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Well-Posedness of Optimization Problems and Measurable Functions

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In this paper we consider the maximization problem and the problems of differentiability of m-convex and of p-convex functionals on Banach lattices of measurable functions. Conceptually, this article is a continuation of investigations in [7-11].

0. Introduction

Let X be a Tykhonov space, $C(X)$ be the set of all real-valued bounded and continuous functions on X with the sup-norm. Let $B_0(X) = C(X)$, and inductively define $B_\alpha(X)$ for each ordinal $\alpha < \omega_1$ to be the space of bounded pointwise limits of sequences of functions in $\cup\{B_\xi(X) : \xi < \alpha\}$. The functions in

$$B_{\omega_1}(X) = \cup\{B_\alpha(X) : \alpha < \omega_1\}$$

are called bounded Baire functions (see: 4, 5, 6, 12, 16, 25, 28).

Foe every $\alpha < \omega_1$ the space $B_\alpha(X)$ is a Banach lattice.

Every bounded function $f : X \longrightarrow R$ determines a maximization problem which we denote by (X, f): find $x_0 \in X$ such that $f(x_0) = \max\{f(x) : x \in X\}$. A maximization problem (X, f) is called Tykhonov well-posed if every maximizing sequence $\{x_n : n \in N = \{1, 2, \ldots\}\}$, i.e. $\lim f(x_n) = \sup\{f(x) : x \in X\}$, converges in X. Hence, if x_0 is a solution of the Tykhonov well-posed maximization problem, i.e. $f(x_0) = \max\{f(x) : x \in X\}$, and $\{x_n : n \in N\}$ is a maximizing sequence, then $\lim f(x_n) = f(x_0)$ and $x_n \to x_0$.

Let E be a Banach sublattice of $B_{\omega_1}(X)$ and $B_1(X) \subseteq E$.

In the present paper we concider the following questions.

QUESTION 1. When do the Tykhonov well-posed problems (X, f), where $f \in E$, form a "big" (in the Baire category sence) subset of the space E?

QUESTION 2. When is the sup-norm in E Fréchet differentiable at the points of a dense subset of E?

QUESTION 3. When is the sup-norm in E Gâteaux differentiable at the points of a dense subset or of a dense G_δ-subset of E?

The case $E = C(X)$ was examined in [2, 5-11, 13-15, 17-24, 26-33].

The Question 1 for the spase E of upper semicontinuous functions on X was considered in [32].

In particular, we consider the following questions.

QUESTION 4. When is $C(X)$ a weakly Asplund space?

QUESTION 5. When is $C(X)$ a GDS?

QUESTION 6. Let $C(X)$ be a GDS. Is $C(X)$ a weak Asplund space?

QUESTION 7. Is the class $\{X : X$ is compact and $C(X)$ is a weak Asplund space$\}$ finitly or countably multiplicative?

QUESTION 8. Is the class $\{X : X$ is compact and $C(X)$ is a GDS $\}$ finitly or countably multiplicative?

Well-posedness of optimization problems contains the following components: existence of the solution; uniqueness of the solution; continuous dependence of the solution on the data determining the problem. Generic well-posedness means that the "majority" (in some concrete sence) of the problems from a given class of problems are well-posed (see: [21, 3, 7-11, 13-15, 17-20, 22, 24, 26, 27, 29-33, 35-36]).

We state that the solution of the Question 2 and 3 depends only on the Baire topology (in general, on the topology T_E) on the space X and the solution of the Question depends on the prescribed topology on X too.

All spaces are considered to be Tykhonov. We shall use the notation and terminology from [16, 31, 34]. In particular, βX is the Stone-Čech compactification of the space X, $Cl\,X$ or $Cl_X H$ denotes the clousure of a set H in X, $\chi(x, X)$ is the pseudocharacter of a point x in X, $N = \{1, 2, \ldots, n, \ldots\}$. The pseudocharacter of a point x in X is countable if $\{x\}$ is a G_δ-set of X.

The symbol R will denote the field of real numbers. The vector spaces are considered over R. A normed complete vector space is called a Banach space.

A Banach algebra is a Banach space E which is also a ring such that it satisfies the following conditions:

1. $(\alpha x)y = x(\alpha y) = \alpha(xy)$ and $xy = yx$ for $\alpha \in R$, $x, y \in E$;
2. $||\alpha x|| = |\alpha|\,||x||$, $||xy|| \leq ||x||\,||y||$ for $\alpha \in R$, $x, y \in E$;
3. In E there exists a unit element $1 \in E$ such that $||1|| = 1$ and $x\,1 = x$ for every $x \in E$.

Let E be a Banach algebra. For every $\alpha \in R$ we consider $\alpha = \alpha 1 \in E$, i.e. $R \subseteq E$ and $||\alpha|| = |\alpha|$ for every $\alpha \in R \subseteq E$.

A Banach lattice is a Banach algebra E which is also a lattice satisfying the following conditions:

4. if $|x| \leq |y|$, then $||x|| \leq ||y||$, where $|x| = x \vee (-x) = x^+ + x^-$, $x^+ = x \vee 0$ and $x^- = -(x \wedge 0)$;
5. if $|x| \leq |y|$, and $x, y, z \in E$, then $x + z \leq y + z$;
6. if $|x| \leq |y|$, and $\alpha > 0$, then $\alpha x \leq \alpha y$.

A functional $\varphi : E \longrightarrow R$ on a Banach lattice E is convex if

$$\varphi(\alpha x + (1-\alpha)y) \leq \alpha\,\varphi(x) + (1-\alpha)\varphi(y)$$

for every $x, y \in E$ and $\alpha \in [0,1]$.

A functional $\psi : E \longrightarrow R$ is sublinear if $\psi(x+y) \leq \psi(x) + \psi(y)$ and $\psi(\alpha x) = \alpha\psi(x)$ for every $x, y \in E$ and $\alpha > 0$.

Every sublinear functional is convex (see [31], [34]).

1. On m-convex and p-convex functionals

Fix a Banach lattice E.

A functional $\varphi : E \longrightarrow R$ is called m-convex (monotonically convex) if it satisfies the following conditions:

M1. $\varphi(x + \alpha) = \varphi(x) + \alpha$ for $x \in E$ and $\alpha \in R$;

M2. $\varphi(nx) = n\varphi(x)$ for $x \in E$ and $n \in N$;

M3. $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$ for $x, y \in E$;

Proposition 1.1. *Every m-convex functional $\varphi : E \longrightarrow R$ on E satisfies the folloing conditions:*

M4. $\varphi(0) = 0$.

M5. $\varphi(\alpha) = \alpha$ *for* $\alpha \in R \subseteq E$;

M6. If $x, y \in E$ *and* $x \leq y$, *then* $\varphi(x) \leq \varphi(y)$;

Proof. Let $b = \varphi(0)$. Then $b = \varphi(0) = \varphi(20) = 2\varphi(0) = 2b$ and $b = 0$. If $\alpha \in R \subseteq E$, then $\varphi(\alpha) = \varphi(\alpha + 0) = \varphi(0) + \alpha = \alpha$. If $x, y \in E$ and $x \leq y$, than $y = x \vee y$ and $\varphi(x) \leq \varphi(x) \vee \varphi(y) = \varphi(x \vee y) = \varphi(y)$. The proof is complete. ∎

A functional $\varphi : E \longrightarrow R$ on a Banach lattice E is called p-convex (positive convex) if it satisfies the following conditions:

P1. $\varphi(x) = \varphi(|x|)$ for all $x \in E$;

P2. $\varphi(nx) = n\varphi(x)$ for $x \in E$ and $n \in N$;

P3. If $x, y \in E$, $x \geq 0$ and $y \geq 0$, then $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$;

P4. If $x \in E$, $\alpha \in R$, $x \geq 0$ and $\alpha \geq 0$, then $\varphi(x + \alpha) = \varphi(x) + \alpha$;

Proposition 1.2. *Every p-convex functional* $\varphi : E \longrightarrow R$ *satisfies the followihg conditions:*

P5 $\varphi(0) = 0$;

P6 $\varphi(\alpha) = |\alpha|$ *for* $\alpha \in R \subseteq E$;

P7 If $x, y \in E$ *and* $0 \leq x \leq y$, *then* $\varphi(x) \leq \varphi(y)$;

Proof. Similar to the proof of Proposition 1.1. ∎

Proposition 1.3. *Let* $\varphi : E \longrightarrow R$ *be an m-convex functional on* E. *Then the functional* $\Phi(x) = \varphi(|x|)$ *is p-convex.*

Proof. Obvious ∎

2. Spaces to functions

Let $B(S)$ be the Banach lattice of all bounded functions from the non-empty set S into R with the Wierstrass-Chebyshev norm $||f|| = max\{|f(x)| : x \in S\}$.

If S is a subspace of $B(S)$, then T_E is the topology on S generated by E and it has a base consisting of all sets of the form $\cap\{f_i^{-1}U_i : i = 1, 2, \ldots, n\}$ where $n \in N$, $f_1 \ldots, f_n \in E$ and $U_1, \ldots, U_n$ are open subsets of R. The topology

T_E is the coarsest topology on S such that all functions of E are continuous. The space E separates the set S if for each pair of distinct points x, y of S there exists a function $f \in E$ such that $f(x) \neq f(y)$. If E separates the set S, than (S, T_E) is a Tykhonov space.

Let a subspace E of $B(S)$ separate the set S. For every $f \in E$ we denote $i_S(f) = inf\{f(x) : x \in S\}$ and $m_S(f) = max\{f(x) : x \in S\}$. Then $f(S) \subseteq [i_S(f), m_S(f)] = R(f)$ and the mapping $\pi_E : S \longrightarrow R^E$, where $\pi_E(x) = \{f(x) : f \in E\}$, is an embeding of (S, T_E) in R^E. The closure $c_E S$ of the set $S = \pi_E(S)$ in R^E is a compactification of the space (S, T_E) and $c_E S$ is a subset of the set $\Pi\{R(f) : f \in E\}$.

A subspace E of $B(S)$ will be called a complete Banach lattice of functions on a set S if E contains all constant functions, separates the set S and E is a Banach sublattice of the Banach lattice $B(S)$.

Let the subspaces E and F of $B(S)$ separate the set S. The symbol $c_E S \geq c_F S$ means that there exists a continuous mapping $h : c_E S \longrightarrow c_F S$ such that $h(x) = x$ for every $x \in S$.

Property 2.1. *Let the subspace E of $B(S)$ separate the set S. Then $c_E S$ is the smallest compactification on the space (S, T_E) such that all functions of E are continuously extandable over $c_E S$.*

Proof. Obviuos. ∎

Property 2.2. *Let $F \subseteq E \subseteq B(S)$ and F separate the set S. Then $c_E S \geq c_F S$.*

Proof. Obviuos. ∎

Property 2.3. *Let E be a complete Banach lattice of functions on a set S. Then the operator $u : C(c_E S) \longrightarrow B(S)$, where $u(f) = f|S$, is an isomorphism of the Banach lattice $C(c_E S)$ onto the Banach lattice E.*

Proof. Follows from Property 2.1 and the Wierstrass-Stone Theorem ([16],p. 191; [34], p. 115). ∎

Property 2.4. *Let $e_E(f)$ be a continuous extension of the function $f \in E$ over $c_E S$ and E be a complete Banach lattice of functions on a set S. Then $e_E : E \xrightarrow{onto} C(c_E S)$ is an isomorphism.*

Proof. Follows from Property 2.3. ∎

3. On m-convex and p-convex functionals over spaces of functions

Fix a non-empty set S and a complete Banach lattice F of finctions on a set S.

Proposition 3.1. *Let $\emptyset \neq Y \subseteq S$. Then:*

1. *The functional*
$$m_Y : F \longrightarrow R,$$
where $m_Y(f) = \sup\{f(y) : y \in Y\}$, is sublinear and m-covex.

2. *The functional*
$$n_Y : F \longrightarrow R,$$
where $n_y(f) = \sup\{\,|f(y)| \,:\, y \subseteq Y\}$, is sublinear and p-convex.

Proof. Obvious. ∎

The family γ of subset of S is multiplicative if for all $H_1, H_2 \in \gamma$ we have $H_1 \cap H_2 \in \gamma$ and $H_1 \neq \emptyset$.

Corollary 3.2. *Let γ be a multiplicative family of subsets of S. Then:*

1. *The functional*
$$m_\gamma : F \longrightarrow R,$$
where $m_\gamma(f) = \inf\{\, m_Y(f) : Y \in \gamma\}$, is sublenear and m-convex.

2. *The functional*
$$n_\gamma : F \longrightarrow R,$$
where $n_\gamma(f) = \inf\{n_Y(f) : Y \in \gamma\}$, is sublinear and p-convex.

Proposition 3.3. *Let S be a compact space, $F = C(S)$, γ be a multiplicatiive family of closed subsets of S and $Y = \cap\{H : H \in \gamma\}$. Then $Y \neq \emptyset$, $m_Y = m_\gamma$ and $n_Y = n_\gamma$.*

Proof. It is clear that $m_\gamma(f) \geq m_Y(f)$ and $n_\gamma \geq n_Y(f)$ for every $f \in F$. Let $m_\gamma(h) > m_Y(h)$ for some $h \in F$. Denote $g = h - i_S(h) + 1$. Then $g > 0$ and $n_\gamma(g) = m_\gamma(g) > m_Y(g) = n_Y(g)$. Fix an open subset U of S such that $Y \subseteq U \subseteq \{x \in S : g(x) \leq m_\gamma(g)\}$. Then $H \setminus U \neq \emptyset$ for every $H \in \gamma$ and $Y \supseteq \cap\{H \setminus U : H \in \gamma\} \neq \emptyset$. Hence $Y \setminus U = \emptyset$. This is contradiction. The proof is complete. ∎

Lemma 3.4. *Let $\varphi : F \longrightarrow R$ be an m-convex functional. Then $i_S(f) \leq \varphi(f) \leq m_S(f)$ for every $f \in F$.*

Proof. Follows from the inequalities $i_S(f) \leq f \leq m_S(f)$ and Properties M5 and M6. ∎

Lemma 3.5. *Let $\varphi : F \longrightarrow R$ be an p-convex functional. Then $0 \leq \varphi(f) \leq ||f||$ for every $f \in F$.*

Proof. Follows from Properties P1, P5 and P7. ∎

Lemma 3.6. *Let S be a compact space, $F = C(S)$, $\varphi : F \longrightarrow R$ be an m-convex functional, $h \in F$ and $H = \{x \in S : h(x) \leq \varphi(x)\}$. Then $H \neq \emptyset$ and $\varphi(f) \leq m_H(f)$ for every $f \in F$.*

Proof. Let $g = (h - \varphi(h)) \vee 0$. Then $g \geq 0$, $\varphi(g) = 0$ and $H = g^{-1}(0)$. Suppose that $m_H(f_1) < \varphi(f_1)$ for some $f_1 \in F$. Let $f_1 > 0$, $m_H(f_1) = 0$ and $\varphi(f_1) = b$. There exists $\varepsilon > 0$ and an open set U such that $H \subseteq U$ and $b > \varepsilon + m_U(f_1)$. For some $k \in N$ we have $kg(x) > f_1(x) + b$ for every $x \in S \setminus U$. It is clear that $\varphi(kg) = m_H(kg) = \varphi(g) = 0$. Let $\delta = max\{f_1(x) : x \in U\} = m_U(f_1)$. Then $\varphi(\delta) = \delta < b$. By construction $f_2 = \delta \vee kg > f_1$, $\varphi(f_2) = \varphi(\delta \vee kg) = \delta \vee 0 = \delta < b = \varphi(f_1)$. From Property M6 it follows, that $\varphi(f_2) \geq \varphi(f_1)$. The proof is complete. ∎

Lemma 3.7. *Let $\varphi : F \longrightarrow R$ be an m-convex functional, $\delta > 0, h \in F$ and $H = \{x \in S : h(x) < \varphi(h) + \delta\}$. Then $H \neq \emptyset$ and $\varphi(f) \leq m_H(f)$ for every $f \in F$.*

Proof. Let $X = c_F S$, $Y = \{x \in X : e_F(h)(x) \leq \varphi(h)\}$. Then $Y \subseteq Cl_X H$ and $m_Y(e_F(f)) \leq m_H(f)$ for every $f \in F$. From Lemma 3.6 it follows that $\varphi(f) \leq m_Y(e_F(f))$ for every $f \in F$. The proof is complete. ∎

Lemma 3.8. *Let $\varphi : F \longrightarrow R$ be a p-convex functional, $\delta > 0$, $h \in F$ and $H = \{x \in S : |h(x)| \leq \varphi(x) + \delta\}$. Then $H \neq \emptyset$ and $\varphi(f) \leq m_H(f)$ for every $f \in F$.*

Proof. It is sufficient to consider the case $S = c_F S$ and $F = C(c_F S)$. Let $g = (|h| \vee \varphi(h)) - \varphi(h)$. Then $\varphi(g) = 0, g \geq 0$ and $\emptyset \neq Y = g^{-1}(0) = \{x \in S : g(x) \leq \varphi(g)\} \subseteq H$. By construction, $n_Y(g) = \varphi(g) = 0$. Suppose that $n_Y(f_1) < \varphi(f_1) = d$ for some $f_1 \in F$ and $f_1 > 0$. There exists $\varepsilon > 0$ and an open set U such that $Y \subseteq U$ and $d > \varepsilon + m_U(f_1)$. For some $k \in N$ we have $kg(x) > f_1(x) + d$ for every $x \in S \setminus U$. Let $b = m_U(f_1) = n_U(f_1)$. It is clear, that $\varphi(kg) = n_Y(kg) = 0$.

By construction, $f_2 = b \vee kg > f_1$ and $n_Y(f_2) = n_Y(b \vee kg) = b = \varphi(b \vee kg) = b < d = \varphi(f_1)$. From Property P7 it follows that $\varphi(f_2) > \varphi(f_1)$. Hence $\varphi(f) \leq n_Y(f) \leq n_H(f)$ for every $f \in F$. ■

Proposition 3.9. *Let $\varphi : F \longrightarrow R$ be an m-convex functional. Then $\varphi = m_\eta$ for some multiplicative family η of closed subsets of the space (S, T_F).*

Proof. For every $f \in F$ and $\delta \in R$ we denote $Y_{f,\delta} = \{x \in S : f(x) \leq \varphi(f) + \delta\}$. If $\delta < \varepsilon$, then $Y_{f,\delta} \subseteq Y_{f,\varepsilon}$. It is clear, that $Y_{f,\delta} \neq \emptyset$ for every $\delta > 0$. Let $Z_f = \{x \in X = c_F S : e_F(f)(x) \leq \varphi(f)\}$. Then $Z_f = \cap\{Cl_X(Y_{f,\delta}) : \delta > 0\}$. We consider the set $Z = \cap\{Z_f : f \in F\}$, the family $\xi = \{Y_{f,\delta} : f \in F, \delta > 0\}$ and the family $\eta = \{H_1 \cap H_2 \cap \ldots \cap H_n : H_i \in \xi, i \leq n, n \in N\}$. If $\delta < \varepsilon$ and $h = f \vee g$, then $Y_{h,\delta} \subseteq Y_{f,\delta} \cap Y_{g,\varepsilon}$. Hence for every $H \in \eta$ there exists $f \in F$ and $\delta > 0$ such that $Y_{f,\delta} \subseteq H$. In particular, the family η is multiplicative and $Z \neq \emptyset$. By Proposition 3.3 we have $m_Z(e_F(f)) = m_\eta(f)$ for every $f \in F$. From construction of the family η it follows that $\varphi(f) \geq m_\eta(f)$ for every $f \in F$. By virtue of Lemma 3.7, we have $\varphi(f) \leq m_H(f)$ for every $f \in F$ and $N \in \eta$. Hence, $\varphi(f) = m_\eta(f)$ for all $f \in F$. The proof is complete. ■

Corollary 3.10. *Let $\varphi : F \longrightarrow R$ be an m-convex functional. Then the functional φ is convex and there exists a closed subset $S(\varphi)$ of the space $c_F S$ such that $\varphi(f) = m_{S(\varphi)}(e_F(f))$ for every $f \in F$.*

Corollary 3.11. *(N.S.Kukushkin [23]). Let X be a compact space and $\varphi : C(X) \longrightarrow R$ be an m-convex functional. Then $\varphi = m_Y$ for some closed subset Y of X.*

Proposition 3.12. *Let $\varphi : F \longrightarrow R$ be a p-convex functional. Then $\varphi = n_\xi$ for some multiplicative family ξ of the closed subsets of the space (S, T_F).*

Proof. Analogous to the proof of Proposition 3.9. ■

Corollary 3.13. *Let $\varphi : F \longrightarrow R$ be a p-convex functional. Then the functional φ is convex and there exists a closed subset $S(\varphi)$ of the space $c_F S$ such that $\varphi(f) = n_{S(\varphi)}(e_F(f))$ for every $f \in F$.*

Proposition 3.14. *The following assertions are equivalent:*

1. (s, T_F) is a compact space.

2. *For every m-convex functional $\varphi : F \longrightarrow R$ there exists a subset Y of S such that $\varphi = m_Y$.*

3. *For every m-convex functional $\varphi : F \longrightarrow R$ there exists a closed subset Y of the space (S, T_F) such that $\varphi = m_Y$.*

4. *For every p-convex functional $\varphi : F \longrightarrow R$ there exists a subset Y of S such that $\varphi = n_Y$.*

5. *For every p-convex functional $\varphi : F \longrightarrow R$ there exists a closed subset Y of the space (S, T_F) such that $\varphi = n_Y$.*

Proof. Let $Z \subseteq S$ and Y be a closure of Z in (S, T_F). Then $m_Z = m_Y$ and $n_Z = n_Y$.

Suppose that the space (S, T_F) is not compact. Fix a point $x_0 \in c_F S \setminus S$. Consider the m-convex functional $\varphi(f) = e_F(f)(x_0)$ and the p-convex functional $\psi(f) = |e_F(f)(x_0)|$. Then $\varphi \neq m_Y$ and $\psi \neq n_Y$ for every non-empty subset Y of S. The proof is complete. ■

QUESTION 3.15. Let E be a Banach lattice and $\varphi : E \longrightarrow R$ be an m-convex or a p-convex functional. Is it true that φ is sublinear or convex?

4. Differentiability of functionals

Fix a non-empty set S, a complete Banach lattice F of functions on the set S and a functional $\varphi : F \to R$.

Denote by $G(\varphi, F)$ the set of points of Gâteaux differentiability of the functional φ and by $F(\varphi, F)$ the set of points of Fréchet differentiability of the functional φ.

Let φ be an m-convex functional. From Corollary 3.10 it follows that there exists a unique closed subset $S(\varphi)$ of $c_F S$ such that $\varphi(f) = m_{S(\varphi)}(e_F(f)) = \sup\{e_F(f)(x) : x \in S(\varphi)\}$ for every $f \in F$. For every $f \in F$ we denote

$$S(\varphi, f) = \{x \in S(\varphi) : e_F(f)(x) = \varphi(f)\},$$

$$\varphi\prime(f) = \sup\{e_F(f)(x) : x \in S(\varphi) \setminus S(\varphi, f)\}.$$

Let φ be a p-convex functional. From Corollary 3.13 it follows that there exists a unique closed subset $S(\varphi)$ of $c_F S$ such that $\varphi(f) = \sup\{|e_F(f)(x)| : x \in S(\varphi)\} = n_{S(\varphi)}(f)$ for every $f \in F$. For every $f \in F$ we put

$$S(\varphi, f) = \{x \in S(\varphi) : |e_F(f)(x)| = \varphi(f)\}$$

$$\varphi''(f) = \sup\{|e_F(f)(x)| : x \in S(\varphi) \setminus S(\varphi, f)\}.$$

From results of [8-10], Property 2.3 and Corollaries 3.10 and 3.13 we have the following assertions.

Proposition 4.1. *The m-convex functional $\varphi : F \to R$ is Gâteaux differentiable at $f \in F$ if and only if $S(\varphi, f)$ is a singleton set.*

Proposition 4.2. *The p-convex functional $\varphi : F \longrightarrow R$ is Gâteaux differentiable at $f \in F$ if and only if $S(\varphi, f)$ is a singleton set.*

Proposition 4.3. *The m-convex functional $\varphi : F \longrightarrow R$ is Fréchet differentiable at $f \in F$ if and only if $\varphi'(f) < \varphi(f)$ and $S(\varphi, f)$ is a singleton set.*

Proposition 4.4. *The p-convex functional $\varphi : F \longrightarrow R$ is Fréchet differentiable at $f \in F$ if and only if $\varphi''(f) < \varphi(f)$ and $S(\varphi, f)$ is a singleton set.*

Proposition 4.5. *Let $\varphi : F \longrightarrow R$ be an m-convex or a p-convex functional. Then the following statements are equivalent:*

1. *The set $G(\varphi, F)$ is dense in F.*
2. *The set $\Omega(\varphi, F) = \{f \in F : S(\varphi, f)$ is a singleton set$\}$ is dense in $S(\varphi)$.*
3. *There exists a dense first-countable subspace of $S(\varphi)$.*

Proposition 4.6. *Let $\varphi : F \longrightarrow R$ be an m-convex or a p-convex functional. Then the following statements are equivalent:*

1. *The set $G(\varphi, F)$ contains a dense G_δ-subset of F.*
2. *The space $S(\varphi, F)$ contains a dense subset which is completely metrizable.*

Proposition 4.7. *Let $\varphi : F \longrightarrow R$ be an m-convex or a p-convex functional. Then the following statements are equivalent:*

1. *The set $F(\varphi, F)$ is dense in F.*
2. *The set $F(\varphi, F)$ is dense and open in F.*

3. The set of the isolated points of the space $S(\varphi)$ is dense in $S(\varphi)$.

E x a m p l e 4.8. Let Y and $Z = [0,1]\backslash Y$ be dense subspaces of the space $[0,1]$. Let $X = \beta Y$, $S = \beta Y \setminus Y$ and $F = \{f|S : f \in C(X)\}$. Then $c_F S = X$ and the set Y is dense in $c_F S$. Hence the set $G(m_S, F) \cap G(n_S, F)$ is dense in F and $F(m_S, F) = F(n_S, F) = \emptyset$. If Y is a G_δ-set in $[0,1]$, then $G(m_S, F)$ and $G(n_S, F)$ are G_δ-subsets of F. If Y is the space of rational numbers of $[0,1]$, then $G(m_S, F) \cup G(n_S, F)$ does not contain a dense G_δ-subset of F.

E x a m p l e 4.9. Let X be an infinite discrete space, Z be an infinite subset of $X, Y = Cl_{\beta X} Z \setminus X$ and $F = C(X)$. Consider the functionals $\varphi(f) = m_Y(e_F(f))$ and $\psi(f) = n_Y(e_F(f))$. Then $S(\varphi) = S(\psi) = Y$. From Propositions 4.2 and 4.3 $G(\varphi, F) = G(\psi, F) = \emptyset$.

5. Baire topologies and well-posed maximization problems

For each topological space X let PX be the set X with the topology generated by the G_δ-sets in X. The topology of the space PX is called the Baire topology of the space X. The family $\{f^{-1}U : f \in B_1(X), U$ is a closed subset of $R\}$ of Baire sets of class 1 and the family $\{f^{-1}U : f \in B_{\omega_1}(X), U$ is a closed subset of $R\}$ of all Baire sets of the space X form the bases for the topological space PX. If $B_1(X) \subseteq F \subseteq B_{\omega_1}(X)$ and F is a Banach lattice, then F is a complete Banach lattice of Barie functions on the space X and T_F is the topology of the space PX.

Let $b_\alpha PX = c_{B_\alpha(X)}X$ for every $\alpha \leq \omega_1$. The space $b_\alpha PX$ is called the Baire compactification of PX of class α.

If X contains a non-empty perfect compact subspace and $\alpha < \beta$, then $b_\alpha PX < b_\beta PX$ and $b_\alpha PX \neq b_\beta PX$.

The sequence $\{H_n : n \in N\}$ of subsets of the space X is called point-convergent in X if $H = \cap\{H_n : n \in N\}$ is a singleton subset and for every open sey $U \supseteq H$ in X we have $H_n \subseteq U$ for some $n \in N$.

Fix a space X, a complete Banach lattice L of functions on the space X and a non-empty closed subspace Y of the space (X, T_L). Every $f \in L$ determines a maximization problem (Y, f) : "find $y_0 \in Y$ such that $f(y_0) = \sup\{f(y) : y \in Y\}$". Such a point y_0 will be called a solution of (Y, f). The maximization problem (Y, f) is Tykhonov well-posed if every maximizing sequence $\{y_n \in Y : n \in N\}$, i.e. $\lim f(y_n) = \sup\{f(y) : y \in Y\}$, converges to a solution of (Y, f).

The metric characterization of the Tykhonov well-posedness was obtained in [17] by M. Furi and A. Vignoli.

Proposition 5.1. *(see [10], Proposition 1.5). For the maximization problem (Y, f) in the space X the following assertions are equivalent:*

1. *The problem (Y, f) is Tykhonov well-posed and $y_0 \in Y$ is a solution of (Y, f).*

2. *The sequence $\{H_n(f, Y) = Y \cap (f^{-1}[m_Y(f) - 2^{-n}, m_Y(f)]) : n \in N\}$ is convergent in X and $\{y_0\} = \cap\{H_n(f, Y) : n \in N\}$.*

Proof. Let U be an open subset of the space X, $H = \cap\{H_n(f, Y) : n \in N\} \subseteq U$ and $y_n \in Y \cup (H_n(f, Y) \setminus U)$. Then $\{y_n : n \in N\}$ is a maximizing sequence of (Y, f), $\lim f(y_n) = f(y_0)$ and $\lim y_n \neq y_0$. The implication 2. $\longrightarrow$ 1. is obvious. ∎

The compact set Φ of X is a Baire set if and only if Φ is a G_δ-set in X. Hence Proposition 5.1 implies.

Corollary 5.2. *Let $Y \subseteq X$, $f \in B_{\omega_1}(X)$, (Y, f) be a Tykhonov well-posed problem and $y_0 \in Y$ be a solution of (Y, f). Then $\{y_0\}$ is a G_δ-subset of the subspace Y of the space X.*

Example 5.3. Let (X, d) be a metric space, $d(x, y) < 1$ for every $x, y \in X$, H be a non-empty subset of X and $f_H(x) = 1 - \inf\{d(x, y) : y \in H\}$. Then $Cl\, H$ is a set of solutions of the maximizing problem (X, f_H). The problem (X, f_H) is Tykhonov well-posed if and only if H is a singleton set. It is clear that $f_H \in C(X)$.

Example 5.4. Let x_0 be a non-isolated point of a space X and the character $\chi(x_0, X)$ of a point x_0 in X be countable. Fix an ordinal number $0 < \alpha < \omega_1$. Then tnere exist a countable base $\{U_n : n \in N\}$ for X at the point x_0 and a sequence of non-empty Baire sets $\{V_n : n \in N\} \subseteq \{f^{-1}(0) : f \in B_\alpha(X)\}$ of class α such that $V_n \subseteq U_n \setminus U_{n+1}$ and $U_n \setminus V_n \neq \emptyset$ for every $n \in N$. We consider the function $g : X \longrightarrow R$ such that $g(x_0) = 1$, $g^{-1}(0) = X \setminus \cap\{V_n \cap \{x_0\} : n \in N\}$ and $g^{-1}(1 - 2^{-n}) = V_n$ for every $n \in N$. Then $g \in B_\alpha(X) \setminus C(X)$, the maximization problem (X, g) is Tykhonov well-posed and x_0 is a solution of (X, g).

Example 5.5. Let $x_0 \in X$, $g(x_0) = 1$ and $g^{-1}(0) = X \setminus \{x_0\}$. Then the maximization problem (X, g) is Tykhonov well-posed and x_0 is a solution of (X, g). If the pseudocharacter of a point x_0 in X is countable, then $g \in B_1(X)$. If the pseudocharacter of a point x_0 in X is uncountable, then $g \in B(X) \setminus B_{\omega_1}(X)$.

Proposition 5.6. *Let X be a space, L be a Banach sublattice of $B(X)$ and $B_1(X) \subseteq L$. Then:*

1. *Every compact subset of the space $S = (X, T_L)$ is finite.*

2. *If $\{x_n : n \in N\}$ is a convergent sequence of the space S, then there exists $n \in N$ such that $x_m = x_n$ for every $m > n$.*

3. *If Y is an infinite countable subset of S, then the set Y is closed and descrete in the space S and Y contains an infinite subset Z such that the closure of Z in $c_L(X) = bX$ is homeomorphic to the Stone-Čech compactification βZ of the descrete space Z. Moreover, if $B_2(X) \subseteq L$, then the closure of Y in $c_L(X)$ is homeomorphic to the Stone-Čech compactification βY of the descrete space Y.*

4. *If Y is an infinite subspace of the space S and Z is a closure of Y in $c_L(X)$, then for every point $z \in Z \setminus Y$ the character $\chi(z, Z)$ of the point z in the space Z is uncountable.*

Proof. Let $Y = \{y_n : n \in N\}$ be an infinite subset of S and $y_n = y_m$ implies $n = m$. For every $n \in N$ there exists a closed G_δ-subset Φ_n of the space X such that:

(i). $y_n \in \Phi_n$ for every $n \in N$.

(ii). If $n < m$, then $\Phi_n \cap \Phi_m = \emptyset$.

Then $\{\Phi_n : n \in N\}$ is a descrete family of open and closed subsets of the spaces PX and S. The mapping $id_X : S \longrightarrow PX$, where $id_X(x) = x$ for all $x \in X = S = PX$, is continuous. Let $H \subseteq N$. We consider the function $f_H : X \longrightarrow R$, where $f_H^{-1}(1) = \cup\{\Phi_n : n \in H\}$ and $f_H^{-1}(0) = X \setminus \cup\{\Phi_n : n \in H\}$. It is clear that $f_H \in B_2(X)$. Hence if $B_2(X) \subseteq L$, then every pair of disjoint subsets of Y has disjoint closures in bX and the closure of Y in bX is homeomorphic to βY.

We put $Z(X) = \{f^{-1}(0) : f \in C(X)\}$ and $CZ(X) = \{X \setminus U : U \in Z(X)\}$.

There exists a sequence $\{U_m \in CZ(X) : m \in N\}$ such that:

(iii). $U_m \cap Y \neq \emptyset$ and $Y \setminus U_m$ is an infinite set for every $m \in N$.

(iv). $U_m \cap U_n = \emptyset$ if $m \neq n$.

Then $\{U_m : m \in N\}$ is a descrete family of open and closed subsets of PX and S and $U = \cup\{U_m : m \in N\}$ is open and closed in PX and S. Fix the points $z_n = y_{m_n} \in U_n \cap Y$. Let $Z = \{y_n : n \in N\}$. If $H \subseteq N$, then we consider the function $g_H : X \longrightarrow R$, where $g_H^{-1}(1) = \cup\{U_m : m \in H\}$ and $g_H^{-1}(0) = X \setminus g_H^{-1}(1)$. By construction, $g_H \in L$ for every subset H of N. Hence every pair of disjoint closed subsets of Z has disjoint closures in bX and βZ is homeomorphic to the closure of Z in bX. The assertions 1 and 3 are proved. The assertions 2 and 4 follow from the assertion 3. The proof is complete. ∎

Corollary 5.7. *Let X be an infinite space, L be a Banach sublattice of $B(X)$ and $B_1(X) \subseteq L$. Then there exists an m-convex functional $\varphi : L \longrightarrow R$ and a p-convex functional $\psi : L \longrightarrow R$ for which $G(\varphi, L) = G(\psi, L) = \emptyset$.*

Proof. There exists an infinite subset Y of X such that the closure H of Y in $c_L X$ is homeomorphic to βN and Y is a closed subset of (X, T_L). Let $Z = H \setminus Y$. Then Z is a compact subset of $c_L X$ and Z is a homeomorphic to $\beta N \setminus N$. Consider the functionals $\varphi(f) = m_Z(e_L(f))$ and $\psi(f) = n_Z(e_L(f))$. Then $S(\varphi) = S(\psi) = Z$. From Proposition 4.2 it follows that $G(\varphi, L) = G(\psi, L) = \emptyset$.
∎

Proposition 5.8. *Let $L \subseteq F \subseteq B(S)$, L be a complete Banach lattice of functions on S, $X = c_L S$, $Y = c_F S$ and $\hat{f} \in F$ for every $f \in L$, where $\hat{f}(x) = 0$ if $x \in f^{-1}[i_S(f), 0]$ and $\hat{f}(x) = 1$ if $f(x) > 0$. Then for every open F_δ-set U of the space X the set $Cl_Y(U \cap S)$ is open in Y.*

Proof. Let U be an open F_δ-subset of X. Then there exists a continuous function $g : X \longrightarrow [0,1]$ such that $X \setminus U = g^{-1}(0)$. Let $f = g|S$. Then $f \in L$ and $\hat{f} \in F$. The function $e_F(\hat{f}) = h$ is continuous on Y, $h(Y) \subseteq \{0,1\}$ and $h^{-1}(1) = Cl_Y(U \cap S)$. The proof is complete. ∎

Corollary 5.9. *Let X be a space and $\alpha < \beta$. Then for every open F_δ-subset U of $b_\alpha PX$ the set $Cl_{b_\beta PX}(U \cap X)$ is open in $b_\beta PX$.*

Corollary 5.10. *Let X be a space. Then the closure of every open F_σ-subset of $b_{w_1} PX$ is open.*

Example 5.11. Let Y_1 and Y_2 be countable dense subsets of the Čech complete space X, $Y_1 \cap Y_2 = \emptyset$, $Y = Y_1 \cup Y_2$ and $L = B_1(X)$. Then Y is a descrete closed subspace of the space PX and $Cl_{c_L X} Y_1 \cap Cl_{c_L X} Y_2 \neq \emptyset$. Hence the closure of Y in $c_L X$ is not homeomorphic to the Stone-Čech compactification βY of the descrete space Y and the last part of assertion 3 from Proposition 5.6 is not valid for $L = B_1(X)$.

QUESTION 5.12. Let $B_1(X) \subseteq L$. Is every convergent sequence of the space $c_L X$ trivial?

6. The solution of the main questions in the spaces of Baire functions

Fix a Tikhonov space X and a Banach lattice L of functions on a space X such that $B_1(X) \subseteq L \subseteq B_{\omega_1}(X)$.

Lemma 6.1. *Let $Y \subseteq X$. Then $G(m_Y, L) = F(m_Y, L)$ and $G(n_Y, L) = F(n_Y, L)$.*

Proof. Let Z be the closure of Y in $c_L X$. Then $m_Y(f) = m_Z(e_L(f))$ and $n_Y(f) = n_Z(e_L(f))$ for every $f \in L$. Let $f \in L$ and $S(m_Y, f)$ be a singleton subset. Then $S(m_Y, f)$ is a G_δ-set of Z. By Proposition 5.6 we have $S(m_Y, f) \subseteq Y$. Hence $S(m_Y, f)$ is an open subset of the space Z, the set $H = Z \setminus S(m_Y, f)$ is closed in Z and $f(y) < m_Y(f)$ for every $y \in H$. Therefore $m'_Y(f) = m_H(e_L(f)) < m_Y(f)$ and from Propositions 4.1 and 4.3 it follows that $f \in G(m_Y, L) \cap F(m_Y, L)$.

If $S(n_Y, F)$ is a singleton subset, then $S(n_Y, f)$ is an open subset of Z and $n''_Y(f) < n_Y(f)$. The Propositions 4.2 and 4.4 imply that $f \in G(n_Y, L) \cap F(n_Y, L)$.

These facts yield the following assertion too.

Lemma 6.2. *Let $T(Y, L) = \{(Y, f) : f \in L,\ (Y, f)$ is a Tykhonov well-posed maximization problem$\}$. Then $G(m_Y, L) \subseteq T(Y, L)$.*

Theorem 6.3. *For every non-empty subspace Y of the space PX the following assertions are equivalent:*

1. *The set $G(m_Y, L)$ is dense in L.*
2. *The set $G(n_Y, L)$ is dense in L.*
3. *The set $G(m_Y, L)$ is dense and open in L.*
4. *The set $G(n_Y, L)$ is dense and open in L.*
5. *The set $F(m_Y, L)$ is dense and open in L.*
6. *The set $G(n_Y, L)$ is dense and open in L.*
7. *The set $T(Y, L)$ is dense in L.*
8. *The set $T(Y, L)$ contains an open and dense subset of L.*
9. *Every non-empty G_δ-subset of Y contains a singleton G_δ-subset of Y.*

10. The set of isolated points of Y is dense in Y.

Proof. The implications $1 \longrightarrow 3 \longrightarrow 5 \longrightarrow 10 \longrightarrow 1$ and $2 \longrightarrow 4 \longrightarrow 6 \longrightarrow 10 \longrightarrow 2$ follow from Lemma 6.1 and Proposition 4.1 – 4.7. The implication $3 \longrightarrow 8$ follows from Lemma 6.1. The implications $8 \longrightarrow 7$ and $9 \longrightarrow 10 \longrightarrow 9$ are obvious.

Let $f \in L$, $(Y, f) \in T(Y, Y)$ and $y_0 \in Y$ be the solution of the problem (Y, f). Then $\{y_0\}$ is a G_δ-subset of the space Y. There exists a function $g \in B_1(X)$ such that $g(y_0) = 1$ and $g(y) = 0$ for every $y \in Y \setminus \{y_0\}$. Let $f_n = f + 2^{-n}g$. Then $f = lim f_n$ and $f_n \in F(m_Y, L)$ for every $n \in N$. Hence $F(m_Y, L)$ is a dense subset of $T(Y, L)$. This proves the implication $7 \longrightarrow 5$. The proof is complete. ■

Corollary 6.4. *The following statements are equivalent:*

1. *The set $G(m_X, L)$ is dense in L.*
2. *The set $G(n_X, L)$ is dense in L.*
3. *The set $T(X, L)$ is dense in L.*
4. *The set $G(m_X, L)$ is dense and open in L.*
5. *The set $G(n_X, L)$ is dense and open in L.*
6. *The set $F(m_X, L)$ is dense and open in L.*
7. *The set $F(n_X, L)$ is dense and open in L.*
8. *The set $T(X, L)$ contains an open and dense subset of L.*
9. *Every non-empty G_δ-subset of the space X contains a singleton G_δ-subset of X.*
10. *Every non-empty G_δ-subset of the space PX contains an isolated point of the space PX.*
11. *The set of isolated points of PX is dense in PX.*

Corollary 6.5. *Let Y be a non-empty subspace of the space PX. Then $G(m_Y, L) \subseteq G(m_Y, B_{\omega_1}(X))$ and $G(n_Y, L) \subseteq G(n_Y, B_{\omega_1}(X))$.*

Example 6.6. If X is a first countable non-descrete space, then the space PX is descrete and $T(X, B_\alpha(X)) \setminus G(m_X, B_{\omega_1}(X)) \neq \emptyset$ for every $\alpha > 0$.

Example 6.7. Let $X = [0,1]$ and $B_1(X) \subseteq L$. Then $G(m_X, C(X)) \subseteq T(X, L)$ and $C(X) \cap G(m_X, L) \neq \emptyset$.

CONSTRUCTION 6.8. Fix a space Y and the compactification bY of Y. For every ordinal number α on the set $\Omega(\alpha) = \{\beta < \alpha : \beta$ is an ordinal number$\}$ consider the topology induced by the order. Consider the subspace $\Pi(Y, bY) = (bY \times \Omega(\omega_1)) \cup (Y \times \{\omega_1\})$ of the space $bY \times \Omega(\omega_1 + 1)$. The space $bY \times \Omega(\omega_1)$ is pseudocompact and $\beta\Omega(\omega_1) = \Omega(\omega_1 + 1)$. Hence, by the I.Gliksberg Theorem ([16], p.298), $\beta\Pi(Y, bY) = bY \times \Omega(\omega_1 + 1)$. The subspace $Y \times \{\omega_1\} = Y$ is closed in $X = \Pi(Y, bY)$ and $S(m_Y) = S(n_Y) = bY \times \{\omega_1\}$.

Example 6.9. There exist a Tykhonov space X and a closed subspace Y of X such that:

1. *Y is homeomorphic to the space Q of rational numbers of the space $Z = [0,1]$.*
2. *$S(m_Y) = S(n_Y) = Z$.*
3. *$F(m_Y, C(X)) = F(n_Y, C(X)) = \emptyset$.*
4. *$G(m_Y, C(X))$ and $G(n_Y, C(X))$ are dense G_δ-subsets of $C(X)$.*
5. *The set $T(Y, C(X))$ is dense in $C(X)$ and it is of the first Baire category in $C(X)$.*

Let $Y = Q$, $bY = Z = [0,1]$ and $X = \Pi(Y, bY)$. The space X and its subspace $Y = Y \times \{\omega_1\}$ are constructed

Example 6.10. There exist a Tykhonov space X and a descrete closed subspace Y of X such that:

1. *$F(m_Y, C(X)) = T(Y, C(X)) \subseteq G(m_Y, C(X))$.*
2. *$F(m_Y, C(X)) \neq G(m_Y, C(X))$.*
3. *$F(m_Y, C(X))$ is an open and dense subset of $C(X)$.*
4. *$S(m_Y)$ is a metrizable compact space.*

Suppose that bY is a metrizable compactification of a countable descrete space $Y = N$ and $Y = Y \times \{\omega_1\} \subseteq \Pi(Y, bY) = X$. The subspace Y is closed and descrete. Hence $F(m_Y, C(X)) = T(Y, C(X)) = \{f \in C(X) : S(m_Y, f)$ is a singleton subset of $Y\}$. By construction, $c_{C(X)}X = bY \times \Omega(\omega_1 + 1)$ and $S(m_Y) = bY \times \{\omega_1\}$. There exists a function $f \in C(X)$ such that $S(m_Y, f)$ is a singleton subset of $bY \setminus Y$. Therefore $f \in G(m_Y, C(X)) \setminus F(m_Y, C(X))$.

7. The space of Borel measurable functions

Let X be a Tykhonov space.

The field of the Borel subsets of X is the smallest σ-field of subsets containing all open and all closed subsets of X.

The function $f : X \longrightarrow R$ is Borel measurable if $f^{-1}H$ is a Borel subset of X for every open subset H of R. Every Baire measurable function is Borel measurable. If X is a perfectly normal space, then every Borel measurable function is Baire measurable.

Denote by $M(X)$ the set of all Borel measurable functions of the space X. It is clear that $M(X)$ is a complete Banach lattice of functions on the set X and $B_{\omega_1}(X) \subseteq M(X)$.

Let MX be the set X with the topology $T_{M(X)}$. The space MX is descrete. For every countable space Y of the space MX the closure of Y in $c_{M(X)}(X)$ is homeomorphic to the Stone-Čech compactification βY of the descrete space Y.

Theorem 7.1. *Let Y be a non-empty subspace of the space X. Then:*

1. $G(m_Y, M(X)) = F(m_Y, M(X)) \subseteq T(Y, M(X))$.
2. $G(n_Y, M(X)) = F(n_Y, M(X))$.
3. *The sets $F(m_Y, M(X))$ and $F(n_Y, M(X))$ are open and dense in the space $M(X)$.*

Proof. The space MX is descrete. Let Z be the closure of the set Y in $c_{M(X)}X$. If H is a singleton G_δ-subset of the space Z, then $H \subseteq Y$. The set Y is descrete and dense in Z. The proof is complete. ■

Corollary 7.2. *Let L be a complete Banach lattice of functions on a space X, $M(X) \subseteq L$ and Y be a non-empty subspace of the space MX. Then:*

1. *The space (X, T_L) is descrete.*
2. $G(m_Y, L) = F(m_Y, L) \subseteq T(Y, L)$.
3. $G(n_Y, L) = F(n_Y, L)$.
4. *The sets $F(m_Y, L)$ and $F(n_Y, L)$ are open and dense in L.*
5. *If the set Y is countable, then the Stone-Čech commpactification βY of the descrete space Y coincides with the closure of Y in $c_L X$.*

Proposition 5.8 yields.

Corollary 7.3. *The closure of every open F_σ-subset of the space $c_{M(X)}X$ is open.*

Remark 7.4. *Let $X = [0,1]$. Then $PX = MX$ is a descrete space, $M(X) = B_{\omega_1}(X)$, $c_{M(X)}X \neq \beta MX$ and for every countable subspace Y of MX the Stone-Čech compactification βY of Y coincides with the closure of Y in $c_{M(X)}X$.*

8. The case of Lindelöf spaces

Proposition 8.1. *Let PX be a Lindelöf space. Then:*

1. *X is a Lindelöf space.*
2. *$B_1(X) = B_{\omega_1}(X) = C(PX)$.*
3. *The set $f(X)$ is countable for every $f \in C(PX)$.*

Proof. It is clear that $B_{\omega_1}(X) \subseteq C(PX)$ for every space X. Let $f \in C(PX)$. For every point $x \in X$ there exists a closed G_δ-subset Φx of the space X such that $x \in \Phi x \subseteq f^{-1}(f(x))$. The open cover $\{\Phi x : x \in X\}$ of the space PX contains some countable subcover $\{\Phi x_n : n \in N\}$. Hence $f(X) = \{f(x_n) : n \in N\}$ is a countable set and $f^{-1}U = \cup\{\Phi x_n : f(x_n) \in U\}$ is a F_σ-set for every subset U of R, i.e. $f \in B_1(X)$. The proof is complete. ∎

Interesting results in this direction were obtained in [18] by J.E.Jayne. In [18, 25] it was proved that the scattered space X is Lindelöf if and only if the space PX is Lindelöf. A space X is scattered if every non-empty subspace Y of X contains at least one isolated point.

QUESTION 8.2. Let X be a Lindelöf space, PX be a paracompact space and $C(PX) = B_{\omega_1}(X)$. Is it true that PX is a Lindelöf space ?

QUESTION 8.3. Let X be a metric space and $C(PX) = B_{\omega_1}(X)$. Is it true that X is a σ-descrete space ?

9. The case of locally compact non-compact space

Fix a locally compact non-compact space X.

Let $f : X \longrightarrow R$ be any function. The limit of f as $x \in X$ tends to infinity is the element $b = lim\{f(x) : x \longrightarrow \text{infinity}\}$ satisfying the following condition : for every $\varepsilon > 0$ there exists a compact subset Φ of X such that $|f(x) - b| < \varepsilon$ for every $x \in X \setminus \Phi$.

By $C_b(X)$ we denote the set of all continuous functions f on X tending to the number $b \in R$ as x tends to infinity. Let $C_\infty(X) = \cup\{C_b(X) : b \in R\}$. The space $C_\infty(X)$ is a complete Banach lattice of continuous functions on X. The space $C_0(X)$ is a Banach space, a ring without unity and a lattice.

The compactification $c_{C_0(X)}X = c_{C_\infty(X)}X$ is the one-point Alexandrov compactification of the space X (see [16, 31, 34]).

If $L \subseteq C(X)$ and $F = L \cup C_0(X)$, then $c_L X = c_F X$ is the Constantinesku–Cornea comactification of the space X generated by L (see [4], Chapter 13).

By $C'_{ks}(X)$ we denote the set of all continuous functions $f \in C(X)$ satisfying the following condition: there exist a compact subset Φ of X and a disjoint family $\{U_\mu : \mu \in M\}$ of open subsets of X such that $X \setminus \Phi \subseteq \cup\{U_\mu : \mu \in M\}$ and $f|U_\mu$ is a constant function for every $\mu \in M$. Let $G_{ks}(X)$ be the closure of $C'_{ks}(X)$ in $C(X)$. Then $C_\infty(X) \subseteq C_{ks}(X)$, $C_{ks}(X)$ is a complete Banach lattice of continuous functions on X and $ksX = c_{C_{ks}(X)}X$ is the Kerekjarto-Stoilow compactification of the space X (see [4], Chapter 13).

E x a m p l e 9.1. Let N be the descrete space of the natural numbers and αN be the one-point Alexandrov compactification of the space N. Then $c_0 = C_0(N), l_\infty = C(N) = B(N)$, $c_{C_\infty(N)}N = \alpha N$ and $C(\alpha N) = C_\infty(N)$.

Proposition 9.2. *If $dim(\beta X \setminus X) = 0$, then $ksX = \beta X$.*

P r o o f. Let A and B be a pair of completely separated subsets of the space X. Then the sets $A_1 = Cl_{\beta X}A$ and $B_1 = Cl_{\beta X}B$ are disjoint and there exist the open F_σ-subsets V, W of the space βX such that $A \subseteq V$, $B \subseteq W$ $\beta X \setminus X \subseteq V \cup W$ and $Cl V \cap Cl W = \emptyset$. Then there exists a continuous function $f : \beta X \longrightarrow [0,1]$ such that $V \subseteq f^{-1}(0)$ and $W \subseteq f^{-1}(1)$. Then $g = f|X \in C'_{ks}X$, $A \subseteq g^{-1}(0)$ and $B \subseteq g^{-1}(1)$. The proof is complete. ∎

E x a m p l e 9.3. The space ksR is homeomorphic to the space [0,1] and $ksR \setminus R$ is the two-point set.

E x a m p l e 9.4. If $n > 1$, then ksR^n is the one-point Alexandrov compactification of the space R^n.

E x a m p l e 9.5. (see [8,9] and [31], E x a m p l e 1.21). Let $N_n = \{m \in N : m > n\}$ and $\xi = \{N_n : n \in N\}$. Then ξ is a multiplicative family of closed subsets of the descrete space N, $n_\xi(f) = lim\{sup\{|f(m)| : m > n\} : n \longrightarrow \text{infinity}\} = \lim n_{N_m}(f)$ and $S(n_\xi) = \beta N \setminus N$. Hence the functional $n_\xi : l_\infty = C(N) \longrightarrow R$ is nowhere Câteaux diferentiable.

Theorem 9.6. *Let Y be a non-empty compact subset of the space X. Then $G(n_Y, C(Y)) = \{f|Y : f \in G(n_Y, C_0(X))\}$, $G(m_Y, C(Y)) = (f|Y : f \in G(m_Y, C_0(X))\}$, $F(n_Y, C(Y)) = \{f|Y : f \in F(n_Y, C_0(X))\}$, $F(m_Y, C(Y)) = \{f|Y : f \in F(m_Y, C_0(X))\}$.*

Proof. Follows from the equality $C(Y) = \{f|Y : f \in C_0(X)\}$. ∎

Proposition 9.7. *Let Y be a non-compact subset of the space X, $L = \{f \in C_0(X) : f \leq 0$ and $M_Y(f) = \{y \in Y : m_Y(f) = f(y)\} \neq \emptyset\}$ and $H = \{f \in C_0(X) : f \leq 0\}$. Then $H \cap F(m_Y, C_0(X)) = \emptyset$ and $L \cap G(m_Y, C_0(X)) = \emptyset$.*

Proof. Fix $f \in H$. Let $\varphi : C_0(X) \longrightarrow R$ be the Fréchet differential of m_Y at f. By construction, $m_Y(g) = 0$ and $\varphi(g) = 0$ for every $g \in H$. Hence $\varphi(g) = 0$ for every $g \in C_0(X)$. There exist a sequence $Z = \{y_n \in Y : n \in N\}$ and a sequence $\{f_n \in C_0(X) : n \in N\}$ such that $Z \setminus F \neq \emptyset$ for every compact subset F of X, $f_n(y_n) = |f(y_n)| + 2^{-n} = ||f_n||$ and $-2^{-2n} < f(y_n) \leq 0$ for every $n \in N$. It is clear that

$$\lim \frac{m_Y(f + f_n) - m_Y(f)}{||f_n||} \geq 2^{-1}$$

Therefore $f \notin F(m_Y, C_0(X))$.

Let $g \in L$. Fix $y_0 \in M_Y(g)$. There exists a continous function $h : X \longrightarrow [0, 1]$ such that $h(y_0) = 1$ and $f \in C_0(X)$. Then $m_Y(g + th) = 0$ if $t < 1$ and $m_Y(g + th) = t$ if $t > 0$. Hence

$$\lim \frac{m_Y(g + th) - m_Y(g)}{t} = \begin{cases} 0 & \text{if} \quad t < 0, \\ 1 & \text{if} \quad t > 0 \end{cases}$$

and $g \notin L$. The proof is complete. ∎

Theorem 9.8. *Let Y be a non-empty closed subspace of the space $X, f \in C_0(X)$ and $M_Y(f) = \{y \in Y : f(y) = m_Y(f)\}$. Then:*

1. *$\{f \in C_0(X) : M_Y(f) = \emptyset$ or $m_Y(f) > 0$ and $M_Y(f)$ is a singleton subset of $Y\} = G(m_Y, C_0(X)\}$.*

2. *$\{f \in C_0(X) : M_Y(|f|)$ is a singleton subset $\} = G(n_Y, C_0(X))$.*

3. *$\{f \in C_0(X) : M_Y(f)$ is an open singleton subset of the space Y and $m_Y(f) > 0\} = F(m_Y, C_0(X))$.*

4. $\{f \in C_0(X) : M_Y(|f|)$ *is an open singleton subset of the space* Y *and* $n_Y(f) > 0\} = F(n_Y, C_0(X))$.

Proof. Obviuos. ■

Corollary 9.9. *Let* Y *be a non-empty closed subspace of the space* X. *Then:*

1. *The set* $G(n_Y, C_0(X)) \cap G(m_Y, C_0(X))$ *is dense in* $C_0(X)$ *if and only if* Y *contains a dense first-countable subspace.*
2. *The sets* $G(n_Y, C_0(X))$ *and* $G(m_Y, C_0(X))$ *contain a dense* G_δ*-subset of* $C_0(X)$ *if and only if* Y *contains a dense subspace which is completely metrizable.*

Corollary 9.10. *Let* Y *be a non-empty closed subspace of the space* X. *Then the following statements are equivalent:*

1. *The set* $F(n_Y, C_0(X)) \cup F(m_Y, C_0(X))$ *is dense in* $C_0(X)$.
2. *The sets* $F(n_Y, C_0(X))$ *and* $F(m_Y, C_0(X))$ *are open and dense in* $C_0(X)$.
3. *The set of the isolated points of the space* Y *is dense in the space* Y.

Example 9.11. Let Γ be a descrete infinite space and $X = R \times \Gamma$. Then $ksX = \beta(ksR \times \Gamma)$. The families $\xi = \{\Phi \subseteq X : \Phi$ is closed in X and $Cl_X(X \setminus \Gamma)$ is compact $\}$ and $\eta = \{R \times (\Gamma \setminus H) : H$ is a finite subset of $\Gamma\}$ are multiplicative. Let $Y = ksX \setminus X$ $Z = ksX \setminus (ksR \times \Gamma)$ and $S = Y \setminus Z$. Then S is a descrete dense subspace of the space Y, the character of Z is not countable at any point of Z, $S(m_\xi) = S(n_\xi) = Y$ and $S(m_\eta) = S(n_\eta) = Z$. Hence the sets $F(m_\xi, C_{ks}(X))$ and $F(n_\xi, C_{ks}(X))$ are open and dense in $C_{ks}(X)$ and $G(n_\eta, C_{ks}(X)) = G(m_\eta, C_{ks}(X)) = \emptyset$.

By $C_\omega(X)$ we denote the set of all continuos functions $f \in C(X)$ satisfying the following condition: there exists a σ-compact closed subset Φ of X such that $f|(X \setminus \Phi)$ is a constant function. Then $C(X) \subseteq C_\omega(X)$ and $C_\omega(X)$ is a complete Banach lattice of continuous functions on X. Let $\beta\omega X = c_{C_\omega(X)}X$.

If X is a noramal space, then for every Lindelöf closed subspace Y of X the closure of Y in $\beta\omega X$ is the Stone-Čech compactification βY of the space Y.

For every normal space X the set $\beta\omega_0 X = \cup\{Cl_{\beta\omega X}Y : Y$ is a Lindelöf closed subspace of $X\} = \cup\{Cl_{\beta\omega X}Y : Y$ is a σ-compact closed subspace of $X\}$

is open in $\beta\omega X$, $\beta\omega X \setminus \beta\omega_0 X$ is a singleton set, $X \subseteq \beta\omega_0 X \subseteq \beta X$ and $\beta\omega_0 X$ is open in βX.

Theorem 9.12. *Let X be a paracompact locally compact non-compact space, Y be a non-compact closed subspace of X and $Z = Cl_{\beta\omega X} Y \setminus Y$. Then:*

1. *If $z \in Z$, then the character $\chi(z, Z)$ is uncountable.*
2. $G(m_Z, C(\beta\omega X)) = G(n_Z, C(\beta\omega X)) = \emptyset$.
3. *If Y is metrizable, then $G(m_Y, C_\omega(X))$ and $G(n_Y, C_\omega(X))$ are dense G_δ-subsets of $C_\omega(X)$.*
4. *If the set of the isolated points of Y is dense in Y, then $F(m_Y, C_\omega(X))$ and $F(n_Y, C_\omega(X))$ are dense open subsets of the space $C_\omega(X)$.*
5. *If X is descrete, then for every open F_δ-subset U of the space $\beta\omega X$ the set $Cl_{\beta\omega X} Y$ is open in $\beta\omega X$.*
6. $\beta\omega X = Cl_{\beta\omega X} Y$.

Proof. If $z \in Z \cap \beta\omega_0 X$, then $z \in Cl_{\beta\omega X} H$ for some closed Lindelöf subspace H of Y and $\chi(z, Z)$ is uncountable.

Let $\{z\} = Z \setminus \beta\omega_0 X$. If $\chi(z, Z)$ is countable, then Y is a Lindelöf space and $Z \subseteq \beta\omega_0 X$. The assertions $1 - 4$ are proved. The assertions 5 and 6 are obviuos. ■

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