Theorem 3 of J.- M. D e K o n i n c k [3]. The rest of the proof is on of the same lines, with the appropriate modifications, so that there is no need to repeat all the details.

6. Quotients of $B(n,Q) - \beta(n,Q)$

There exist several asymptotic formulas involving the functions $P(n,Q)$, $\beta(n,Q)$ and $B(n,Q)$ that appear interesting. In particular, one could try to evaluate sums of the six different quotients $f(n)/g(n)$, where

$$f, g \in \{P(n,Q), \beta(n,Q), B(n,Q)\}.$$

In the case of $\{P(n), \beta(n), B(n)\}$, such evaluations were made by P. Erdős and A. Ivić [6]. In the case of sums of $B(n)/\beta(n)$ one has

$$\sum_{2\le n\le x} \frac{B(n)}{\beta(n)} = x + O\{x\exp(-C(\log x\log\log x)^{1/2})\} \qquad (C>0), \tag{6.1}$$

and T. Xuan [16] further refined (6.1).

We shall deal here only with the analogue of (6.1) for $B(n,Q)/\beta(n,Q)$ and prove

Theorem 6.

$$\sum_{n\le x}{}' \frac{B(n,Q)}{\beta(n,Q)} = x + O\left(\frac{x\log\log x}{(\log x)^{\delta}}\right), \tag{6.2}$$

where $\sum'$ denotes summation over those n for which $\beta(n,Q)\ne 0$.

Proof. Note first that, by using Lemma 5 of J.-M. De Koninck [3], we have

$$\begin{aligned}\sum_{n\le x}{}' 1 &= \sum_{n\le x,(n,Q)>1} 1 = [x] - \sum_{n\le x,(n,Q)=1} 1\\ &= x(1+O(\log^{-\delta} x)).\end{aligned}$$

To prove (6.2) it clearly suffices to prove

$$\sum_{n\le x}{}' \frac{(B(n,Q)-\beta(n,Q)}{\beta(n,Q)} \ll \frac{x\log\log x}{(\log x)^{\delta}}. \tag{6.3}$$

Again let $n = qs$, $(q,s)=1$, where $q=q(n)$ is squarefree and $s=s(n)$ is squarefull. We have

$$\sum_{n\le x}{}' \frac{(B(n,Q)-\beta(n,Q)}{\beta(n,Q)} \le \sum{}' + \sum{}'',$$

where in $\sum'$ we have $q \le x^{1/2}$, and in $\sum''$ we have $s \le x^{1/2}$, since $n = sq > x$ is impossible. To estimate $\sum'$ and $\sum''$ note first that, in view of $\sum_{s\le x} 1 \ll x^{1/2}$, we have

$$\begin{aligned}\sum_{n\le x}(B(s)-\beta(s)) &= \sum_{p^a s\le x, a\ge 2} p = \sum_{p^a\le x, a\ge 2} p \sum_{s\le x/p^a} 1 \\ &\ll \sum_{p^a\le x, a\ge 2} p\left(\frac{x}{p^a}\right)^{1/2} \ll \frac{x}{\log x}.\end{aligned} \tag{6.4}$$

Thus using (6.4) we obtain

$$\begin{aligned}\sum{}' &\ll \sum_{q\le x^{1/2}}{}' \sum_{s\le x/q}{}' \frac{B(s)-\beta(s)}{\beta(sq,Q)} \ll \sum_{q\le x^{1/2}}{}' \frac{1}{\beta(q,Q)} \frac{x}{q\log(\frac{x}{q})} \\ &\ll \frac{x}{\log x}\sum_{n\le x^{1/2}}{}' \frac{1}{nP(n,Q)} \ll \frac{x}{\log x}\left(\frac{1}{\log^\delta x} + \int_2^x \frac{dt}{t\log^\delta t}\right) \ll \frac{x}{\log^\delta x}\end{aligned}$$

since $0 < \delta < 1$, and where (1.4) and partial summation were used. Similarly using (1.4), (6.4) and partial summation it follows that

$$\begin{aligned}\sum{}'' &\ll \sum_{s\le x^{1/2}}{}'(B(s)-\beta(s)) \sum_{q\le x/s}{}' \frac{1}{\beta(q,Q)} \\ &\ll \sum_{s\le x^{1/2}}{}'(B(s)-\beta(s)) \frac{x}{s\log^\delta(\frac{x}{s})} \ll \frac{x}{\log^\delta x}\sum_{s\le x^{1/2}} \frac{B(s)-\beta(s)}{s} \\ &\ll \frac{x}{\log^\delta x}\left(\frac{1}{\log x} + \int_2^x \frac{dt}{t\log t}\right) \ll \frac{x\log\log x}{\log^\delta x}.\end{aligned}$$

Combining the bounds for $\sum'$ and $\sum''$ we obtain the assertion of Theorem 6.

References

[1] K. Alladi, P. Erdős. On an Additive Arithmetic Function. *Pacific Math.*, **71**, 1977, 275–294.

[2] D. A. Goldston, K. S. McCurley. Sieving the Positive Integers by Large Primes. *J. Number Theory*, **28**, 1988, 94–115.

[3] J.-M. De Koninck. Sur les Plus Grands Facteurs Premiers d'un Entier. *Monatshefte Math.*, **116**, 1993, 13–37.

[4] J. -M. De Koninck, A. Ivić. Topics in Arithmetical Functions. Mathematics Studies. **43**, North Holland, Amsterdam, 1980.

[5] J. -M. De Koninck, A. Ivić. The Distribution of Average Prime Divisor of an Integer. *Archiv. Math.*, **43**, 1984, 37–43.

[6] J. -M. De Koninck, A. Ivić. Arithmetic Functions Defined on Sets of Primes of Positive Density. *Math. Balkanica*, **10**, 1994, in print.

[7] P. Erdős, A. Ivić. Estimates of Sums Involving the Largest Prime Factor of an Integer and Certain Related Additive Functions. *Studia Scien. Math. Hungarica*, **15**, 1980, 183–199.

[8] P. Erdős, A. Ivić, C. Pomerance. On Sums Involving Reciprocals of the Largest Prime Factor of an Integer. *Glasnik Matematički*, **21**, 1986, 27–44.

[9] A. Ivić. Sums of Reciprocals of the Largest Prime factor of an Integer. *Archiv. Math.*, **36**, 1981, 57–61.

[10] A. Ivić. The Riemann Zeta-function. John Wiley & Sons, New York, 1985.

[11] A. Ivić. The Distribution of Quotients of Small and Large Additive functions. *Bollettino Unione Mat. Ital.*, **7 2–B**, 1988, 79–97.

[12] A. Ivić. On the k-th Prime Factor of an Integer. *Univ. u Novom Sadu Zb. Rad. Prirodno. -Mat. Fak. Ser. Mat.*, **20**, 1990, 63–73.

[13] A. Ivić, C. Pomerance. Estimates of Certain Sums Involving the Largest Prime Factor of an Integer. in "Topics in Classical Number Theory", Colloq. Math. Soc. J. Bolyai, **34**, 1984, 769–789.

[14] A. Ivić, G. Tenenbaum. Local Densities over Integers Free of Large Prime Factors. *Quart. J. Math.*,(Oxford), (2)**37**, 1986, 401–417.

[15] R. Warlimont. Sieving by Large Prime Factors. *Monatshefte Math.*, **109**, 1990, 247–256.

[16] T. Xuan. On Sums Involving Reciprocals of Certain Large Additive Functions. Publications Inst. Math. (Belgrade), **45**(59), 1989, 41–55 and II, ibid. **46**(60), 1989, 25–32.

Katedra Matematike RGF-a
Universiteta u Beogradu
Djušina 7,
11000 Beograd
SERBIA (YUGOSLAVIA)

Received 05.06.1993
Revised version 04.08.1994