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Normed Unit Groups and Direct Factor Problem for Commutative Modular Group Algebras

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In the theory of the group algebras the following major problem has not been solved yet: Is it true that the abelian group G is a direct factor of the group of normalized units of the algebra KG, for some field K of characteristic $p > 0$? In this work the current problem is solved when G is an algebraically compact abelian p-group. The complement of the group G is being described when the field K is perfect.

1. Introduction.

Let G be an abelian p-group, K be a field, $\text{char}K = p > 0$, R be a commutative (abelian) ring with identity, $U(R)$ be its multiplicative group, RG be the group algebra of the group G over the ring R, and $U(RG)$ and $V(RG)$ be the unit group and the group of normalized units (i.e. of augmentation 1 – the coefficients sum to 1) in the algebra (ring) RG, respectively.

In this paper the group $V(KG)$ is being examined when it is topologically complete (quasi complete) or algebraically compact p-group. A full system of invariants of $V(RG)$ is given when G is an algebraically compact p-group and R is an arbitrary ring of characteristic $p > 0$. The direct factor problem for this group class is being solved. An invariant system for the topologically complete group $V(KG)$ is also obtained. Some topologically pure subgroups of $V(KG)$ are being described and the problem for the basic subgroups of the normed unit group $V(KG)$ is being discussed, too.

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§I. Topologically complete abelian p-groups.

A subgroup H of the abelian p-group G is said to be topologically pure in G, if it and its topologically closure (cover) $H_G^- \stackrel{def}{=} \bigcap_{n<\omega} (HG^{p^n})$ ([4], p.42) in the group G are pure in G.

Then the reduced abelian p-group G is called topologically complete (quasi complete; [5], p.57) iff every its pure subgroup is topologically pure in G. The abelian p-group G is called separable iff $G^{p^\omega} \stackrel{def}{=} \bigcap_{n<\omega} G^{p^n} = 1$. Certainly every topologically complete group is separable.

Thus, because $H_G^-/H = \bigcap_{n<\omega} (G/H)^{p^n} = (G/H)^{p^\omega}$, then H_G^- is pure in G iff H is pure in G and $(G/H)^{p^\omega}$ is a divisible group. Finally H is topologically pure in G iff $(G/H)^{p^\omega}$ is a divisible group and H is pure in G ([4], p.137, Lemma 26.1).

A subgroup C of the abelian p-group M is called balanced if it is both nice and isotype ([5], p.94).

Lemma 1 . *If C is balanced in the p-group M, then for every ordinal number τ:*

$$(M/C)^{p^\tau} \cong M^{p^\tau}/C^{p^\tau} \tag{1}$$

P r o o f. We have that C is nice in M and therefore $M^{p^\tau} \to (M/C)^{p^\tau} = (CM^{p^\tau})/C$ is the natural map with kernel $C \cap M^{p^\tau} = C^{p^\tau}$. The lemma is proved. ■

Lemma 2 *(May [7]). If H is isotype in G, then $GV(KH)$ is balanced in $V(KG)$.*

Lemma 3. *The abelian p-group G is balanced in $V(KG)$.*

P r o o f 1. We have $G \cap V^{p^\tau}(KG) = G \cap V(K^{p^\tau} G^{p^\tau}) = G \cap G^{p^\tau} = G^{p^\tau}$, where the first equality follows from [3]. Hence G is isotype in $V(KG)$. Let $\overline{K}$ be the algebraic closure of K. Following May [6], G is nice in $V(\overline{K}G)$, hence

G is balanced in $V(\overline{K}G)$. Finally from ([5], p.96), G is balanced in $V(KG)$, since $V(KG) \leq V(\overline{K}G)$. The lemma is true. ■

P r o o f 2. Follows immediately from Lemma 2 at $H = 1$. The proof is completed. ■

Lemma 4. *Let $F \leq R$ and $B \leq A$, A is an abelian group. Thus*

(j) $RA = FB$ iff $R = F$ and $A = B$.

(jj) $V(RA) = 1$ iff $A = 1$.

(jjj) If $B \neq 1$, char $R = p$ and A is p-torsion, then $V(RA) = V(FB)$ iff $R = F$ and $A = B$.

(jjjj) $V(RA) = V(RB)$ iff $A = B$.

P r o o f. Evidently. ■

Lemma 5. *The group $V(RG)$ is separable iff G is separable.*

P r o o f. Let $G^{p^\omega} = 1$. Consequently (see [3]), $V^{p^\omega}(RG) = V(R^{p^\omega}G^{p^\omega}) = 1$ by Lemma 4, where $R^{p^\omega} \stackrel{def}{=} \bigcap_{n<\omega} R^{p^n}$. The lemma is proved. ■

Lemma 6. *Let $H \leq G$. The group $V(RH)$ is isotype in $V(KG)$ iff H is isotype in G.*

P r o o f. Certainly from Lemma 4, $H \cap G^{p^\tau} = H^{p^\tau}$ iff $V(RH) \cap V^{p^\tau}(RG) = V(RH) \cap V(R^{p^\tau}G^{p^\tau}) = V(R^{p^\tau}(H \cap G^{p^\tau})) = V(R^{p^\tau}H^{p^\tau}) = V^{p^\tau}(RH)$. The lemma is true. ■

We know that, if C is a nice subgroup of the separable abelian p-group M, then M/C is separable. Indeed $(M/C)^{p^\omega} = (CM^{p^\omega})/C = 1$ (see [5], p.91, Lemma 79.2), since $M^{p^\omega} = 1$.

Theorem 7. *Let G be a separable abelian p-group and $H \leq G$. The group $V(KH)$ is topologically pure in $V(KG)$ iff H is topologically pure in G.*

P r o o f. Let H be pure in G. Then H is isotype in G. We will prove that:

$$(V(KG)/V(KH))^{p^\omega} \cong (G/H)^{p^\omega}. \tag{2}$$

Evidently, because $H = G \cap V(KH)$, then $(G/H)^{p^\omega} \cong (GV(KH)/V(KH))^{p^\omega}$. By Lemma 2 and ([5], p.96) we see that $GV(KH)/V(KH)$ is balanced in $V(KG)/V(KH)$. Hence from Lemma 1,

$$(V(KG)/V(KH))^{p^\omega}/(GV(KH)/V(KH))^{p^\omega} \cong (V(KG)/GV(KH))^{p^\omega} = 1,$$

since $GV(KH)$ is nice in $V(KG)$ and by Lemma 5, $V(KG)$ is separable. Finally $(V(KG)/V(KH))^{p^\omega} = (GV(KH)/V(KH))^{p^\omega} \cong (G/H)^{p^\omega}$. Obviously from Lemma 6, H is pure in G iff $V(KH)$ is pure in $V(KG)$. Thus the theorem follows immediately in view of formula (2). This proves the theorem. ■

Proposition 8. *Let A be an abelian group and R be a commutative ring with identity. If $A = B \times C$, then:*

$$V(RA) \cong V(RB) \times V(PC).$$

$$V(RA)/V(RB) \cong V(P(A/B)), \text{ where } P = RB.$$

Proof. Bacause $A = B \times C$, then $RA = (RB)C = PC$. Therefore $U(RA) = U(PC)$, i.e. $V(RA) \times U(R) = V(PC) \times U(P) = V(PC) \times V(RB) \times U(R)$. Hence $V(PC) \times V(RB) \cong V(RA) \times U(R)/U(R) \cong V(RA)$ and $V(P(A/B)) \cong V(PC) \cong V(RA){\times}U(R)/V(RB){\times}U(R) \cong V(RA)/V(RB)$. This completes the proof of the proposition.

The abelian ring R of prime characteristic p with identity is called perfect (p-perfect) if $R^p = R$ (i.e., every element is a p-th power), $R^p = \{r^p \mid r \in R\}$. ■

Lemma 9.

(j) RG is perfect iff R is perfect and G is divisible.

(jj) $V(RG)$ is divisible iff R is perfect and G is divisible.

Proof. (j) Indeed $RG = (RG)^p = R^pG^p$ iff $R^p = R$ and $G^p = G$ from Lemma 4.

(jj) We may assume $G \neq 1$. If $R = R^p$ and $G = G^p$, then $V^p(RG) = V(R^pG^p) = V(RG)$ (see [3]), i.e. $V(RG)$ is divisible. Conversely, let now $V(RG) = V^p(RG)$, i.e. $V(RG) = V(R^pG^p)$ [3]. Consequently by Lemma 4, $R = R^p$ and $G = G^p$. The lemma is proved. ∎

Theorem 10. *Let G be an abelian p-group, $H \leq G$, H be a divisible group (more generally, a direct factor of G) and R be at commutative ring of prime characteristic p with identity. The group $V(RH)$ is topologically pure in $V(RG)$ iff H is topologically pure in G and R^{p^ω} is perfect when $(G/H)^{p^\omega} \neq 1$ (and H^{p^ω} is divisible in the more general case).*

Proof. We know that H is a direct factor of G. Hence from Proposition 8, we have:

$$(5) \qquad (V(RG)/V(RH))^{p^\omega} \cong V^{p^\omega}((RH)(G/H)) = V((RH)^{p^\omega}(G/H)^{p^\omega}),$$

where the second equality follows owing to [3], since char$RH = p$. Moreover $(RH)^{p^\omega} = R^{p^\omega}H^{p^\omega} = RH$ is a perfect ring and thus the statement follows by virtue of (5), Lemma 6 and Lemma 9. So, the theorem is true. ∎

If A is an abelian p-group, then under a final rank of A we must understand the cardinal number fin $r(A) = \inf_{n<\omega} r(A^{p^n})$ ([4], p.177).

Proposition 11. *Let G be an abelian p-group. If the group $V(KG)$ is topologically complete, then the group G is topologically complete and* fin $r(G) \leq 2^{\aleph_0}$.

Proof. 1. Because every topologically complete group is separable, then G is a separable group. Suppose H is pure in G, i.e. $V(KH)$ is pure in $V(KG)$ from Lemma 6. Hence $V(KH)$ is topologically pure in $V(KG)$, i.e. H is topologically pure in G by Theorem 7. Therefore G is topologically complete. Let us assume that fin $r(G) > 2^{\aleph_0}$. By ([5], p.60, Theorem 74.8) G is torsion complete. Since G is pure in $V(KG)$ by Lemma 3, then $V(KG) = G \times M$ ([5], p.25, Theorem 68.4). But G is an unbounded group and according ([10] or [1]), $V(KG)$ is not a torsion complete group. Thus using ([5], p.60, Corollary 74.6) we derive that M is a bounded group. Therefore $V(KG)$ is a torsion complete group, because G is torsion complete ([5], p.29, Exercise 8), and this is a contradiction. This proves the proposition. ∎

Proof. 2. Since Lemma 3 is true, the proof is analogous to this in ([1], Proposition 3). This proves the proposition. ∎

Propositios 11 implies that if $V(KG)$ is topologically complete, then G is topologically complete, but it is not unbounded torsion complete. Moreover, the following is actual:

Remark 12. *For R a ring of char $R = p$, we have obtained that $V(RG)$ is topologically complete (in particular, torsion complete; see also [2]) iff G is bounded, but the proof we will give elsewhere.*

From Proposition 11 follows that if G is an unbounded torsion complete group, then $V(KG)$ is not a topologically complete group. So, the next has a key role:

Problem 13. *If G is a torsion complete abelian p-group, then what is the structure of the group $V(KG)$? Is it true that $V(KG)$ is a pure complete group ?*

§II. Algebraically compact abelian p-groups.

If W is an abelian group, then suppose W_d and W_r are the maximal divisible subgroup of W and the reduced part of W, respectively, and L is the maximal perfect subring of the ring R. The group W is called reduced if $W_d = 1$.

Lemma 14. *The group $V(RG)$ is reduced iff G is reduced.*

Proof. The maximal divisible subgroup of $V(RG)$ is $V(LG_d)$ [8], i.e. $V(RG)_d = V(LG_d)$. But $G_d = 1$, i.e., $V(RG)_d = 1$ by Lemma 4. Hence $V(RG)$ is a reduced group. The lemma is shown. ∎

The abelian ring R of prime characteristic p with identity is said to be weakly perfect iff $R^{p^i} = R^{p^{i+1}}$ for some $i \in \mathbb{N}$, i.e. R^{p^i} is perfect for this $i \in \mathbb{N}$.

Apparently every perfect ring is weakly perfect and the weakly perfect ring R is perfect when it is a ring without nilpotent elements. Really let $a \in R$. Thus $a^{p^i} = b^{p^{i+1}}$, $b \in R$ and $(a - b^p)^{p^i} = 0$, i.e. $a = b^p$ and $R = R^p$.

Lemma 15. *The group $V(RG)$ is bounded iff G is bounded.*

Proof. Clearly $G^{p^k} = 1$ for some $k \in \mathbb{N}$ iff $V^{p^k}(RG) = V(R^{p^k}G^{p^k}) = 1$ by Lemma 4 (see [3]). So, the lemma is verified. ∎

We know that every divisible and every bounded groups are algebraically compact ([4], p.187). Therefore following Lemma 9 and Lemma 15, we obtain

Theorem 16. *Let G be an abelian p-group and R be an abelian ring with identity of prime characteristic p without nilpotent elements. Thus*

(j) If G is reduced, then $V(RG)$ is algebraically compact iff G is algebraically compact.

(jj) If G is not reduced , then $V(RG)$ is algebraically compact iff G is algebraically compact and R is perfect.

Proof. (j) From Lemma 14, $V(RG)$ is reduced. Therefore ([4], p.199, Corollary 40.3), $V(RG)$ is algebraically compact iff $V(RG)$ is bounded, i.e. iff G is bounded by Lemma 15.

(jj) Let $G = G_d \times G_r$. Hence by Proposition 8, we have the isomorphism

$$V(RG) \cong V(RG_d) \times V(PG_r), \quad P = RG_d. \tag{6}$$

We know that ([4], p.199, Corollary 40.3) if G is algebraically compact, then G_r is bounded, i.e. in view of Lemma 15, $V(PG_r)$ is bounded. Besides $V(RG_d)$ is divisible owing to Lemma 9. Finally ([4], p.189, Corollary 38.3), $V(RG)$ is an algebraically compact p-group.

Let now $V(RG)$ be algebraically compact. Consequently ([4], p.189, Corollary 38.3), $V(RG_d)$ and $V(PG_r)$ are algebraically compact. But $V(PG_r)$ is reduced and immediately it is bounded, i.e. G_r is bounded. Hence G is algebraically compact. Let L be the maximal perfect subring of the ring R. Then $V(RG_d) = V(LG_d) \times V(RG_d)_r$, where $V(LG_d) = V(RG_d)_d$. Let us assume that does exist an element $0 \neq x \in R : x^{p^s} \notin L$, for each $s \in \mathbb{N}_0 =$

$\mathbb{N} \cup \{0\}$. Denote

$$(7) \qquad Z(p^\infty) = \langle a_0, a_1, \ldots, a_n, \ldots \mid a_0 = 1,\, a_n^p = a_{n-1},\, \forall n \in \mathbb{N} \rangle.$$

Evidently the elements a_n have orders p^n, i.e. $o(a_n) = p^n$, i.e. $a_n^{p^n} = 1$, and $a_n^{p^s} \neq 1$ if $0 \leq s < n$. Now we set the sequences $(x_n)_{n=1}^\infty$, $x_n = 1 + x(1 - a_n)$ and $(y_n)_{n=1}^\infty$, $y_n = x_n V(LG_d)$. Hence $x_n \in V(RG_d)$ and $y_n \in V(RG_d)/V(LG_d)$ since $G_d \cong \prod_m Z(p^\infty)$, where m is a cardinal number. Besides $x_n^{p^n} = 1$ and $x_n^{p^s} = 1 + x^{p^s}(1 - a_n^{p^s}) \neq 1$ if $0 \leq s < n$, because $a_n^{p^s} \neq 1$ and ($x^{p^s} = 0$ iff $x = 0$). Thus $o(x_n) = p^n$. Analogiclly $o(y_n) = p^n$. Infact, $y_n^{p^n} = x_n^{p^n} V(LG_d) = 1$ and let $y_n^{p^s} = x_n^{p^s} V(LG_d) = V(LG_d)$, i.e. $x_n^{p^s} = 1 + x^{p^s}(1 - a_n^{p^s}) \in V(LG_d)$, i.e. $(1 + x^{p^s}) - x^{p^s} a_n^{p^s} \in V(LG_d)$. Finally $x^{p^s} \in L$, but this is not true. Therefore $y_n^{p^s} \neq 1$ if $0 \leq s < n$ and the sequence $(y_n)_{n=1}^\infty$ is unbounded. Consequently $V(RG_d)_r \cong V(RG_d)/V(LG_d)$ is unbounded and $V(RG_d)$ is not an algebraically compact group which is a contradiction. Then for each $\alpha \in R$, there exists $n_\alpha \in \mathbb{N} : \alpha^{p^{n_\alpha}} \in L$. Finally R is perfect since $\alpha^{p^{n_\alpha}} \in L = L^{p^{n_\alpha}}$, i.e. $\alpha^{p^{n_\alpha}} = b^{p^{n_\alpha}}$, $b \in L$, i.e. $\alpha = b$, i.e. $R = L$. The theorem is verified. ∎

Now we shall deneralize the above theorem using a different method of proof,namely:

Theorem 17. *Let G be an abelian p-group and R be a commutative ring with identity and prime characteristic p.*

(j) If G is reduced, then $V(RG)$ is algebraically compact iff G is algebraically compact.

(jj) If G is not reduced, then $V(RG)$ is algebraically compact iff G is algebraically compact and R is weakly perfect.

P r o o f. We well-know that, $V(RG) = V(LG_d) \times V(RG)_r$, where $V(LG_d)$: $V(RG)_d$ (see [9]) and L is the maximal perfect subring of R. Furthermore, $V(RG)$ is algebraically compact iff $V(RG)/V(LG_d)$ is bounded (cf. [4]), i.e. iff $V(R^{p^i} G^{p^i}) = V(LG_d)$ for some $i \in \mathbb{N}$. This equality is equivalent by Lemma 4 to $R^{p^i} = L$ and $G^{p^i} = G_d$, when $G_d \neq 1$, i.e. to $R^{p^i} = R^{p^{i+1}}$ and

$1 = (G/G_d)^{p^i} \cong G_r^{p^i}$. Hence $V(RG)$ is algebraically compact iff (jj) holds (see [4]), when G is not reduced.

If G is reduced, then $V(RG)$ is reduced by Lemma 14 and so $V(RG)$ is bounded, i.e. G is bounded by Lemma 15, i.e. G is algebraically compact (see [4]). So, the theorem is true. ∎

A full describtion of the algebraic compact p-components of $U(RG)$ when R is an arbitrary abelian ring with identity and prime characteristic p and G is an arbitrary abelian group, is obtained in [3]. Let K be a perfect field. Thus by W. May [6], $V(KG)$ is simply presented iff G is simply presented.

Problem 18. *Let G be an abelian p-group.*

(j) If G is reduced, then $V(RG)$ is simply presented iff G is simply presented.

(jj) If G is not reduced, then $V(RG)$ is simply presented iff G is simply presented and R is weakly perfect.

Corollary 19. *Let G be an abelian p-group and R be a perfect commutative ring of prime characteristic p with identity. The group $V(RG)$ is algebraically compact iff G is algebraically compact.*

Theorem 20. (DIRECT FACTOR). *Let G be an algebraically compact abelian p-group. Then G is a direct factor of $V(RG)$. If R is a weakly perfect ring, then the complement is an algebraically compact p-group.*

Proof. Analogous to Lemma 3 it follows that, G is pure in $V(RG)$. Thus G is a direct factor of $V(RG)$ ([4], p.187), i.e., $V(RG) = G \times M$. Let $G = G_d \times G_r$. Hence G_r is bounded ([4], p.199, Corollary 40.3). By Proposition 8, we conclude $V(RG) \cong V(RG_d) \times V(PG_r)$, where $P = RG_d$. Using Lemma 15, $V(PG_r)$ is bounded. Let $R^{p^i} = R^{p^{i+1}}$ for some $i \in \mathbb{N}$. Therefore $R^{p^i} = (R^{p^i})^p$, i.e. $R^{p^i} = L$ is the maximal perfect subring of R. Fufther $V(RG_d) = V(LG_d) \times V(RG_d)_r$, since $V(RG_d)_d = V(LG_d)$. Consequently $V(RG_d) = V(R^{p^i}G_d) \times V(RG_d)_r$ and $V^{p^i}(RG_d) = V(R^{p^i}G_d) = V(R^{p^i}G_d) \times V^{p^i}(RG_d)_r$. Hence $V^{p^i}(RG_d)_r = 1$, i.e. $V(RG_d)_r$ is bounded.

Finally by ([4], p.189, Corollary 38.3), $V(RG_d)$ and $V(RG)$ are algebraically compact. Besides M is algebraically compact as a direct factor of $V(RG)$ ([4], p.187). This completes the proof of the theorem.

Theorem 17 implies that if G is unbounded algebraically compact and R is not perfect then $V(RG)$ is not algebraically compact.

Problem 21. *If G is an algebraically compact abelian p-group then whether when $V(RG)$ is a simply presented p-group ? What is the structure of $V(RG)$ in this case? Besides we note that the following are fulfilled: $V(RG) = 1+I(RG;G)$ and $V(RG)[p] = 1 + I(RG; G[p]) \oplus M[R(p); \sqcap(G/G[p]$, where $I(RG; G[p])$ is a relative augmentation ideal of RG with respect to $G[p]$; $M(R(p); \sqcap(G/G[p]) \stackrel{def}{=} \{\sum_{g\in G[p]} r_g(1-g) | r_g \in R(p)\}$ and the other notations are standard.*

§III. Basic subgroups of abelian p-groups.

Theorem 22. *Let G be an abelian p-group and $B \leq G$. If $V(KB)$ is a basic subgroup of $V(KG)$, then B is a basic subgroup of G. The oposite statement is completely wrong.*

Proof. Certainly by Lemma 6 it is enough to prove only that if $V(KG)/V$ is a divisible group, then G/B is a divisible group and that the opposite statement is not true.

Evidently because $B = G \cap V(KB)$, then $G/B \cong GV(KB)/V(KB) \leq V(KG)/V(KB)$. But $V(KB)$ is pure in $V(KG)$, i.e. from Lemma 6, B is pure in G. Therefore by Lemma 2, $GV(KB)$ is pure in $V(KG)$, i.e. $GV(KB)/V(KB$ is pure in the divisible group $V(KG)/V(KB)$. Finally G/B is a divisible group.

Let now B is nice in G. Therefore B is a balanced basic subgroup of G, i.e. B is a direct factor of G, i.e. $B = G_r$ ([5], p.99, Exercise 9 and p.92, Exercise 2). Then by virtue of Proposition 8, we obtain:

$$V(KG)/V(KB) \cong V((KB)(G/B)). \tag{8}$$

If we assume that $V(KG)/V(KB)$ is a divisible group, then $V((KB)(G/B))$ is a divisible group, i.e. G/B is a divisible group and KB is a perfect ring owing to Lemma 9, as $G \neq B$. Again by Lemma 9 follows that, B is a divisible group and this is a contradiction ($B \neq 1$). Hence even if we have a perfect ring, the group $V(KG)/V(KB)$ may not be divisible. This proves the theorem. ∎

Theorem 23. *Let G be a reduced abelian p-group and $B \leq G$. The group $V(KB)$ is a basic subgroup of $V(KG)$ iff $B = G$ and G is a direct product of cyclic groups.*

Proof. Let $V(KB)$ be basic in $V(KG)$. Then by Theorem 22, B is basic in G. If $B = 1$, then $V(KG)$ is a divisible group and from Lemma 14, $V(KG) = 1$, i.e. $G = 1$ and so $B = G$. Thus let $B \neq 1$. Certainly $V(KG)/V(KB) \to V(KG)/GV(KB)$ is an epimorphism and since $V(KG)/V(KB)$ is a divisible group, then $V(KG)/GV(KB)$ is a divisible group. Lemma 14 implies that, $V(KG)$ is a reduced group and consequently $V(KG)/GV(KB)$ is a reduced group by Lemma 2 and ([5], p.92, Exercise 2). Finally $V(KG) = GV(KB)$. Thus $G = B$. Indeed let $1 \neq g \in G$ and $g \notin B$, i.e. $g \in G \setminus B$. If $1 \neq b \in B$, then $x_{gb} = 1 + g - b \in V(KG)$ is a canonic element and $1 + g - b = g'.(\alpha_1 b_1 + \cdots + \alpha_t b_t) \in GV(KB)$, $g' \in G$, $b_1, \ldots, b_t \in B$; $\alpha_1, \ldots, \alpha_t \in K$. Hence $g' \in B$ and $g \in B$, i.e. $G = B$. But besides, [8] and [2,3] imply that $V(KG)$ is a direct product of cyclic p-groups iff G is a direct product of cyclic p-groups. The theorem is true. ∎

Lemma 24. *Let A be an abelian p-group and $B \leq C \leq A$ as C is pure in A. If B is a basic subgroup of A, then B is a basic subgroup of C.*

Proof. Really B is pure in C, since B is pure in A. Let now A/B be divisible. But C/B is pure in A/B ([4], p.137, Lemma 26.1) and hence C/B is divisible, so B is basic in C. So, the lemma is true. ∎

Theorem 25. *Let G be an abelian p-group and $1 \neq B \leq G_r$. The group $V(KB)$ is a basic subgroup of $V(KG)$ iff $B = G$ and G is a direct product of cyclic groups.*

Proof. By Theorem 22, B is basic in G. Let $G = G_d \times G_r$. Hence G_r is pure in G and owing to Lemma 6, $V(KG_r)$ is pure in $V(KG)$. Thus from Lemma 24, $V(KB)$ is basic in $V(KG_r)$, i.e. Theorem 23 does imply that $B = G_r$. Finally $G = G_d \times B$. Similarly to Theorem 22, if $B \neq G$, then

$$V(KG)/V(KB) \cong V((KB)(G/B)) \tag{9}$$

and therefore Lemma 9 is applicable to obtain $V(KG)/V(KB)$ is not a divisible group. This is a contradiction and the theorem is true. ■

R e m a r k 26. Suppose $B' \leq G_r$. Futhermore B' is basic in G iff B' is basic in G_r. In fact, by Lemma 24 the necessary is valid. Now, let B' be basic in G_r. Hence B' is pure in G, since G_r is pure in G. Moreover, the groups G_r/B' and $G/G_r \cong G/B'/G_r/B'$, are divisible, therefore G/B' is divisible [4] and we are done.

As a final, we shall announce that if B is basic in G and R is perfect, then $1 + I(RG; B)$ is basic in $V(RG)$(cf. [3]; this is proved also by N.Nachev). But if $R^{p^i} = R^{p^{i+1}} (i \in N_0)$ and R^{p^i} has no nilpotens, then $1 + I(RG; B) + R(p^i)G$ is basic in $V(RG)$. The proof will be given elsewhere.

References

[1] P. V. D a n c h e v. Sylow p-subgroups of commutative group algebras, *C. R. Acad. Bulg. Sci.*, 46, 1993, **5**, 13–14.

[2] P. V. D a n c h e v. Units in abelian group rings of prime characteristic, *C. R. Acad. Bulg. Sci.*,48, 1995, **8**, 5-8.

[3] P. V. D a n c h e v. Topologically pure and basic subgroups in commutative group rings, *C. R. Acad. Bulg. Sci.*, **48**, 1995, 9-10, 7-10.

[4] L. F u c h s. Infinite Abelian Groups, v.1, *M."Mir"*, 1974. (In Russian).

[5] L. F u c h s. Infinite Abelian Groups, v.2, *M."Mir"*, 1977. (In Russian).

[6] W. M a y. Modular group algebras of simply presented abelian groups, *Proc. Amer. Math. Soc.*, **104**, 1988, 2, 403–409.

[7] W. M a y. The direct factor problem for modular abelian group algebras, *Contemp. Math.*, **93**, 1989, 303–308.

[8] T. Zh. M o l l o v. The multiplicative groups of modular group algebras of primary abelian groups of arbitrary cardinality, *Publ. Math. Debrecen*, **18**,

1971, 9–21. (In Russian).
[9] T. Zh. M o l l o v, N. A. N a c h e v. Ulm-Kaplansky invariants of the group of normalized units of modular group rings of primary abelian groups, *Serdika*, **6**, 1980, 258–263. (In Russian).
[10] N. A. N a c h e v. Torsion completeness of the group of normalized units in modular group rings, *C. R. Acad. Bulg. Sci.* (to appear).

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