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Representation of Commutants and Multipliers Related to the Nonlocal Sturm-Liouville Integro-Differential Operator, which Integral Part is of Volterra Type[1]

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The paper deals with the representation of the commutants of the nonlocal Sturm-Liouville integro-differential operator, whose integral part is of Volterra type, as well as with the convolutional structure and multipliers of its root function (eigen and associated function) expansion.

0. Preliminaries

Throughout the paper all functional spaces $L^p, 1 \leq p \leq \infty$, C, BV, AC etc. are considered on the segment $[0,a]$, $a > 0$. Also we denote by $AC^1 = \{f \in AC : f' \in AC\}$, $BV^1 = \{f \in AC : f' \in BV\}$, $BV_0^1 = \{f \in BV^1 : f(0) = 0\}$, $AC_0^1 = \{f \in AC^1 : f(0) = 0\}$, and let BV_{norm} be the subspace of these functions $f \in BV$, which are normalized by the condition $f(t+0) = f(t), t \in [0,a)$, $f(a-0) = f(a)$. Also $=_{\text{a.e.}}$ denote coincidence almost everywhere with respect to the Lebesgue measure in $[0,a]$.

Consider the integro-differential expression

$$(1) \qquad ly = y''(t) - q(t)y(t) + \int_0^t V(t,u)y(u)du, \quad t \in [0,a]$$

where $q \in L^1$, $V(t,u) \in C(G), G = \{(t,u) : 0 \leq u \leq t \leq a\}$. Let $\chi_0(f) = \alpha_0 f(0) + \beta_0 f'(0)$, $|\alpha_0| + |\beta_0| \neq 0$ and let $\chi(f) = \int_0^a f'\,d\varphi + \int_0^a f\,d\psi$ with $\varphi, \psi \in$

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BV be arbitrary nonzero continuous linear functional in C^1, which is linearly independent with χ_0.

The *nonlocal Sturm-Liouville integro-differential operator* is said to be the operator D generated by the expression (1) in the space $X = L^1$ with domain $X_D = \{f \in AC^1 : \chi_0(f) = \chi(f) = 0\}$. More general, with the same letter D we denote the operator generated by the same expression (1) in the spaces $X = L^p, 1 \leq p \leq \infty$ or $X = C$; $X = BV$ with domain $X_D = \{f \in AC^1 : lf \in X, \ \chi_0(f) = \chi(f) = 0\}$.

The present paper is devoted to representation of the $(X,X)-$*commutant* of the operator D for several classes of functionals χ_0, χ when X is a space of the mentioned types, i.e. to representations of the $(X,X)_{D,\text{com}}$ consisting of the continuous linear operators $M : X \to X$ with $M(X_D) \in X_D$ and $MD = DM$ in X_D (see [1], p. 10). Another aim of the present paper is to obtain representation formulas for the *coefficient multipliers* and *multiplier sequences* of the root function expansion of the operator D. This expansion have been studied by N. S. B o z h i n o v in [2]. Our results have been partially announced without proofs in [3], [4].

Let $y(\lambda,t)$ be the solution of the problem $ly = \lambda y, y(\lambda,0) = \beta_0, y'(\lambda,0) = -\alpha_0$. Also, for each $f \in L^1$ and each $\lambda \in \mathbf{C}$ let $\eta = R_\lambda^{(0)} f \in AC^1$ be the unique solution of the Cauchy problem $l\eta - \lambda\eta = f, \eta(0) = 0, \eta'(0) = 0$. (The existence and uniqueness of the solutions of these problems is trivially proved reducing them to Volterra integral equations of second kind. It is clear that "the initial resolvent" $R_\lambda^{(0)}$ maps L^1 in AC^1 and that for each $f \in L^1$ the function $R_\lambda^{(0)} f$ and $y(\lambda,t)$ are $\mathbf{C} \to \mathbf{C}$ entire function of λ.)

It is easily proved that the operator D has a point spectrum $\sigma(D)$, which coincides with the set of zeroes $\{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ of the entire function $E(\lambda) = \chi_t\{y(\lambda,t)\}$, if zeroes exist. For each λ of the resolvent set $\rho(D) = \mathbf{C} \setminus \sigma(\mathbf{D}) = \{\lambda \in \mathbf{C} : \mathbf{E}(\lambda) \neq \mathbf{0}\}$ the resolvent $R_\lambda = (D - \lambda I)^{-1}$ of the operator D is represented by the equality

$$R_\lambda f = R_\lambda^{(0)} f - \frac{\chi\{R_\lambda^{(0)} f\}}{E(\lambda)} y(\lambda,t), \quad f \in L^1. \tag{2}$$

1. Convolutional representations of the commutants of the operator D in some functional spaces

In [2] N.S. B o z h i n o v proves that the operator D has a continuous convolution $*$ in the space X, which is a continuous convolution in each space $X = L^p, 1 \leq p \leq \infty$; C; BV as well, i.e. $*$ is such a continuous bilinear commutative and associative operation in X, that X_D is an ideal of the algebra

$(X, *)$ and $D(f * g) = (Df) * g$ for all $f \in X_D, g \in X$ (see [1], p.10). This convolution represents the resolvent R_λ in the space X by the equality

$$R_\lambda f = \left\{ -\frac{y(\lambda, t)}{E(\lambda)} \right\} * f, \quad f \in X, \text{where } \lambda \in \mathbf{C},\ \mathbf{E}(\lambda) \neq \mathbf{0}, \tag{3}$$

and it is uniquelly determined by the last property. This shows that the Sturm-Liouville integro-differential operator D is an operator with convolutional multiplier resolvent in the sense of [1], p.65. In [2] N.S. B o z h i n o v proves that the spectrum $\sigma(D) = \emptyset$, if and only if $\operatorname{supp} \chi = \{0\}$, i.e. if and only if χ is of the form $\chi(f) = \alpha f(0) + \beta f'(0)$ with $\alpha, \beta \in \mathbf{C}$, $\alpha\beta_0 \neq \alpha_0\beta$. In the opposite case, i.e. if and only if $\operatorname{supp} \chi \neq \{0\}$, the spectrum $\sigma(D)$ is always an infinite countable set. However, we obtain representation of the commutants by other conditions, which do not depend, if the spectrum $\sigma(D)$ is infinite set, or not.

We use a transmutation operator of the form $(Tf)(t) = f(t) + \int_0^t K(t, u) \times \times f(u)\, du$, $t \in [0, a]$, considered in [5], [6]. The operator T is a continuous authomorphism of the space L^1 and it is a continuous authomorphism of the spaces $X = L^p, 1 \leq p \leq \infty$; C; BV; AC as well. The operator T "transforms" the integro-differential operator D into the operator $\partial = d^2/dt^2$ considered in L^1 with domain $X_\partial = \{f \in AC^1 : \tilde{\chi}_0(f) = 0, \tilde{\chi}(f) = 0\}$, where $\tilde{\chi} = \chi \circ T \in (C^1)^*$, and where $\tilde{\chi}_0(f) = f'(0)$, if $\beta_0 \neq 0$ or $\tilde{\chi}_0(f) = f(0)$, if $\beta_0 = 0$. The operator T "transforms" D into ∂ in the sense that $T : L^1 \to L^1$, $T(X_\partial) = X_D$, and $DT = T\partial$ in X_∂. Also we note that $y(\lambda, t) = T_t(ch\sqrt{\lambda}\, t)$, if $\beta_0 \neq 0$ and $y(\lambda, t) = T_t(sh\sqrt{\lambda}\, t/\sqrt{\lambda})$, if $\beta_0 = 0$ and that the convolution, introduced by N.S. B o z h i n o v in [2] has the form $f * g = T(T^{-1}f *_i T^{-1}g)$, $f, g \in L^1$, where $*_i$ is Dimovski convolution (see [1], 3.1.1) with $i = 1$, if $\beta_0 \neq 0$ and $i = 2$, if $\beta_0 = 0$.

We note also that from the results of N.S. B o z h i n o v ([1], 3.1.1) and the properties of the operator T the next three theorems follows:

Theorem 1.1. *Let $\chi_0(f) = \alpha_0 f(0) + \beta_0 f'(0)$, $|\alpha_0| + |\beta_0| \neq 0$ and let $\chi(f) = \int_0^a f'\, d\varphi + \int_0^a f\, d\psi$ with $\varphi, \psi \in BV$ be arbitrary nonzero continuous linear functional in C^1, which is linearly independent with χ_0. Then:*

a) The spaces $X = L^p$, $1 \leq p \leq \infty$; C; BV, AC are ideals without annihilators of the algebra $(L^1, *)$, and the convolution $*$ is $X \times L^1 \to X$ continuous bilinear operation.

b) If $\beta_0 \neq 0$, then the convolution $*$ is is $BV^1 \times L^1 \to AC^1$, $BV^1 \times L^p \to \{f \in AC^1 : lf \in L^p\}, 1 \leq p \leq \infty$, $C \times BV^1 \to \{f \in AC^1 : lf \in C\}$ and $BV^1 \times BV^1 \to \{f \in AC^1 : lf \in BV^1\}$ continuous operation.

c) If $\beta_0 = 0$, then the convolution $*$ is $BV_0^1 \times L^1 \to AC^1$, $BV_0^1 \times L^p \to \{f \in AC^1 : lf \in L^p\}, 1 \leq p \leq \infty$, $C_0 \times BV_0^1 \to \{f \in AC^1 : lf \in C, f(0) =$

$\chi(f) = 0\}$ *and* $BV_0^1 \times BV_0^1 \to \{f \in AC^1 : lf \in BV_0^1, f(0) = \chi(f) = 0\}$ *continuous operation.*

d) In both cases $\beta_0 \neq 0$ *or* $\beta_0 = 0$ *the operation* $*_\lambda = (D-\lambda)(.*.)$ *is* $L^1 \times BV^1 \to L^1$, $L^p \times BV^1 \to L^p, 1 \leq p \leq \infty$ *continuous operation. It is also* $C \times BV^1 \to C$ *and* $BV^1 \times BV^1 \to BV^1$ *continuous operation, if* $\beta_0 \neq 0$, *and* $C_0 \times BV_0^1 \to C_0$, $BV_0^1 \times BV_0^1 \to BV_0^1$ *continuous operation, if* $\beta_0 = 0$.

Theorem 1.1 ′. *Le t* $\chi_0(f) = \alpha_0 f(0) + \beta_0 f'(0)$, $|\alpha_0| + |\beta_0| \neq 0$ *and let* $\chi(f) = \int_0^a f\, d\psi$ *with* $\psi \in BV$ *be arbitrary nonzero continuous linear functional in* C, *which is linearly independent with* χ_0. *Then the convolution* $*$ *is is* $L^1 \times L^1 \to AC$, $BV \times L^1 \to AC^1$, $BV \times L^p \to \{f \in AC^1 : lf \in L^p\}, 1 \leq p \leq \infty$ *and* $BV \times BV \to \{f \in AC^1 : lf \in BV\}$ *continuous operation. The operation* $*_\lambda = (D-\lambda)(.*.)$ *is* $L^1 \times BV \to L^1$, $L^p \times BV \to L^p, 1 \leq p \leq \infty$, *and* $BV \times BV \to BV$ *continuous operation.*

Theorem 1.1 ″. *Let* $\chi_0(f) = \alpha_0 f(0) + \beta_0 f'(0)$, $|\alpha_0| + |\beta_0| \neq 0$ *and let the functional* χ *is of the form* $\chi(f) = \int_0^a f\gamma$ *with* $\gamma \in BV$. *Then the convolution* $*$ *is is* $L^1 \times L^1 \to AC^1$, $C \times C \to C^2$, $L^p \times L^p \to \{f \in AC^1 : lf \in L^p\}, 1 \leq p \leq \infty$ *continuous operation. The operation* $*_\lambda = (D-\lambda)(.*.)$ *is* $L^1 \times L^1 \to L^1$, $L^p \times L^p \to L^p, 1 \leq p \leq \infty$, *and* $C \times C \to C$ *continuous operation.*

The next theorem shows that for the mentioned spaces X the commutant $(X,X)_{D,\text{com}}$ coincides with the corresponding set $(X,X)_*$ consisting of the (X,X)-*multipliers* of the algebra $(X,*)$, i.e. with the set of these operators $M : X \to X$ for which $Mf * g = f * Mg$, when $f, g \in X$ (see [1], p.10).

Theorem 1.2. *Let* $\chi_0(f) = \alpha_0 f(0) + \beta_0 f'(0)$, *and let* $\chi(f) = \int_0^a f'\, d\varphi + \int_0^a f\, d\psi$ *with* $\varphi, \psi \in BV$ *be arbitrary nonzero continuous linear functional in* C^1, *which is linearly independent with* χ_0. *Then the equalities* $(L^p, L^p)_{D,\text{com}} = (L^p, L^p)_*, 1 \leq p \leq \infty; (C,C)_{D,\text{com}} = (C,C)_*$, $(BV,BV)_{D,\text{com}} = (BV,BV)_*$ *hold.*

P r o o f: For $X = L^p, 1 \leq p \leq \infty$; C; BV; AC it is not difficult to see that $(X,X)_{D,\text{com}} = (X,X)_{R_\lambda,\text{com}}$ for arbitrarily fixed $\lambda \in \rho(D)$, and that $(X,X)_* \subset (X,X)_{R_\lambda,\text{com}}$. We shall prove that $(X,X)_{R_\lambda,\text{com}} \subset (X,X)_*$. Let $M \in (X,X)_{R_\lambda,\text{com}}$, i.e. $MR_\lambda f = R_\lambda Mf$ for $f \in X$ and for arbitrary $\lambda \in \mathbf{C}$ with $E(\lambda) \neq 0$. Then from (3) we obtain that $M\{y(\lambda,t) * f\} = y(\lambda,t) * Mf$ for $\lambda \in \mathbf{C}$, $E(\lambda) \neq 0$, but by the continuity it follows that the last equality is true for each $\lambda \in \mathbf{C}$ as well. Now, if $\beta_0 \neq 0$, then $M\{T_t(ch\sqrt{\lambda}\,t) * f\} = \{T_t(ch\sqrt{\lambda}\,t)\} * Mf$ for $\lambda \in \mathbf{C}$. If $\beta_0 = 0$, then $M\{T_t(sh\sqrt{\lambda}\,t/\sqrt{\lambda}) * f\} = \{T_t(sh\sqrt{\lambda}\,t/\sqrt{\lambda})\} * Mf$ for $\lambda \in \mathbf{C}$. By differentiation with respect to λ and putting $\lambda = 0$ we get $M(Tt^{2k} * f) = (Tt^{2k}) * Mf$ for $k = 0, 1, 2, \ldots$, if $\beta_0 \neq 0$, and $M(Tt^{2k+1} * f) = (Tt^{2k+1}) * Mf$ for $k = 0, 1, 2, \ldots$, if $\beta_0 = 0$. Since T is a continuous authomorphism of L^1, then the span of the set $\{Tt^{2k}\}_{k=0}^{\infty}$ as well as the span of the set $\{Tt^{2k+1}\}_{k=0}^{\infty}$ are dense in the space $L^1 = L^1[0,a]$. On the other hand, M is $X \to X$ continuous, X is an ideal in L^1 and the convolution $*$ is $X \times L^1 \to X$ continuous. Then using the continuity we get that $M(g * f) = g * Mf$ for all $f, g \in X$. Hence $M \in (X,X)_*$ and the inclusion is proved. ∎

This theorem shows that the problem for representation of the commutant $(X,X)_{D,\mathrm{com}}$ of the operator D for the mentioned spaces is reduced to the problem for representation of the multipliers of the convolution $*$ in the corresponding spaces.

Hereafter we suppose that the complex number $\lambda \in \mathbf{C}$ is arbitrarily fixed that $E(\lambda) \neq 0$ and let us denote $r_\lambda = -y(\lambda,t)/E(\lambda)$. Also, by I_λ we denote the mapping $I_\lambda :\ m \in \mathcal{M} \to I_\lambda(m)$, where $I_\lambda(m)f = (D-\lambda)(m * f),\ f \in X$ and m belongs to certain space $\mathcal{M}$. By $*_\lambda$ we denote the formal operation $f *_\lambda g = (D-\lambda)(f * g)$, where f, g belong to suitable spaces, which will be described later in all particular cases . Using these denotations in the next statements we denote shortly by $(\mathcal{M}, *_\lambda) \cong_{I_\lambda} (X,X)_*$ that the mapping I_λ is an algebraical and topological isomorphism between the algebras $(\mathcal{M}, *_\lambda)$, $(X,X)_*$.

Theorem 1.3. *Let $\chi_0(f) = \alpha_0 f(0) + \beta_0 f'(0)$ and let $\chi(f) = \int_0^a f'\,d\varphi + \int_0^a f\,d\psi$ with $\varphi, \psi \in BV$, where the representing function $\varphi \in BV$ be normalized by $\varphi(t-0) = \varphi(t)$ for $t \in (0,a)$, and let φ has at least one point of discontinuity $t_0 \in (0,a]$, if $\beta_0 \neq 0$ and $t_0 \in [0,a]$, if $\beta_0 = 0$. Then :*

a) An operator $M \in (L^1, L^1)_*$, if and only if M is represented in the form

$$(4)\qquad Mf = (D-\lambda)(m * f), \quad f \in L^1$$

with unique $m \in BV^1$, if $\beta_0 \neq 0$ and $m \in BV_0^1$, if $\beta_0 = 0$, where $m =_{\text{a.e.}} Mr_\lambda$ in both cases.

b) The relation of algebraical and topological isomorphism $(BV^1, *_\lambda) \cong_{I_\lambda} (L^1, L^1)_*$ holds, if $\beta_0 \neq 0$ and $(BV_0^1, *_\lambda) \cong_{I_\lambda} (L^1, L^1)_*$ holds, if $\beta_0 = 0$.

c) The factorization relation $L^1 = L^1 * L^1$ holds.

d) An operation $\tilde{*} :\ L^1 \times L^1 \to L^1$ is a continuous convolution in L^1 for the operator D, if and only if it is represented in the form

$$(5)\qquad f \,\tilde{*}\, g = (D-\lambda)(m * f * g), \quad f, g \in L^1$$

with unique $m \in BV^1$, if $\beta_0 \neq 0$ and $m \in BV_0^1$, if $\beta_0 = 0$, where $m =_{\text{a.e.}} (D-\lambda)(r_\lambda \tilde{*} r_\lambda)$.

Proof. a) The convolution $*$ introduced by N.S. Bozhinov has the form $f * g = T(T^{-1}f *_i T^{-1}g)$, $f, g \in L^1$, where $*_i$ is Dimovski convolution (see [1], 3.1.1) with $i = 1$, if $\beta_0 \neq 0$ and $i = 2$, if $\beta_0 = 0$. Let us consider the functional $\tilde{\chi} = \chi \circ T \in (C^1)^*$. It is not difficult to see that this functional has the form $\tilde{\chi}(f) = \int_0^a f'\, d\tilde{\varphi} + \int_0^a f\, d\tilde{\psi}$, $f \in C$, where its representation function $\tilde{\varphi} \in BV$ has the same jumps as φ. Let $r_\lambda = -y(\lambda, t)/E(\lambda)$. From (3) we have that $M \in (L^1, L^1)_*$ implies the equalities $Mr_\lambda * f = r_\lambda * Mf = R_\lambda Mf \in AC^1$ for each $f \in L^1$, and using a theorem of N.S. B o z h i n o v ([1], 3.1.2) for Dimovski convolutions $*_i, i = 1, 2$ we get that $Mr_\lambda =_{\text{a.e.}} m \in BV^1$ or $Mr_\lambda =_{\text{a.e.}} m \in BV_0^1$ in the cases $\beta_0 \neq 0$ or $\beta_0 = 0$ respectively. Therefore $Mf = (D - \lambda)(m * f)$, $f \in L^1$ with m in corresponding spaces BV^1 or BV_0^1 and the first part of a) is proved.

Conversely, from the $BV^1 \times L^1 \to AC^1$ or $BV_0^1 \times L^1 \to AC_0^1$ continuity of the convolution $*$, established in Theorem 1.1 b), c), it is easy to see that for each $m \in BV^1$ or $m \in BV_0^1$ respectively, formula (4) defines $L^1 \to L^1$ continuous operator M. To prove that $M \in (L^1, L^1)_*$ we note that from (3) one may prove easily that for all $f \in L^1$, $g \in X_D$ we have $f * g \in X_D$ and $D(f * g) = (Df) * g$. Moreover, it can be proved that for each $m \in BV^1$ or $m \in BV_0^1$ respectively, we have $m * f, m * g \in X_D$ for arbitrary $f, g \in L^1$, and we get $(Mf) * g = [(D - \lambda)(m * f)] * g = (D - \lambda)[(m * f) * g] = (D - \lambda)[f * (m * g)] = f * [(D - \lambda)(m * g)] = f * (Mg)$.

b) follows from a general result of N.S. B o z h i n o v ([1], Theorem 1.1.2).

c) follows from the existence of L^1-bounded approximate identity for the Dimovski convolution $*_i$ in L^1 when $\tilde{\varphi}$ has at least one jump (see [1], Theorem 3.1.7).

■

Theorem 1.4. *Let $\chi_0(f) = \alpha_0 f(0) + \beta_0 f'(0)$ with $\beta_0 \neq 0$, $\chi(f) = f(t_0) + \int_0^a f\gamma$ with $\gamma \in BV$, $t_0 \in [0, a]$. Then an operator $M \in (L^1, L^1)_*$, if and only if M is represented in the form*

$$Mf = (D - \lambda)(m * f), \quad f \in L^1 \tag{6}$$

with unique $m \in BV_{\text{norm}}$, $m =_{\text{a.e.}} Mr_\lambda$. The relation of algebraical and topological isomorphism $(BV_{\text{norm}}, *_\lambda) \cong_{I_\lambda} (L^1, L^1)_*$ holds.

P r o o f. Now it is not difficult to see that the functional $\tilde{\chi}$ has the similar form $\tilde{\chi}(f) = f(t_0) + \int_0^a f\tilde{\gamma}$ with $\tilde{\gamma} \in BV$, $t_0 \in [0, a]$. Let $r_\lambda = -y(\lambda, t)/E(\lambda)$. From (3) we have that $M \in (L^1, L^1)_*$ implies the equalities $Mr_\lambda * f = r_\lambda * Mf = R_\lambda Mf \in AC^1$ for each $f \in L^1$, and using a proposition of N.S. B o z h i n o v ([1], 3.1.2, lemma 7) for Dimovski convolution $*_1$ we get that $Mr_\lambda =_{\text{a.e.}} m \in BV_{\text{norm}}$. Therefore $Mf = (D - \lambda)(m * f)$, $f \in L^1$ with $m \in BV_{\text{norm}}$ and the first part of a) is proved.

Conversely, according to Theorem 1.1′ for such a functional χ the convolution $*$ is $L^1 \times BV \to AC^1$ continuous operation and the operation $*_\lambda = (D-\lambda)(.*.)$ is $L^1 \times BV \to L^1$, continuous operation. Then for each $m \in BV$ formula $Mf = (D-\lambda)(m*f)$, $f \in L^1$ defines $L^1 \to L^1$ continuous operator M. To prove that $M \in (L^1, L^1)_*$ we recall that for all $f \in L^1$, $g \in X_D$ we have $f*g \in X_D$ and $D(f*g) = (Df)*g$. Moreover, now it can be proved that for each $m \in BV$ we have $m*f, m*g \in X_D$ for arbitrary $f, g \in L^1$, and we get $(Mf)*g = [(D-\lambda)(m*f)]*g = (D-\lambda)[(m*f)*g] = (D-\lambda)[f*(m*g)] = f*[(D-\lambda)(m*g)] = f*(Mg)$. The relation of algebraical and topological isomorphism follows from a general result of N.S. B o z h i n o v ([1], Theorem 1.1.2). ■

Theorem 1.5. *Let $\chi_0(f) = \alpha_0 f(0) + \beta_0 f'(0)$ and $\chi(f) = \int_0^a f\gamma$ with $\gamma \in BV$. Then:*

a) An operator $M \in (C, C)_*$ if and only if M is represented in the form

$$Mf = (D-\lambda)(m*f), \quad f \in C \tag{7}$$

with unique $m \in C$, $m = Mr_\lambda$.

b) An operator $M \in (L^p, L^p)_*$, $1 \le p \le \infty$, if and only if M is represented in the form

$$Mf = (D-\lambda)(m*f), \quad f \in L^p \tag{8}$$

with unique $m \in L^p$, $m =_{\text{a.e.}} Mr_\lambda$.

c) The relations of algebraical and topological isomorphism $(C, *_\lambda) \cong_{I_\lambda} (C,C)_*, (L^p, *_\lambda) \cong_{I_\lambda} (L^p, L^p)_*$, $1 \le p \le \infty$ hold.

d) Let $X = C$ or $X = L^p$, $1 \le p \le \infty$. Then an operation $\tilde{*}: X \times X \to X$ is a continuous convolution in X for the operator D, if and only if it is represented in the form

$$f \,\tilde{*}\, g = (D-\lambda)^2(m*f*g), \quad f, g \in X \tag{9}$$

with unique $m \in X$, $m =_{\text{a.e.}} r_\lambda \tilde{*} r_\lambda$. (If $X = C$ the last equality holds everywhere in $[0,a]$.)

P r o o f. According to theorem 1.1″, for such a functional χ the convolution $*_\lambda = (D-\lambda)(.*.)$ is $L^1 \times L^1 \to L^1$, $C \times C \to C$, $L^p \times L^p \to L^p$ continuous operation for $1 \le p \le \infty$. Hereafter the proof is similar as those of theorems 1.3 and 1.4. ■

Theorem 1.6. *Let $\chi_0(f) = \alpha_0 f(0) + \beta_0 f'(0)$ and let $\chi(f) = \int_0^a f\, d\psi$ with $\psi \in BV$ be an arbitrary continuous linear functional in C, which is linearly independent with χ_0 . Then:*

a) An operator $M \in (BV, BV)_*$ if and only if M is represented in the form

$$Mf = (D - \lambda)(m * f), \quad f \in BV \tag{10}$$

with unique $m \in BV_{\text{norm}}$, $m =_{\text{a.e.}} Mr_\lambda$. The relation of algebraical and topological isomorphism $(BV_{\text{norm}}, *_\lambda) \cong_{I_\lambda} (BV, BV)_*$ holds.

b) An operation $\tilde{*} : BV \times BV \to BV$ is a continuous convolution in BV for the operator D, if and only if it is represented in the form $f \,\tilde{*}\, g = (D - \lambda)^2(m * f * g)$, $f, g \in BV$ with unique $m \in BV_{\text{norm}}$, $m =_{\text{a.e.}} r_\lambda \tilde{*} r_\lambda$.

Proof. Now according to Theorem 1.1′, for such a functional χ the convolution $*_\lambda = (D - \lambda)(. * .)$ is $BV \times BV \to BV$ continuous operation. Hereafter the proof is similar as those of Theorems 1.3 and 1.4. ■

2. Convolutional structure and multipliers of the root expansion of the operator D

Hereafter we suppose that the condition $\operatorname{supp} \chi \neq \{0\}$ is always satisfied. As we mentioned before in [2] N.S. Bozhinov proved that this condition is necessary and sufficient the spectrum $\sigma(D)$ to be an infinite countable set $\sigma(D) = \{\lambda_k\}_{k=0}^{\infty}$ (in the opposite case the spectrum is always an empty set). Let $m_0, m_1, m_2, \ldots$ be the corresponding multiplicities of the zeros $\lambda_0, \lambda_1, \lambda_2, \ldots$ To each zero λ_k one may correspond the one-dimensional eigensubspace generated by the eigenfunction $y(\lambda_k, t)$. So, for each λ_k the corresponding root projection $P_{\lambda_k} = -\frac{1}{2\pi i} \int\limits_{\Gamma_k} R_\lambda \, d\lambda$ has the form

$$P_{\lambda_k} f = \frac{1}{2\pi i} \int\limits_{\Gamma_k} \frac{y(\lambda, t)\chi\{R_\lambda^{(0)} f\}}{E(\lambda)} \, d\lambda, \quad f \in L^1, \tag{11}$$

and it maps the space L^1 on the m_k-dimensional root subspace $H_{\lambda_k} = \ker(D - \lambda_k I)^{m_k}$ of the operator D corresponding to λ_k (here Γ_k is a circle with a center λ_k enclosing only the eigenvalue λ_k among all the eigenvalues of the operator D). The root subspace H_{λ_k} is generated by the basis of root functions (eigen and associated functions) $\left\{ \frac{1}{s!} \frac{\partial^s}{\partial \lambda^s} y(\lambda_k, t) : 0 \leq s \leq m_k - 1 \right\}$ of the operator D and the projection P_{λ_k} is represented with respect to this basis in the form

$$P_{\lambda_k} f = \sum_{s=0}^{m_k - 1} A_{m_k - 1 - s}^k(f) \frac{1}{s!} \frac{\partial^s}{\partial \lambda^s} y(\lambda_k, t), \quad f \in L^1, \tag{11'}$$

where

$$(12)\qquad A_s^k(f) = \frac{1}{2\pi i}\int_{\Gamma_k} \frac{(\lambda-\lambda_k)^{m_k-1-s}\chi\{R_\lambda^{(0)}f\}}{E(\lambda)}\,d\lambda, \qquad f\in L^1,$$

$0\le s\le m_k-1$, $k=0,1,2,\dots$ are the coefficient functionals of the projection P_{λ_k} with respect to this basis. Let $\{P_{\lambda_k}\}_{k=0}^{\infty}$ be the orthogonal projection system of the root projections of the operator D (i.e. $P_{\lambda_k}P_{\lambda_s}=0$ for $k\ne s$). In [2] N.S. Bozhinov proved that the condition $a\in\operatorname{supp}\chi$ is necessary and sufficient for the totality of the projection system $\{P_{\lambda_k}\}_{k=0}^{\infty}$ in the space L^1, i.e. the equalities $P_{\lambda_k}f=0$, $k=0,1,2,\dots$ for some $f\in L^1$ (i.e. $A_s^k(f)=0$, $0\le s\le m_k-1$, $k=0,1,2,\dots$) imply that $f=0$ almost everywhere in $[0,a]$. In other words, the condition $a\in\operatorname{supp}\chi$ is necessary and sufficient a uniqueness theorem to be valid for the root function expansion $f\sim\sum\limits_{k=0}^{\infty}P_{\lambda_k}f$ of the functions of L^1, i.e. for the formal expansion

$$(13)\qquad f\sim\sum_{k=0}^{\infty}\sum_{s=0}^{m_k-1}A_{m_k-1-s}^k(f)\frac{1}{s!}\frac{\partial^s}{\partial\lambda^s}y(\lambda_k,t),$$

of the operator D. Then the coefficient functionals A_s^k of this expansion generates the "traditional" integral transformation

$$(14)\qquad f\in L^1 \longrightarrow \tilde f = \{A_0^k(f),\dots,A_{m_k-1}^k(f)\}_{k=0}^{\infty},$$

which is injective according to the totality and maps the space L^1 in the algebra $(\mathcal{X},*_{\mathcal{X}})$ consisting of the cellular sequences of the form $\xi=\{\xi_0^k,\dots,\xi_{m_k-1}^k\}_{k=0}^{\infty}$ provided with the inner Cauchy convolution $\xi *_{\mathcal{X}}\eta=\{\sum\limits_{i=0}^{s}\xi_{s-i}^k\eta_i^k : 0\le s\le m_k-1\}_{k=0}^{\infty}$ (see [1], 2.2, 3.3.1). Following the ideas in [1], p.30 we introduce the outer Cauchy convolution $\xi *_{\mathcal{X}}\mathcal{U}=\{\sum\limits_{i=0}^{s}\xi_{s-i}^k u_i^k(t) : 0\le s\le m_k-1\}_{k=0}^{\infty}$ defined for arbitrary $\xi=\{\xi_0^k,\dots,\xi_{m_k-1}^k\}_{k=0}^{\infty}\in\mathcal{X}$ and arbitrary root function system $\mathcal{W}=\{w_0^k(t),\dots,w_{m_k-1}^k(t)\}_{k=0}^{\infty}$, where $w_0^k,\dots,w_{m_k-1}^k$ is a chain of root functions in the root subspace H_{λ_k} for $k=0,1,2,\dots$

Hereafter we suppose that the condition $a\in\operatorname{supp}\chi$ is always satisfied.

In the preliminaries we marked that the operator D is an operator with convolutional multiplier resolvent with respect to a continuous convolution $*$ in the space L^1 in the sense of [1], p.65, introduced by N. S. Bozhinov in [2]. Using this, the next theorem is true, which makes clear the convolutional structure of the root expansion (14) and the integral transformation (15).

Theorem 2.1. *Let λ_k be m_k-multiple zero of the entire function $E(\lambda)$ and let Γ_k be a circle enclosing only λ_k among the zeros of $E(\lambda)$. Then:*

a) The projection P_{λ_k} is the unique continuous projection, mapping L^1 onto H_{λ_k} and commuting with D. It is the unique nontrivial continuous projection mapping L^1 in H_{λ_k} and commuting with D. The projection P_{λ_k} is convolutionally represented in the form

$$(15)\qquad P_{\lambda_k} f = f * u^k_{m_k-1}, \quad f \in L^1, \quad \text{where } u^k_{m_k-1} = \frac{1}{2\pi i}\int_{\Gamma_k} \frac{y(\lambda,t)}{E(\lambda)}\, d\lambda$$

is an associated function of highest order. The functions

$$(16)\qquad u^k_s(t) = \frac{1}{2\pi i}\int_{\Gamma_k} \frac{(\lambda-\lambda_k)^{m_k-1-s} y(\lambda,t)}{E(\lambda)}\, d\lambda, \quad 0 \le s \le m_k - 1$$

form a "good" root basis in H_{λ_k} with respect to the convolution $*$, i.e.

$$(17)\qquad u^k_p * u^k_s = \begin{cases} 0 & , \ p+s < m_k - 1 \\ u^k_{p+s-m_k+1} & , \ p+s \ge m_k - 1 \end{cases}, \quad 0 \le p, s \le m_k - 1 \ .$$

b) If $E(\lambda) = (\lambda-\lambda_k)^{m_k} \sum_{l=0}^{\infty} \alpha^k_l (\lambda-\lambda_k)^l$ is the Taylor expansion of $E(\lambda)$ around λ_k, then

(18)

$$\frac{1}{s!}\frac{\partial^s}{\partial\lambda^s} y(\lambda_k, t) = \sum_{l=0}^{s} \alpha^k_{s-l} u^k_l(t), \ u^k_s = \sum_{l=0}^{s} \beta^k_{s-l} \frac{1}{l!}\frac{\partial^l}{\partial\lambda^l} y(\lambda_l, t), \ 0 \le s \le m_k - 1;$$

for $0 \le s \le m_k - 1$, where $\beta^k_s : 0 \le s \le m_k - 1$ are the first m_k coeficients of the Taylor expansion of the function $(\lambda-\lambda_k)^{m_k}/E(\lambda)$ around λ_k.

c) The representation

$$(19)\qquad P_{\lambda_k} f = \sum_{l=0}^{m_k-1} C^k_{m_k-1-l}(f) u^k_l(t), \quad f \in L^1,$$

holds with respect to the "good" basis in H_{λ_k} ,where

$$(20)\qquad C^k_s(f) = \frac{1}{2\pi i}\int_{\Gamma_k} \frac{\chi\{R^{(0)}_\lambda m\}}{(\lambda-\lambda_k)^{s+1}}\, d\lambda.$$

The relations

$$(21)\qquad C^k_s = \sum_{l=0}^{s} \alpha^k_{s-l} A^k_l, \quad A^k_s = \sum_{l=0}^{s} \beta^k_{s-l} C^k_l, \quad 0 \le s \le m_k - 1;$$

$$C_s^k(f*g) \;=\; \sum_{l=0}^{s} C_{s-l}^k(f)C_l^k(g), \quad f,g\in L^1; \tag{22}$$

$$A_s^k(f*g) = \sum_{l=0}^{s} \alpha_{s-l}^k \sum_{j=0}^{l} A_{s-j}^k(f)A_j^k(g), \quad f,g\in L^1; \tag{23}$$

hold for $0 \le s \le m_{k-1}$ *(*A_s^k *are defined by (12)).*

P r o o f. Since $P_{\lambda_k} = -\frac{1}{2\pi i}\int\limits_{\Gamma_k} R_\lambda\, d\lambda$, formula (15) follows from (3) by contour integration under the convolution sign. Other statements of the theorem are proved in similar way as the proof of a theorem of N.S. B o z h i n o v ([1], pp.193 - 195).

Let us consider the "good" transformation

$$f\in L^1 \;\longrightarrow\; \hat f = \{C_0^k(f),\ldots,C_{m_k-1}^k(f)\}_{k=0}^{\infty}\in \mathcal{X} \tag{24}$$

generated by the root function system $\mathcal{U} = \{u_0^k(t),\ldots,u_{m_k-1}^k(t)\}_{k=0}^{\infty}$ of the "good" root basis in L^1. Let $\alpha = \{\alpha_0^k,\ldots,\alpha_{m_k-1}^k\}_{k=0}^{\infty}\in\mathcal{X}$, $\beta = \{\beta_0^k,\ldots,\beta_{m_k-1}^k\}_{k=0}^{\infty}\in \mathcal{X}$.

The previous theorem shows that the "good" transformation (24) and the "good" root system $\mathcal{U}$ are related to the "traditional" transformation (14) and the "traditional" root system $\mathcal{V} = \left\{\frac{1}{s!}\frac{\partial^s}{\partial\lambda^s}y(\lambda_k,t) :\ 0\le s\le m_k-1\right\}_{k=0}^{\infty}$ with the equalities

$$\mathcal{V} = \alpha *_{\mathcal{H}} \mathcal{U},\;\; \mathcal{U} = \beta *_{\mathcal{H}} \mathcal{V};\quad \tilde f = \beta *_{\mathcal{X}} \hat f,\;\; \hat f = \alpha *_{\mathcal{X}} \tilde f,\;\; f\in L^1 \tag{25}$$

and that the convolution $*$ is a coefficient convolution of the root expansion in the sense of [1], 1.2.6, i.e.

$$(f*g)\hat{} = \hat f *_{\mathcal{X}} \hat g,\;\; (f*g)\tilde{} = \alpha *_{\mathcal{X}} \tilde f *_{\mathcal{X}} \tilde g,\;\; f,g\in L^1 \tag{26}$$

Following the ideas of [1], 1.2.4 we call (X,X)-*coefficient multiplier* every operator $M: X\to X$ for which there is a *multiplier sequence* $\mu = \{\mu_0^k,\ldots,\mu_{m_k-1}^k\}_{k=0}^{\infty}\in \mathcal{X}$ such that

$$(Mf)\tilde{} = \;\mu *_{\mathcal{X}} \tilde f \;\text{ for }\; f\in X. \tag{27}$$

In [1], 1.2.6 N.S. B o z h i n o v proves that this defintion does not depend on the choice of the transformation and that the equality

$$(Mf)\hat{} = \;\mu *_{\mathcal{X}} \hat f \;\text{ for }\; f\in X \tag{27'}$$

holds with the same multiplier sequence μ. Also, with $(X,X)\,\mathrm{cm}$ and $(X,X)\,\mathrm{ms}$ we denote the corresponding spaces of coefficient multipliers and mutliplier sequences. (We recall that $(X,X)_*$ and $(X,X)_{D,\mathrm{com}}$ denote the spaces of multipliers of the convolution $*$ and the commutant of the operator D for the space X.) ■

Theorem 2.2. *Let $a \in \operatorname{supp}\chi$. Then, for $X = L^p, 1 \leq p \leq \infty$; C; BV; AC we have the equalities*

$$(X,X)\,\mathrm{cm} = (X,X)_* = (X,X)_{D,\mathrm{com}}\,. \tag{28}$$

Proof. The second equality in (28) follows from Theorem 1.1. To prove the equality $(X,X)\,\mathrm{cm} = (X,X)_*$ we use that the condition $a \in \operatorname{supp}\chi$ implies the totality in L^1 of the projection system $\{P_{\lambda_k}\}_{k=0}^{\infty}$ and the injectivity of both transformations (14), (24) in L^1. Let $M \in (X,X)\,\mathrm{cm}$ and let $h = Mf * g - f * Mg$ for arbitrary $f, g \in X$. Then from (26) and (27′) we have $\hat{h} = (Mf*g)\hat{} - (f*Mg)\hat{} = (Mf)\hat{} *_\chi \hat{g} - \hat{f} *_\chi (Mg)\hat{} = (\mu *_\chi \hat{f}) *_\chi \hat{g} - \hat{f} *_\chi (\mu *_\chi \hat{g}) = 0$, since the Cauchy convolution $*_\chi$ is commutative and associative (see [1], 1.2.2). Then by the injectivity of the transformation (24) we get $h = 0$. Thus we prove the inclusion $(X,X)_* \subset (X,X)\,\mathrm{cm}$. The converse inclusion follows from the fact that (15) implies that $MP_{\lambda_k} = P_{\lambda_k}M$ in X , if $M \in (X,X)_*$. Then $M(H_{\lambda_k}) \subset H_{\lambda_k}$ and $MD = DM$ in H_{λ_k} according to the equality $(X,X)_* = (X,X)_{D,\mathrm{com}}$ established in Theorem 1.2. Now it is not difficult to prove (27′) since $H_{\lambda_k}$is generated by the chain basis $\{u_0^k(t), \ldots, u_{m_k-1}^k(t)\}$ of eigen and associated functions and that $\{Mu_0^k, \ldots, Mu_{m_k-1}^k\}$ form also a chain of eigen and associated functions in H_{λ_k}, since $MD = DM$ in H_{λ_k} . ■

Using this theorem we reduce the problem for the representation of the coefficient mutlipliers and the multiplier sequences to our results in Section 1. There we found algebraical and topological isomorphism I_ν ($\nu \in \rho(D)$ isfixed) between the multiplier space $(X,X)_*$ and a function algebra $\mathcal{M}$ for various spaces X.

Theorem 2.3. *Let $X = L^p, 1 \leq p \leq \infty$; C; BV; AC and let $\mathcal{M}$ be the functional algebra that the relation of algebraical and topological isomorphism $\mathcal{M} \cong_{I_\nu} (X,X)_*$ be established in [3]. Then for the space $(X,X)\,\mathrm{ms}$ the representation*

$$(X,X)\,\mathrm{ms} = \{\lambda_k - \nu,\ 1,\ 0, \ldots,\ 0\}_{k=0}^{\infty}\ *_\chi\ \hat{\mathcal{M}} \tag{29}$$

holds. A sequence $\mu = \{\mu_0^k, \ldots, \mu_{m_k-1}^k\}_{k=0}^{\infty} \in (X, X)_{\mathrm{ms}}$, *if and only if*

$$(30) \qquad \mu_s^k = \frac{1}{2\pi i} \int_{\Gamma_k} \frac{(\lambda - \nu)\chi\{R_\lambda^{(0)} m\}}{(\lambda - \lambda_k)^{s+1}} \, d\lambda,$$

$0 \le s \le m_k - 1, \quad k = 0, 1, 2, 3, \ldots$ *with some* $m \in \mathcal{M}$.

P r o o f. The isomorphism $\mathcal{M} \cong_{I_\nu} (X, X)_*$ means that $M \in (X, X)_*$, iff $Mf = (D - \nu)(m * f)$, $f \in X$ with unique $m \in \mathcal{M}$ (here $\nu \in \rho(D)$ is arbitrarily fixed). Then from (27′) and (26) we have $\mu *_{\mathcal{X}} \hat{f} = (Mf)^{\hat{}} = (D - \nu)^{\check{}} *_{\mathcal{X}} (m * f)^{\hat{}} = (D - \nu)^{\check{}} *_{\mathcal{X}} (\hat{m} *_{\mathcal{X}} \hat{f}) = ((D - \nu)^{\check{}} *_{\mathcal{X}} \hat{m}) *_{\mathcal{X}} \hat{f}$ for each $f \in X$, where $(D - \nu)^{\check{}} = \{\lambda_k - \nu, \ 1, \ 0, \ldots, \ 0\}_{k=0}^{\infty} \in (\mathcal{X}, *_{\mathcal{X}})$. Hence, $\mu = (D - \nu)^{\check{}} *_{\mathcal{X}} \hat{m}$, since there are $f \in X$ such that $\hat{f}$ is not divizor of zero of the algebra $(\mathcal{X}, *_{\mathcal{X}})$ and since the correspondence $M \in (X, X)_{\mathrm{cm}} \leftrightarrow \mu \in (X, X)_{\mathrm{ms}}$ is a linear isomorphism between the space $(X, X)_{\mathrm{cm}}$ and the space $(X, X)_{\mathrm{ms}}$, where $(X, X)_{\mathrm{ms}} \subset (\mathcal{X}, *_{\mathcal{X}})$. Finally, formula (30) follows from Theorem 1.4.10 in [1]. ■

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