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Difference Methods for Nonlinear First Order Partial Differential Equations with Mixed Initial Boundary Conditions

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This paper deals with the differential-functional problem

$$(i)\qquad \begin{aligned} &D_x z(x,y) = f(x,y,z_{(x,y)}, D_y z(x,y)),\\ &z(x,y) = \varphi(x,y) \text{ for } (x,y) \in E_0 \cup \partial_0 E, \end{aligned}$$

where $D_y z = (D_{y_1} z, \ldots, D_{y_n} z)$ and $E_0 \cup \partial_0 E$ is an initial-boundary set.

The corresponding one-step difference method is of the form

$$(ii)\qquad \begin{aligned} &\delta_0 z^{(m)} = \Phi_h\left(x^{(m_0)}, y^{(m')}, z_{(m)}, \delta z^{(m)}\right),\\ &z^{(m)} = \varphi_h^{(m)} \text{ on } E_0[h] \cup \partial_0 E[h], \end{aligned}$$

where δ_0 and δ are difference operators. We give sufficient conditions for the convergence of a sequence $\{u_h\}$ of solutions of problem (ii) to a solution of (i). We assume that Φ_h satisfies a nonlinear estimate of Perron type with respect to the functional argument. The proof of the convergence of the difference method is based on recurrent inequalities theorems.

We also provide multistep difference methods for (i). A numerical example is given.

1. Introduction

The problems of finite difference approximation for initial and initial-boundary value problems of first order partial differential or differential-functional equations were considered by many authors and under various assumptions. The main problem of this research is to find a suitable difference equation which satisfies some consistency conditions with respect to the original problem and is stable.

The papers [4], [5], [8] initiated the discussion of difference methods for nonlinear equations with first order partial derivatives. Numerical treatment of the Cauchy problem for differential-functional equations can be found in [9], [10]. The problem of finite difference approximations for initial-boundary problems were considered in [2], [3]. The theory of difference methods for weak solutions of quasi-linear hyperbolic systems in two independent variables was treated in [12].

An error estimate implying the convergence of the difference schemes is obtained in [2]-[5], [9], [10] by difference inequalities methods or by simple recurrent inequalities theorems. All these results are proved under the assumption that the right-hand sides of equations satisfy the Lipschitz condition with respect to the functional argument.

In this paper we extend the results of [2] to the case of nonlinear Perron-type estimates with respect to the functional argument. We also provide multi-step difference methods for mixed differential-functional equations.

To illustrate our results, we give an example of the method of the second order for the mixed problem.

In this research we take advantage of the general ideas for finite difference approximations which were introduced in [6], [7], [11].

2. Differential-functional problem.

We denote by $C(X,Y)$ the class of all continuous functions from X into Y, where X and Y are metric spaces.

Let k' and n be given integers with $1 \le k' \le n$.

Let $E = [0,a] \times [-b,b]$ where $a > 0$, $b = (b_1 \dots b_n)$, $b_i > 0$ for $1 \le i \le n$. Suppose that $\tau_0 \in R_+$, $R_+ = [0,+\infty)$, and $\tau = (\tau_1 \dots \tau_n) \in R^n_+$.

We define

$$D = [-\tau_0, 0] \times [0,\tau_1] \times \dots \times [0,\tau_{k'}] \times [-\tau_{k'+1}, 0] \times [-\tau_n, 0]$$

and $c = (c_1 \dots c_n)$, $d = (d_1 \dots d_n)$ with $c_i = -b_i$, $d_i = b_i + \tau_i$ for $1 \le i \le k'$ and $c_i = -b_i - \tau_i$, $d_i = b_i$ for $k'+1 \le i \le n$.

We put $\Omega = E \times C(D,R) \times R^n$ and

$$E^* = [0,a] \times [-b_1,b_1) \times \dots \times [-b_{k'}, b_{k'}) \times (-b_{k'+1}, b_{k'+1}] \times \dots \times (-b_n, b_n].$$

$$E_0 = [-\tau_0, 0] \times [c,d], \quad \partial_0 E = ([0,a] \times [c,d]) \setminus E^*, \quad B = [-\tau_0, a] \times [c,d].$$

For every function $z : B \to R$ and every $(x,y) \in E$ we will consider the function $z_{(x,y)} : D \to R$ defined by $z_{(x,y)}(t,s) = z(x+t, y+s)$, $(t,s) \in D$.

Finally, let $f : \Omega \to R$ and $\varphi : E_0 \cup \partial_0 E \to R$ be given functions.

We take into consideration the following mixed differential-functional problem

$$(1)\qquad \begin{aligned} &D_x z(x,y) = f(x,y,z_{(x,y)}, D_y z(x,y)),\\ &z(x,y) = \varphi(x,y) \quad \text{for } (x,y) \in E_0 \cup \partial_0 E, \end{aligned}$$

where $D_y z(x,y) = (D_{y_1} z(x,y), ..., D_{y_n} z(x,y))$. We consider classical solutions of problem (1). More precisely, a function $z \in C(B,R)$ is a solution of (1) if it admits partial derivatives $D_x z(x,y)$, $D_y z(x,y)$ for every $(x,y) \in E^*$, satisfies the differential-functional equation on E^* and the initial-boundary condition on $E_0 \cup \partial_0 E$.

R e m a r k 1. We wish to recall that in [1], [2] we already took into consideration an analogous mixed problem. Actually, the functional argument here considered is more general than those adopted in [1], [2], however all the results given there can be easily carried over to the present setting. We only state here a result concerning the uniqueness and the continuous dependence of the solutions.

Lemma 1. *Suppose that*

1° the function $f : \Omega \to R$ of the variables (x,y,w,q) is continuous and admits continuous partial derivatives $D_q f = (D_{q_1} f, ... D_{q_n} f)$ on Ω which are constant in sign, more precisely:

$$D_{q_i} f(x,y,w,q) \geq 0 \text{ for } 1 \leq i \leq k', \ D_{q_i} f(x,y,w,q) \leq 0 \text{ for } k'+1 \leq i \leq n,$$

2° there exists a function $\sigma \in C(R_+ \times \mathbb{R}_+, R_+)$ such that
(i) σ is non-decreasing with respect to the second argument
(ii) $\sigma(x,0) = 0$ for $x \in R_+$ and $\eta(x) = 0$ is the unique solution of problem

$$\eta'(x) = \sigma(x, \eta(x)), \ \eta(0) = 0,$$

(iii) f satisfies the following Perron – type equation

$$|f(x,y,w,q) - f(x,y,\bar{w},q)| \leq \sigma(x, \|w - \bar{w}\|_0) \text{ on } \Omega,$$

where $\|\cdot\|_0$ is the supremum norm in $C(D,R)$.

Under these assumptions the solution of (1) is unique and it depends continuously on the initial-boundary conditions.

The proof of Lemma 1 is based on the differential inequalities methods developed in [1].

3. The finite-difference approximation

We now introduce a mesh in the set B. Let $e = (e_0, e_1, ..., e_n)$ be fixed with $e_i > 0$ for $1 \le i \le n$. Assume that for a given $h = (h_0, h_1, ..., h_n) \in (0, e]$, M_0 and $\tilde{M} = (\tilde{M}_1 \dots \tilde{M}_n)$ exist such that $\tilde{M}_i$, $i = 1, ..., n$ and M_0 are integers and $\tilde{M}_i h_i = \tau_i$, $i = 1, ..., n$, $M_0 h_0 = \tau_0$. Denote by I_e the set of all the constants $h \in (0, e]$ having the above property. We define the nodal points as follows

$$y_i^{(m_i)} = m_i h_i,\ i = 1, ..., n,\ x^{(m_0)} = m_0 h_0,$$

where m_i, $0 \le i \le n$, are integers. Let $m = (m_0, m')$, $m' = (m_1 \dots m_n)$. Then $K = (K_1 \dots K_n)$ and $N = (N_1 \dots N_n)$ exist, where K_i and N_i are natural numbers, such that

$$N_i h_i < b_i \le (N_i + 1)h_i,\ -(K_i + 1)h_i < -b_i \le -K_i h_i, \quad \text{for } i = 1, ..., k'$$

$$N_i h_i < b_i < (N_i + 1)h_i,\ -(K_i + 1)h_i \le -b_i < -K_i h_i, \quad \text{for } i = k' + 1, ..., n.$$

Let $M = (M_1 \dots M_m)$ where

$$M_i = N_i + \tilde{M}_i \text{ if } (N_i + 1)h_i > b_i,\ M_i = N_i + \tilde{M}_i + 1 \text{ if } (N_i + 1)h_i = b_i,\ 1 \le i \le k',$$

and

$$-M_i = -K_i - \tilde{M}_i \text{ if } -(K_i + 1)h_i < b_i,\ -M_i = -K_i - \tilde{M}_i - 1$$
$$\text{if } -(K_i + 1)h_i = -b_i, \quad k' + 1 \le i \le n.$$

Let N_0 be a natural number such that $N_0 h_0 \le a < (N_0 + 1)h_0$. We define

$$E_0[h] = \{(x^{(m_0)}, y^{(m')}) : -M_0 \le m_0 \le 0, -K_i \le m_i \le M_i \text{ for } 1 \le i \le k'$$
$$\text{and } -M_i \le m_i \le N_i \text{ for } k' + 1 \le i \le n\}$$

$\partial_0 E[h] = \{(x^{(m_0)}, y^{(m')}) \in B[h] : 0 \le m_0 \le N_0$ and there exists i, $1 \le i \le k'$, such that $N_i < m_i \le M_i$ or there exists i, $k' + 1 \le i \le n$, such that $-M_i \le m_i < -K\}$,

$$E[h] = \{(x^{(m_0)}, y^{(m')}) : -0 \le m_0 \le N_0, -K \le m' \le N\}$$

$$.B[h] = E[h] \cup E_0[h] \cup \partial_0 E[h].$$

We also need the mesh in the set D. To this purpose we put:

$$D[h] = \{(x^{(m_0)}, y^{(m')}) : -M_0 \le m_0 \le 0,\ 0 \le m_i \le \tilde{M}_i \text{ for } 1 \le i \le k',$$
$$-\tilde{M}_i \le m_i \le 0 \quad \text{for } k' + 1 \le i \le n\}.$$

If $z : B[h] \to R$ and $(x^{(m_0)}, y^{(m')}) \in E[h]$ then we define $z_{(m)} : D[h] \to R$ by $z_{(m)}(x^{(s_0)}, y^{(s)}) = z(x^{(m_0+s_0)}, y^{(m'+S)}, (s_1 \dots s_n, (x^{(s_0)}, y^{(s)}) \in D[h]$.

Suppose that r is a fixed natural number and $-K \le m' \le N$. Then we define

$$S[m'] = \{s = (s_1 \dots s_n) : -K_i \le m_i + s_i \le M_i \quad \text{for } i = 1, ..., k',$$
$$-M_i \le m_i + s_i \le N_i \quad \text{for } i = k' + 1, ..., n$$
$$\text{and } s_i \in \{-r, -r+1, ..., 0, 1, ..., r\} \text{ for } i = 1, ..., n\}.$$

We introduce the following difference operators A, $\delta = (\delta_1 \dots \delta_n)$ and δ_0.

If $z : B[h] \to R$ and $-K \le m' \le N$, $0 \le m_0 \le N_0$ then

(2)
$$Az^{(m)} = \sum_{s \in S[m']} a_{(s,m)} z^{(m_0, m'+s)}),$$

and
$$\delta_i z^{(m)} = h_i^{-1} \sum_{s \in S[m']} c^{(i)}_{(s,m)} z^{(m_0, m'+s)}), \; i = 1, ..., n,$$

(3)
$$\delta_0 z^{(m)} = h_0^{-1} [z^{(m_0+1, m')} - Az^{(m)}]$$

where $a_{s,m}$, $c^{(i)}_{s.m'} \in R$ are given. We write $\delta z^{(m)} = (\delta_1 z^{(m)}, ..., \delta_n z^{(m)})$. We will approximate the derivatives $D_x z$ and $D_y z$ by means of $\delta_0 z$ and δz respectively. Note that since the coefficients $a_{(s,m)}$ and $c^{(i)}_{(s,m)}$ depend on m, our approximation will depend on the points of the mesh $E[h]$.

We denote by $F(X, Y)$ the class of all the functions defined on X and taking values in Y where X, Y are sets. Let $\Omega_h = E[h] \times F(D[h], R) \times R^n$. Assume that for each $h \in I_e$ we have

$$\Phi_h : \Omega_h \to R \text{ and } \varphi_h : E_0[h] \cup \partial_0 E[h] \to R.$$

We consider the following difference method for problem (1)

$$Z^{(m_0+1, m')} = Az^{(m)} + h_0 \Phi_h \left(x^{(m_0)}, y^{(m')}, z_{(m)}, \delta z^{(m)} \right),$$

(4)
$$0 \le m_0 \le N_0 - 1, \; -K \le m' \le N$$
$$z^{(m)} = \varphi_h^{(m)} \text{ on } E_0 \cup \partial_0 E[h].$$

It is evident that problem (4) has exactly one solution on $B[h]$.

4. Stability of the difference method

Let $|h| = \max\{h_i : 0 \le i \le n\}$. For $w \in F(D[h], R)$, $z \in F(B[h], R)$ we define

$$\|w\|_h = \max\{|w^{(m)}| : (x^{(m_0)}, y^{(m')}) \in D[h]\}.$$

$$\|z\|_{h,m_0} = \max\{|z(x^{(s_0)}, y^{(s)})| : (x^{(s_0)}, y^{(s)}) \in B[h], s_0 \le m_0\}$$

The following assumptions will be adopted in the whole paper.

Assumption H_0. Suppose that

1° $\Phi_h : \Omega_h \to R$, $h \in I_e$ and for each $(x, y, w) \in E[h] \times F(F[h], R)$ we have $\Phi_h(x, y, w, \cdot) \in C(R^n, R)$,

2° the partial derivatives $(D_{q_1}\Phi_h, ..., D_{q_n}\Phi_h) = D_q\Phi_h$ exist on Ω_h and $D_q\Phi_h(x, y, w, \cdot) \in C(R^n, R^n)$ for $(x, y, w) \in E[h] \times F(D[h], R)$,

3° for $0 \le m_0 \le N_0 - 1$, $-K \le m' \le N$, $s \in S[m']$ we have

$$a_{s,m} + h_0 \sum_{j=1}^{n} h_j^{-1} D_{q_j}\Phi_h(P) C_{s,m}^{(j)} \ge 0, \tag{5}$$

where $P = (x^{(m_0)}, y^{(m')}, w, q) \in \Omega_h$

Assumption H_1. Suppose that $\sigma : [0, a_0) \times R_+ \to R_+$ where $a_0 > a$ satisfies the conditions

1° σ is continuous on $[0, a_0) \times R_+$, $\sigma(x, 0) = 0$ for $x \in [0, a_0)$ and $\eta(x) = 0$, $x \in [0, a_0)$, is the unique solution of the problem

$$\eta'(x) = \sigma(x, \eta(x)),\ \eta(0) = 0,$$

2° if $(x, p), (\bar{x}, \bar{p}) \in [0, a) \times R_+$ and $x \le \bar{x}$, $p \le \bar{p}$ then $\sigma(x, p) \le \sigma(\bar{x}, \bar{p})$.

Assumption H_2. Suppose that

1° for $(x, y, w, q) \in \Omega_h$, $\bar{w} \in F(D[h], R)$ we have

$$|\Phi_h(x, y, w, q) - \Omega_h(x, y, \bar{w}, q)| \le \sigma(x, \|w - \bar{w}\|_h),$$

2° the operators A and δ satisfy the conditions

$$\sum_{s \in S[m']} a_{s,m} = 1 \tag{6}$$

$$\sum_{s \in S[m']} c_{s,m}^{(i)} = 0,\ i = 1, ..., n, \tag{7}$$

where $0 \le m_0 \le N_0 - 1$, $-K \le m' \le N$.

Theorem 1. *Suppose that*

1° Assumption $H_0 - H_2$ are satisfied and u_h is a solution of (4),

2° $v_h : B[h] \to R$ is a given function such that $\gamma_0, \gamma : I_e \to R_+$ exist with the property that

$$|v_h^{(m)} - \varphi_h^{(m)}| \le \gamma_0 \text{ on } E_0[h] \cup \partial_0 E[h],$$

(8)

$$\lim_{h\to 0} \gamma_0(h) = 0,$$

and

$$|\delta_0 v_h^{(m)} - \Phi_h\left(x^{(m_0)}, y^{(m')}, (v_h)_{(m)}, \delta v_h^{(m)}\right)| \le \gamma(h),$$

$$0 \le m_0 \le N_0 - 1, -K \le m' \le N, \tag{9}$$

$$\lim_{h\to 0} \gamma(h) = 0,$$

Under these assumptions a constant $\epsilon_0 > 0$ and a function $\omega_h : [0,a] \to R_+$ exist such that for $|h| \le \epsilon_0$, $h \in I_e$ we have

$$\|u_h - v_h\|_{h,i} \le \omega_h(x^{(i)}), \; i = 0, 1, ..., N_0, \tag{10}$$

and $\lim_{h\to 0} \omega_h(x) = 0$ *uniformly with respect to* $x \in [0,a]$.

Proof. Note that a constant $\epsilon_0 > 0$ exists such that for $|h| \le \epsilon_0$ the solution of

$$\eta'(x) = \sigma(x, \eta(x)) + \gamma(h), \quad \eta(0) = \gamma_0(h) \tag{11}$$

is defined on $[0,a]$ and $\lim_{h\to 0} \omega(x) = 0$ uniformly on $[0,a]$. Moreover, for $-K \le m' \le N$, $0 \le m_0 \le N_0 - 1$ we have

$$\begin{aligned}
& |h_h^{(m_0+1,m')} - v_h^{(m_0+1,m')}| \le \\
\le\; & h_0 |\Phi_h\left(x^{(m_0)}, y^{(m')}, (u_h)_{(m)}, \delta u_h^{(m)}\right) - \Phi_h\left(x^{(m_0)}, y^{(m')}, (v_h)_{(m)}, \delta u_h^{(m)}\right)| + \\
+\; & |Au_h^{(m)} - Av_h^{(m)} + h_0 \Phi_h\left(x^{(m_0)}, y^{(m')}, (v_h)_{(m)}, \delta u_h^{(m)}\right) - \\
& -h_0 \Phi_h\left(x^{(m_0)}, y^{(m')}, (v_h)_{(m)}, \delta v_h^{(m)}\right)| + \\
& +h_0 |\delta_0 v_h^{(m)} - \Phi_h\left(x^{(m_0)}, y^{(m')}, (v_h)_{(m)}, \delta v_h^{(m)}\right)| \le \\
\le\; & h_0 \sigma\left(x^{(m_0)}, \|(u_h)_{(m)} - (v_h)_{(m)}\|_h\right) + \\
+\; & |\textstyle\sum_{s\in S[m']} (u_h^{(m_0,m')} - v_h^{(m_0,m')}[a_{s,m} + h_0 \sum_{i=1}^n h_i^{-1} D_{q_i} \Phi_h(\tilde{P}) c_{s,m}^{(i)}]| + h_0 \gamma(h) \le \\
\le\; & h_0 \sigma\left(x^{(m_0)}, \|(u_h)_{(m)} - (v_h)_{(m)}\|_h\right) + \|u_h - v_h\|_{h,m_0} + h_0 \gamma_0(h),
\end{aligned}$$

where $\tilde{P}$ is an intermediate point. Then we deduce that

$$\|u_h - v_h\|_{h,m_0+1} \leq$$
$$\leq \max\left\{\gamma_0(h), h_0\sigma\left(x^{(m_0)}, \|u_h - v_h\|_{h,m_0}\right) + \|u_h - v_h\|_{h,m_0} + h_0\gamma_0(h)\right\},$$
$$m_0 = 0, 1, ..., N_0 - 1.$$

Let us consider the problem

$$\eta_{i+1} = h_0\sigma(x^{(i)}, \eta_i) + \eta_i + h_0\gamma(h), \quad i = 0, 1, ..., N_0 - 1, \; \eta_0 = \gamma_0(h), \tag{12}$$

an denote by $\tilde{\eta}$ its solution. Then we have

$$\|u_h - v_h\|_{h,i} \leq \tilde{\eta}_i, \quad i = 0, 1, ..., N_0.$$

From Assumption H_1 it follows that ω_h is a convex function. Therefore

$$\omega_h^{(i+1)} \geq \omega_h^{(i)} + h_0\sigma(x^{(i)}, \omega_h^{(i)}) + h_0\gamma(h), \quad i = 0, 1, ..., N_0 - 1,$$

and we obtain

$$\|u_h - v_h\|_{h,i} \leq \omega_h^{(i)}, \quad i = 0, 1, ..., N_0,$$

which completes the proof. ∎

5. Convergence of the difference method

Let us consider now the following additional assumptions

Assumption H$_3$. Suppose that

1° $v \in C(B, R)$ is a solution of problem (1) such that its restriction to the set $E \cup \partial_0 E$ is of class C^2,

2° the operators A and δ satisfy the conditions

$$\sum_{s \in S[m']} s_j c_{s,m}^{(i)} = \delta_{ij}, \quad i, j = 1, ..., n, \; -K \leq m' \leq N, \; 0 \leq m_0 \leq N_0 \tag{13}$$

$$\sum_{s \in S[m']} s_j a_{s,m} = 0, \quad j = 1, ..., n, \tag{14}$$

where δ_{ij} is the Kronecker symbol,

3° two constants $\bar{c}$, $d_0 > 0$ exist such that $h_i \leq d_0 h_0$, $h_i h_j^{-1} \leq d_0$, $i, j = 1, ..., n$, and

$$\sum_{s \in S[m']} |a_{s,m}| \leq \bar{c}, \qquad \sum_{s \in S[m']} |c_{s,m}^{(i)}| \leq \bar{c},$$

$$0 \le m_0 \le N_0, \; -K \le m' \le N, \; i = 1, ..., n.$$

Assumption $\mathbf{H_4}$. Suppose that

1° $f \in C(E \times C(D, R) \times R^n, R)$ and there exists $\beta_0 : I_e \to R_+$ such that

(15)
$$|\Phi_h\left(x^{(m_0)}, y^{(m')}, (v_h)_{(m)}, \delta v_h^{(m)}\right) - f\left(x^{(m_0)}, y^{(m')}, v_{(P[m])}, \delta v^{(m)}\right)| \le \beta_0(h),$$

$$0 \le m_0 \le N_0 - 1, \; -K \le m' \le N,$$

where $P[m] = (x^{(m_0)}, y^{(m)}), v_h = v_{|B[h]}$ and

(16) $$\lim_{h \to 0} \beta_0(h) = 0,$$

2° a function $\gamma_0 : I_e \to R_+$ exists such that

$$|\varphi^{(m)} - \varphi_h^{(m)}| \le \gamma_0(h) \text{ on } E_0[h] \cup \partial_0 E[h], \text{ and } \lim_{h \to 0} \gamma_0(h) = 0.$$

Theorem 2. *Suppose that Assumptions $H_1 - H_4$ are satisfied and u_h is a solution of (4). Then a constant $\epsilon_0 > 0$ and a function $\omega_h : [0, a] \to R_+$ exist such that for $|h| \le \epsilon_0$, $h \in I_e$ we have*

$$\|u_h - v_h\|_{h,i} \le \omega_h^{(i)}, \; i = 0, 1, ..., N_0,$$

and $\lim_{h \to 0} \omega_h(x) = 0$ *uniformly on* $[0, a]$.

Proof. We put $P(m, h, t) = \left(x^{(m_0)}, y_1^{(m_1)} + th_1 s_1, ..., y_n^{(m_n)} + th_n s_n\right)$ and

$$R_0^{(m)}(h) = f\left(x^{(m_0)}, y^{(m')}, v_{(P[m])}, \delta v^{(m)}\right) - \Phi_h\left(x^{(m_0)}, y^{(m')}, (v)_{(m)}, \delta v_h^{(m)}\right),$$

$$R_1^{(m)}(h, t) = -(2h_0)^{-1} \sum_{s \in S[m']} \sum_{i,j=1}^{n} s_i s_j h_i h_j D_{y_i y_j} v(P(m, h, t)),$$

$$R_2^{(m)}(h) = f\left(x^{(m_0)}, y^{(m')}, v_{(P[m])}, Dv_y^{(m)}\right) - f\left(x^{(m_0)}, y^{(m')}, \bar{v}_{(m)}, \delta v^{(m)}\right),$$

$$\begin{aligned} R_3^{(m)}(h, t) = \; & f\left(x^{(m_0)} + th_0, y^{(m')}, v_{(Q[m])}, D_y v(x^{(m_0)} + th_0, y^{(m')})\right) - \\ & -f\left(x^{(m_0)}, y^{(m')}, v_{P[m]}, D_y v^{(m)}\right), \; Q[m] = (x^{(m_0)} + th_0, y^{(m')}). \end{aligned}$$

From Assumption H_3 it follows that for every $h \in I_e$, $0 \le m_0 \le N_0 - 1$, $-K \le m' \le N$, two elements $t, t' \in (0, 1)$ exist such that

$$\delta_0 v_h^{(m)} - \Phi_h\left(x^{(m_0)}, y^{(m')}, (v_h)_{(m)}, \delta v_h^{(m)}\right) =$$

$$= R_0^{(m)}(h) + R_1^{(m)}(h,t) + R_2^{(m)}(h) + R_3^{(m)}(h,t').$$

Moreover, the consistency condition (15), (16) implies the estimate

$$|R_0^{(m)}(h)| \le \beta_0(h), \quad 0 \le m_0 \le N_0 - 1, \ -K \le m' \le N, \ h \in I_e.$$

From the assumptions we deduce that three functions $\beta_1, \beta_2, \beta_3 \to R_+$ exist such that

$$|R_1^{(m)}(h,t)| \le \beta_1(h), \ |R_2^{(m)}(h)| \le \beta_2(h), \ |R_3^{(m)}(h,t)| \le \beta_3(h), \ t \in (0,1),$$

where $0 \le m_0 \le N_0 - 1$, $-K \le m' \le N$, and $\gamma(h) = \sum_{i=0}^{3} \gamma_i(h)$ satisfies (9). In force of Theorem 1 we get the assertion. ■

6. Difference methods of the second order

We start with a remark concerning an error estimate for the difference method (4).

R e m a r k 2. Suppose that all the assumptions of Theorem 2 are satisfied and $\sigma(x,p) = Lp$ where $L > 0$. Then we have

$$\omega_h(x) = [\gamma_0(h) + L^{-1}\gamma(h)]e^{Lx} - L^{-1}\gamma(h), \ x \in [0,a].$$

Suppose that $\varphi_h = \varphi$ on $E_0[h] \cup \partial_0 E[h]$ and that there are ν, $C_0 > 0$ such that $\gamma(h) = C_0|h|^\nu$. Then we have the estimate

$$\|u_h - v_h\|_{h,i} \le L^{-1}C_0[e^{La} - 1]|h|^\nu, \ i = 0,1,\dots,N_0. \tag{17}$$

We will give an example of a difference method satisfying (17) with $\nu = 2$. To this purpose let $T_h : F(B[h],R) \to F(B,R)$ be the operator we introduced in [2] which is defined as follows. Let $z_h \in F(B[h],R)$ and $(x,y) = (y_0,y) \in B$. Here we denote by $y_0 = x$ and $y_0^{(m_0)} = x^{(m_0)}$ for $-M_0 \le m_0 \le N_0$. Then there is $(y_0^{(m_0)}, y^{(m')}) = (x^{(m_0)}, y^{(m')})$ such that $y_0^{(m_0)} \le x \le y_0^{(m_0+1)}$, $y^{(m')} \le y \le y^{(m'+1)}$, with $m'+1 = (m_1+1,\dots,m_n+1)$ and $y_0^{(m_0)}, y^{(m')})$, $(y_0^{(m_0+1)}, y^{(m'+1)}) \in B[h]$. We define

$$(T_h z_h)(y_0,y) = \sum_{s\in S^*} z_h^{(m_0+s_0,m'+s')} \left(\frac{y-y^{(m)}}{h}\right)^s \left(1-\frac{y-y^{(m)}}{h}\right)^{1-s} \tag{18}$$

where $S^* = \{s = (s_0,s') = (s_0,s_1,\dots,s_n) : s_i \in \{0,1\}, \ i = 0,1,\dots,n\}$ and

$$\left(\tfrac{y-y^{(m)}}{h}\right)^s = \prod_{i=0}^n \left(\tfrac{y_i-y_i^{(m_i)}}{h}\right)^{s_i}, \tag{19}$$

$$\left(1-\tfrac{y-y^{(m)}}{h}\right)^{1-s} = \prod_{i=0}^n \left(1-\tfrac{y_i-y_i^{(m_i)}}{h_i}\right)^{1-s_i},$$

and we take $0^0 = 1$ in (19).

Let us prove the following lemma.

Lemma 2. *Suppose that* $z : B \to R$ *is of class* C^2 *and* $z_h = z_{|B[h]}$. *Then there exists* $C \in R_+$ *such that*

$$\|T_h z_h - z\|_{C(B)} \le C|h|^2$$

where $\|\cdot\|_{C(B)}$ *is the supremum norm in the space* $C(B,R)$.

Proof. We have

$$\sum_{s\in S^*} \left(\frac{y-y^{(m)}}{h}\right)^s \left(1-\frac{y-y^{(m)}}{h}\right)^{1-s} = 1,$$

and

$$\sum_{s\in S^*} \left(\frac{y-y^{(m)}}{h}\right)^s \left(1-\frac{y-y^{(m)}}{h}\right)^{1-s} h_i s_i = y_i - y_i^{(m_i)}, \quad i = 0,1,\dots,n,$$

where $y^{(m)} \le y \le y^{(m+1)}$. Therefore

$$\begin{aligned}
&\textstyle\sum_{s\in S^*} z^{(m_0+s_0,m'+s')} \left(\frac{y-y^{(m)}}{h}\right)^s \left(1-\frac{y-y^{(m)}}{h}\right)^{1-s} - z(y_0,y) = \\
= &\textstyle\sum_{s\in S^*} \left(\frac{y-y^{(m)}}{h}\right)^s \left(1-\frac{y-y^{(m)}}{h}\right)^{1-s} \Big[z^{(m)} + \sum_{i=0}^n D_{y_i} z^{(m)} h_i s_i + \\
+ &\textstyle\frac{1}{2}\sum_{i,j=0}^n h_i s_i h_j s_j D_{y_i y_j} z(\tilde{P})\Big] - \\
- &\textstyle\Big[z^{(m)} + \sum_{i=0}^n D_{y_i} z^{(m)}(y_i - y^{(m_i)}) + \frac{1}{2}\sum_{i,j=0}^n D_{y_i y_j} z(\tilde{Q})(y_i - y^{(m_i)})(y_j - y^{(m_j)})\Big] = \\
= &\textstyle\frac{1}{2}\sum_{s\in S^*} \left(\frac{y-y^{(m)}}{h}\right)^s \left(1-\frac{y-y^{(m)}}{h}\right)^{1-s} \times \\
\times &\textstyle\sum_{i,j=0}^n h_i s_i h_j s_j D_{y_i y_j} z(\tilde{P}) - \\
- &\textstyle\frac{1}{2}\sum_{i,j=0}^n D_{y_i y_j} z(\tilde{Q})(y_i - y^{(m_i)})(y_j - y^{(m_j)}).
\end{aligned}$$

Now we have the estimate

$$\begin{aligned}
&|(T_h z_h)(x,y) - z(x,y)| \le \\
&\le \sup\left\{|D_{y_i y_j} z(x,y)| : (x,y) \in B,\ i,j = 0,1,\dots,n\right\} \textstyle\sum_{i,j=0}^n h_i h_j
\end{aligned}$$

and the assertion follows. ■

Let us assume now that $n = 1$, $E = [0,a] \times [-b,b]$, $a > 0$, $b > 0$,

$$D = [-\tau_0, 0] \times [0, \tau_1],\ E_0 = [-\tau_0, 0] \times [-b, b+\tau_1],\ \partial_0 E = [0,a] \times [b, b+\tau_1],$$

and consider the problem

$$
\begin{aligned}
&D_x z(x,y) = F(x,y,z_{(x,y)}) + CD_y z(x,y),\\
&z(x,y) = \varphi(x,y) \text{ for } (x,y) \in E_0 \cup \partial_0 E,
\end{aligned} \tag{20}
$$

where $F : E \times C(D,R) \to R$, $\varphi : E_0 \cup \partial_0 E \to R$ and $C > 0$. Let

$$
\begin{aligned}
x^{(i)} &= ih_0, \quad i = -M_0, -M_0+1, ..., 0, 1, ..., N_0,\\
y^{(j)} &= jh_1, \quad j = -M, -M+1, ..., N, N+1, ..., K,
\end{aligned}
$$

where $M_0 h_0 = \tau_0$, $N_0 h_0 \le a < (N_0+1)h_o$, $Mh_1 = b$, $Nh_1 < b \le (N+1)h_1$, $Kh_1 = b + \tau_1$. Assume that $h_1 = Ch_0$. We will consider the operator T_h given by (18) in the two dimensional case.

Suppose that the function F of the variables (x,y,w) is of class C^1 on $E \times C(D,R)$. We consider the difference method

(21)

$$
\begin{aligned}
&z^{(i+1,j)} = Az^{(i,j)} + h_0 F\left(x^{(i)}, y^{(j)}, (T_h z)_{(P[i,j])}\right) + h_0 C\delta z^{(i,j)} +\\
&+\tfrac{1}{2}h_0^2\left[D_x F\left(x^{(i)}, y^{(j)}, (T_h z)_{(P[i,j])}\right) + CD_y F\left(x^{(i)}, y^{(j)}, (T_h z)_{(P[i,j])}\right)\right] +\\
&+\tfrac{1}{2}h_0 D_w F\left(x^{(i)}, y^{(j)}, (T_h z)_{(P[i,j])}\right)(T_h u_h^*)_{(P[i,j])} +\\
&+\tfrac{1}{2}Ch_0 D_w F\left(x^{(i)}, y^{(j)}, (T_h z)_{(P[i,j])}\right)(T_h v_h^*)_{(P[i,j])},
\end{aligned}
$$

$$
i = 0, 1, ..., N_0 - 1, \; j = -M, -M+1, ..., N,
$$

$$
z^{(i,j)} = \varphi^{(i,j)} \text{ on } E_0[h] \cup \partial_0 E[h],
$$

where

$$
P[i,j] = (x^{(i)}, y^{(j)}, \quad Az^{(i,j)} = \frac{1}{2}[z^{(i,j+1)} + z^{(i,j-1)}],
$$

$$
\delta z^{(i,j)} = (2h_1)^{-1}[z^{(i,j+1)} - z^{(i,j-1)}], \; i = 0, 1, ..., N_0 - 1, \; j = -M+1, ..., N,
$$

and

$$
\begin{aligned}
&Az^{(i,j)} = \tfrac{3}{2}z^{(i,j)} - z^{(i,j+1)} + \tfrac{1}{2}z^{(i,j+2)},\\
&\delta z^{(i,j)} = h_1^{-1}[-\tfrac{3}{2}z^{(i,j)} + 2z^{(i,j)} - \tfrac{1}{2}z^{(i,j+2)}] \quad \text{for } 0 \le i \le N_0 - 1, \; J = -M.
\end{aligned}
$$

The functions u_h^* and v_h^* are defined by

$$
\begin{aligned}
&u_h(x^{(i)}, y^{(j)}) =\\
&= h_0 F\left((x^{(i)}, y^{(j)}, (T_h z)_{(P[i,j])}\right) + \tfrac{1}{2}[z^{(i,j+1)} - z^{(i,j-1)}]\\
&v_h^*(x^{(i)}, y^{(j)}) = \tfrac{1}{2}[z^{(i,j+1)} - z^{(i,j-1)}]
\end{aligned} \tag{22}
$$

where $i = 1, ..., N_0$, $j = -M+1, -M+2, ..., N$, and

$$
\begin{aligned}
&u_h(x^{(i)}, y^{(j)}) =\\
&= h_0 F\left(x^{(i)}, y^{(j)}, (T_h z)_{(P[i,j])}\right) + [-\tfrac{3}{2}z^{(i,j)} + 2z^{(i,j)} - \tfrac{1}{2}z^{(i,j+2)}]\\
&v_h^*(x^{(i)}, y^{(j)}) = -\tfrac{3}{2}z^{(i,j)} + 2z^{(i,j)} - \tfrac{1}{2}z^{(i,j+2)},
\end{aligned} \tag{23}
$$

where $i = 1, ..., N_0$, $j = -M$.

We define u_h^* and v_h^* on the initial-boundary set $E_0[h] \cup \partial_0 E[h]$ by

$$(24) \qquad u_h^*(x^{(i)}, y^{(j)}) = h_0 D_x \varphi(x^{(i)}, y^{(j)}) \quad v_h^*(x^{(i)}, y^{(j)}) = h_0 D_y \varphi(x^{(i)}, y^{(j)}).$$

Theorem 3. *Suppose that*

1° the function $F : E \times C(D, R) \to R$ is of class C^1 and the derivatives $D_x F$, $D_y F$, $D_w F$ satisfy the Lipschitz condition with respect to w,

2° v is a solution of (20), v is of class C^3 on B and $v_h = v_{|B[h]}$,

3° u_h is a solution of (21) and $h_1 = Ch_0$.

Then $C_0 > 0$ exists such that

$$\|u_h - v_h\|_{h,i} \le C_0 |h|^2, \quad i = 0, 1, ..., N_0.$$

Proof. It is easy to see that method (21) satisfies stability condition (5). Now we prove that $C^* \in R_+$ exists such that

$$(25) \qquad |\delta_0 v_h^{(i,j)} - \Phi_h\left(x^{(i)}, y^{(j)}, (v_h)_{(i,j)}, \delta v_h^{(i,j)}\right)| \le C^* |h|^2,$$

$$0 \le i \le N_0 - 1, \; -M \le j \le N,$$

where Φ_h is given by the right-hand side of the difference equation in (21).

For $i = 0, 1, ..., N_0 - 1$, $j = -M + 1, ..., N$ we have

$$(26) \qquad \delta_0 v_h^{(i,j)} = D_x v^{(i,j)} + \frac{1}{2} h_0 D_{xx} v^{(i,j)} - h_1^2 (2h_0)^{-1} D_{yy} v^{(i,j)} + R^{(i,j)}(h),$$

where

$$R^{(i,j)}(h) = \frac{1}{6} D_{xxx} v(P_0) - h_1^3 (12h_0)^{-1} D_{yyy} v(P_1) + h_1^3 (12h_0)^{-1} D_{yyy} v(P_2)$$

and P_0, P_1, P_2 are intermediate points. From (20) and (26) it follows that

$$\begin{aligned}\delta_0 v^{(i,j)} &= F\left(x^{(i)}, y^{(j)}, v_{(P[i,j])}\right) + C D_y v^{(i,j)} + R^{(i,j)}(h) + \\ &+ \tfrac{1}{2} h_0 \left[D_x F\left(x^{(i)}, y^{(j)}, v_{(P[i,j])}\right) + D_w F\left(x^{(i)}, y^{(j)}, v_{(P[i,j])}\right)(D_x v)_{(P[i,j])}\right] + \\ &+ \tfrac{1}{2} h_0 \left[D_y F\left(x^{(i)}, y^{(j)}, v_{(P[i,j])}\right) + D_w F\left(x^{(i)}, y^{(j)}, v_{(P[i,j])}\right)(D_y v)_{(P[i,j])}\right]\end{aligned}$$

Let $Q[v, i, j] = \left(x^{(i)}, y^{(j)}, v_{(P[i,j])}\right)$ and $Q_h[v, i, j] = \left(x^{(i)}, y^{(j)}, (T_h v_h)_{(P[i,j])}\right)$.

A constant $\tilde{C} > 0$ exists such that fir $0 \le i \le N_0 - 1$, $-M + 1 \le j \le N$ we have

$$
(27) \quad \begin{aligned}
&|F(Q[v,i,j]) - F(Q_h[v,i,j])| \le \tilde{C}|h|^2, \\
&|D_x F(Q[v,i,j]) - D_x F(Q_h[v,i,j])| \le \tilde{C}|h|^2, \\
&|D_y F(Q[v,i,j]) - D_y F(Q_h[v,i,j])| \le \tilde{C}|h|^2, \\
&|D_w F(Q[v,i,j]) - D_x F(Q_w[v,i,j])| \le \tilde{C}|h|^2
\end{aligned}
$$

and

$$
(28) \qquad |D_y v^{(i,j)} - \delta v^{(i,j)}| \le \tilde{C}|h|^2, \quad |R^{(i,j)}(h)| \le \tilde{C}|h|^2.
$$

Let U_h^* and V_h^* be given by (22)-(24) where z is replaced by V_h in the right-hand sides. Then we have

$$
\|h_0 D_x v - T_h U_h^*\|_{C(B)} \le \|h_0 D_x v - T_h(h_0 D_x v)_h\|_{C(B)} + \|T_h(h_0 D_x v)_h - T_h U_h^*\|
$$

and

$$
\begin{aligned}
&|(h_0 D_x v)_h(x^{(i)}, y^{(j)}) - (T_h U_h^*)(x^{(i)}, y^{(j)})| \le \\
\le\ &|h_0 F(Q[v,i,j]) + C h_0 D_y v^{(i,j)} - h_0 F(Q_h[v,i,j]) - h_1 \delta v_h^{(i,j)} \le C_1 |h|^2, \\
&\text{for } 1 \le i \le N_0 - 1,\ -M \le j \le N, \\
&|(h_0 D_x v)_h(x^{(i)}, y^{(j)}) - (T_h U_h^*)(x^{(i)}, y^{(j)})| = 0 \quad \text{on } E_0[h] \cup \partial_0 E[h],
\end{aligned}
$$

where $C_1 \in R_+$. In force of Lemma 2 and the above estimates it follows that $C_2 \in R_+$ exists such that

$$
(29) \quad \|(h_0 D_x v)_{(P[i,j])} - (T_h U_h^*)_{(P[i,j])}\|_0 \le C_2 |h|^2, \ 0 \le i \le N_0 - 1, \ -M \le j \le N
$$

In an analogous way we can prove that $C_3 \in R_+$ exists such that

$$
(30) \quad \|(h_0 D_y v)_{(P[i,j])} - (T_h V_h^*)_{(P[i,j])}\|_0 \le C_2 |h|^2, \ 0 \le i \le N_0 - 1, \ -M \le j \le N
$$

From (26)-(30) we deduce that (25) holds for $0 \le i \le N_0 - 1$, $-M + 1 \le j \le N$. We obtain (25) for $j = -M$, $0 \le i \le N_0 - 1$ in analogous way.

Finally the assertion follows by virtue of Theorem 2 (see also Remark 2). ■

7. Multistep difference methods for the mixed problem

Let δ be given by (3) and assume that $A = (A_1 \dots A_k)$ be defined by

$$
A_i z^{(m)} = \sum_{s \in S[m']} a_{s,m}^{(i)} z^{(m_0, m'+s)}, \ i = 1, ..., k, \ 0 \le m_0 \le N_0, \ -K \le m' \le N,
$$

where $a_{s,m}^{(i)}$ are given real numbers. Assume that for every $h \in I_e$ we have $\Phi_h = (\Phi_h^{(1)}, ..., \Phi_h^{(k)}) : \Omega_h \to R^k$. We define

$$\begin{aligned} F_h^{(m)}[z] &= \sum_{i=1}^k \alpha_i A_i z^{(m_0+k-i,m')} + \\ &+ h_0 \sum_{i=1}^k \beta_i \Phi_h^{(i)} \left(x^{(m_0+k-i)}, y^{(m')}, z_{(m_0+k-i,m')}, \delta z^{(m_0+k-i,m')} \right), \end{aligned}$$

where $z : B[h] \to R$ and α_i, $\beta_i \in R$ for $i = 1, ..., k$. Put $E[h,k] = \emptyset$ if $k = 1$ and

$$E[h,k] = \{(x^{(m_0)}, y^{(m')}) : 0 \le m_0 \le k-1, \ -K \le m' \le N\} \text{ for } k > 1$$

Suppose that $\varphi_h : E_0[h] \cup \partial_0 E[h] \cup E[h,k] \to R$ is a given function. We consider the following difference method

$$\text{(31)} \qquad \begin{aligned} &z^{(m_0+k,m')} = F_h^{(m)}[z], \quad m_0 = 0, 1, ..., N_0 - k, \ -K \le m' \le N, \\ &z^{(m)} = \varphi_h^{(m)} \text{ on } E_0[h] \cup \partial_0 E[h] \cup E[h,k]. \end{aligned}$$

Let us prove the following lemma on the difference inequality

$$\text{(32)} \quad \lambda^{(j+k)} \ge \sum_{i=1}^k \alpha_i \lambda^{(j+k-i)} + h_0 L \sum_{i=1}^k |\beta_i| \lambda^{(j+k-i)} + h_0 \gamma(h), \ 0 \le j \le N_0 - k.$$

Lemma 3. *Suppose that $\alpha_i \in R_+$, $\beta_i \in R$ for $i = 1, ..., k$, $L \in R_+$ are given such that*

1° for $B = \sum_{i=1}^k |\beta_i|$ we have $B > 0$ and $\sum_{i=1}^k \alpha_i = 1$,

2° a function $\gamma : I_e \to R_+$ is given such that $\lim_{h \to 0} \gamma(h) = 0$ and $N_0 h_0 \le a$,

3° $0 \le \eta_h^{(0)} \le \eta_h^{(1)} \le ... \le \eta_h^{(k-1)}$ and $\lim_{h \to 0} \eta_h^{(i)} = 0$, $i = 0, 1, ..., k-1$.

Then a function $\tilde{\lambda}_h : \{0, 1, ..., N_0\} \to R_+$ exists such that

1° $\tilde{\lambda}_h^{(i)} \ge \eta_h^{(i)}$, for $i = 0, 1, ..., k-1$, $\tilde{\lambda}_h^{(i+1)} \ge \tilde{\lambda}_h^{(i)}$, for $i = 0, 1, ..., N_0 - 1$,

2° $\tilde{\lambda}_h$ is a solution of (32) and a function $\lambda^* : I_e \to R_+$ exists such that $|\tilde{\lambda}_h^{(i)}| \le \lambda^*(h)$, for $j = 0, 1, ..., N_0$ and $\lim_{h \to 0} \lambda^*(h) = 0$.

Proof. Note that the function $\tilde{\lambda}_h$ defined by

$$\tilde{\lambda}_h^{(j)} = \eta_h^{(j)} \quad \text{for} j = 0, 1, ..., k-1 \quad \text{and}$$

$$\begin{aligned} \tilde{\lambda}_h^{(k+i)} &= \tilde{\lambda}_h^{(k-1)} (1 + h_0 L B)^{i-1} + \gamma(h)[(1 + h_0 L B)^{i+1} - 1](LB)^{-1} \text{ if } L > 0 \\ \tilde{\lambda}_h^{(k+i)} &= \tilde{\lambda}_h^{(k-1)} + (i+1) h_0 \gamma(h) \text{ if } L = 0, \end{aligned}$$

where $i = 0, 1, ..., N_0 - k$, satisfies the thesis of Lemma 3. ■

Assumption H$_5$. Suppose that

1° conditions (7), (13) hold and

$$\sum_{s\in S[m']} a^{(i)}_{s,m} = 1, \quad \sum_{s\in S[m']} s_j a^{(i)}_{s,m} = 0, \quad i = 1,...,k,\ j = 1,...,n,$$

2° there exist $\bar{c}, d_0 > 0$ such that $h_i \le h_0 d_0$, $h_i h_j^{-1} \le d_0$, $i,j = 1,...,n$,
and

$$\sum_{s\in S[m']} |a^{(i)}_{s,m}| \le \bar{c}, \quad \sum_{s\in S[m']} |c^{(j)}_{s,m}| \le \bar{c},$$

$$1 \le i \le k,\ 1 \le j \le n, 0 \le m_0 \le N_0 - k, -K \le m' \le N,$$

3o $\alpha_i \in R_+$, $\beta_i \in R$ for $i = 1,...,k$ and we have

$$\sum_{i=1}^{k} \alpha_i = 1, \quad k - \sum_{i=1}^{k} \alpha_i (k-i) = \sum_{i=1} \beta_i,$$

4° $v : B \to R$ is a solution of (1) and v is of class C^2 on B.

Lemma 4. *If Assumption H_5 is satisfied and Λ is defined by*

$$\Lambda[v,h]^{(j,m')} = v^{(j+k,m')} - \sum_{i=1}^{k} \alpha_i A_i v^{(j+k-i,m')} - h_0 \sum_{i=1}^{k} \beta_i D_x v^{(j+k-i,m')},$$

$$j = 0, 1, ..., N_0 - k, \ -K \le m' \le N,$$

then a function $\gamma^* : I_e \to R_+$ exists such that

$$|\Lambda[v,h]^{(j,m')}| \le h_0 \gamma^*(h), \quad 0 \le j \le N_0 - k, \ -K \le m' \le N, \ \textit{and} \ \lim_{h\to 0} \gamma^*(h) = 0.$$

Proof. Observe that $P_1^{(m)}, P_2^{(m)}, P_3^{(m)} \in R^{1+n}$ exist such that

$$\begin{aligned}
&\Lambda[v,h]^{(m)} = v^{(m)} + kh_0 D_x v^{(m)} + \tfrac{1}{2} k^2 h_0^2 D_{xx} v(P_1^{(m)}) - \\
&- \textstyle\sum_{i=1}^{k} \alpha_i \sum_{s\in S[m']} a^{(i)}_{s,m} [v^{(m)} + (k-i) D_x v^{(m)} + \sum_{j=1}^{n} s_j h_j D_{y_j} v^{(m)} + \\
&+ \tfrac{1}{2}(k-i) h_0 \textstyle\sum_{j=1}^{n} s_j h_j D_{xy_j} v(P_2^{(m)}) + \tfrac{1}{2} \sum_{j,j'=1} s_j h_j s_{j'} h_{j'} D_{y_j y_{j'}} v(P_2^{(m)})] - \\
&- h_0 \textstyle\sum_{i=1}^{k} \beta_i [D_x v^{(m)} + (k-i) h_0 D_{xx} v(P_3^{(m)})].
\end{aligned}$$

Then the assertion is an immediate consequence of Assumption H_5. ■

Assumption H$_6$. Suppose that

1° the functions $\Phi_h^{(i)} : \Omega_h \to R$, $i = 1,...,k$, of the variables (x,y,w,q) satisfy the conditions

(i) for $(x,y,w) \in E[h] \times F(D[h],R)$ we have $\Phi_h^{(i)}(x,y,w,\cdot) \in C(R^n,R)$,

(ii) the partial derivatives $(D_{q_1}\Phi_h^{(i)},\dots,D_{q_n}\Phi_h^{(i)}) = D_q\Phi_h^{(i)}$ exist on Ω_h and $D_q\Phi_h^{(i)}(x,y,q,\cdot) \in C(R^n,R^n)$ where $(x,y,w) \in E[h] \times F(D[h],R)$,

(iii) there exists $L \in R_+$ such that

$$|\Phi_h^{(i)}(x,y,w,q) - \Phi_h^{(i)}(x,y,\bar{w},q)| \leq L\|w-\bar{w}\|_h \quad \text{on } \Omega_h,$$

2° $f : \Omega \to R$ is continuous and there exists $\beta_0 : I_e \to R_+$ such that

$$|\Phi_h\left(x^{(m_0)},y^{(m')},(v_h)_{(m)},\delta v_h^{(m)}\right) - f\left(x^{(m_0)},y^{(m')},v_{(P[m])},\delta v^{(m)}\right)| \leq \beta_0(h),$$

$$m_0 = 0,1,\dots,N_0-1, \ -K \leq m' \leq N,$$

and $\lim_{h\to 0}\beta_0(h) = 0$,

3° for $m_0 = 0,1,\dots,N_0-1$, $-K \leq m' \leq N$, $s \in S[m']$ we have

$$\alpha_i a_{s,m}^{(i)} + h_0\beta_i \sum_{j=1}^{n} h_j^{-1} c_{s,m}^{(j)} D_{q_j}\Phi_h^{(i)}\left(x^{(m_0)},y^{(m')},w,q\right) \geq 0,$$

4° $\varphi_h : E_0[h] \cup \partial_0 E[h] \cup E[h,k] \to R$ and there is $\gamma_0 : I_e \to R_+$ such that $|\varphi_h^{(m)} - \varphi^{(m)}| \leq \gamma_0(h)$ on $E_0[h] \cup \partial_0 E[h] \cup E[h,k]$ and $\lim h \to 0\gamma_o(h) = 0$,

5° $u_h : B[h] \to R$ is a solution of (31).

Theorem 4. *If Assumptions H_5, H_6 are satisfied then there exists a function $\lambda^* : I_e \to R_+$ such that*

$$\|u_h - v_h\|_{h,i} \leq \lambda^*(h), \quad i = 0,1,\dots,N_0, \quad \lim_{h\to 0}\lambda^*(h) = 0. \tag{33}$$

Proof. For $m_0 = 0,1,\dots,N_0-k$, $-K \leq m' \leq N$ we have

$$\begin{aligned} &|v_h^{(m_0+k,m')} - u_h^{(m_0+k,m')}| \leq |F_h^{(m)}[v_h] - F_h^{(m)}[u_h]| + |\Lambda[v,h]^{(m)}| + \\ &+h_0 \textstyle\sum_{i=1}^k |\beta_i| \left| f\left(x^{(m_0+k-i)},y^{(m')},v_{(P[m_0+k-i,m'])},D_y v^{(m_0+k-i,m')}\right) - \right. \\ &\left. -\Phi_h^{(i)}\left(x^{(m_0+k-i)},y^{(m')},(v_h)_{(m_0+k-i,m')},\delta v_h^{(m_0+k-i,m')}\right)\right|. \end{aligned} \tag{34}$$

In force of Assumption H_6 it follows that

$$|F_h^{(m)}[v_h] - F_h^{(m)}[u_h]| \leq |\sum_{i=1}^k \alpha_i[A_i v_h^{(m_0+k-i,m')} - u_h^{(m_0+k-i,m')}]+$$
$$+h_0 \sum_{i=1}^k \beta_i \Phi_h^{(i)} \left(x^{(m_0+k-i)}, y^{(m')}, (v_h)_{(m_0+k-i,m')}, \delta v_h^{(m_0+k-i,m')}\right) -$$
$$-\Phi_h^{(i)} \left(x^{(m_0+k-i)}, y^{(m')}, (v_h)_{(m_0+k-i,m')}, \delta u_h^{(m_0+k-i,m')}\right) |+$$
$$+h_0 \sum_{i=1}^k |\beta_i||\Phi_h^{(i)} \left(x^{(m_0+k-i)}, y^{(m')}, (v_h)_{(m_0+k-i,m')}, \delta u_h^{(m_0+k-i,m')}\right) -$$
$$-\Phi_h^{(i)} \left(x^{(m_0+k-i)}, y^{(m')}, (u_h)_{(m_0+k-i,m')}, \delta u_h^{(m_0+k-i,m')}\right) | \leq$$
$$\leq | \sum_{i=1}^k \sum_{s\in S[m']} \left(v^{(m_0+k-i,m'+s)} - u_h^{(m_0+k-i,m'+s)}\right) |[\alpha_i a_{s,m}^{(i)}+$$
$$+h_0\beta_i \sum_{j=1}^n h_j^{-1} c_{s,m}^{(j)} D_{q_j} \Phi_h^{(i)}(P_i^*)]| + Lh_0 \sum_{i=1}^k |\beta_i| \|v_h - u_h\|_{h,m_0+k-i},$$

where $P_i^* = \left(x^{(m_0+k-i)}, y^{(m')}, (v_h)_{(m_0+k-i,m')}, Q^*\right)$ and Q^* is an intermediate point. Then we have

$$\begin{aligned}|F_h^{(m)}[v_h] \quad -F_h^{(m)}[u_h]| \leq \sum_{i=1}^k \alpha_i \|v_h - u_h\|_{h,m_0+k-i}+\\ +Lh_0 \sum_{i=1}^k |\beta_i| \|v_h - u_h\|_{h,m_0+k-i}.\end{aligned} \tag{35}$$

From condition 2° of Assumption H_6 we deduce that a function $\gamma_* : I_e \to R_+$ exists such that $\lim_{h\to 0} \gamma_*(h) = 0$ and for $0 \leq m_0 \leq N_0 - k$, $-K \leq m' \leq N$ we have

$$\begin{aligned}\sum_{i=1}^k |\beta_i| \Big| f\left(x^{(m_0+k-i)}, y^{(m')}, v_{(P[m_0+k-i,m'])}, D_y v_h^{(m_0+k-i,m')}\right) -\\ -\Phi_h^{(i)} \left(x^{(m_0+k-i)}, y^{(m')}, (v_h)_{(m_0+k-i,m')}, \delta v_h^{(m_0+k-i,m')}\right)\Big| \leq \gamma_*(h).\end{aligned} \tag{36}$$

Estimates (34)-(36) and Lemma 4 imply

$$\|v_h - u_h\|_{h,k+j} \leq \max\Big\{\gamma_0(h), \sum_{i=1}^k \alpha_i \|v_h - u_h\|_{h,j+k-i}+$$
$$+Lh_0 \sum_{i=1}^k |\beta_i| \|v_h - u_h\|_{h,j+k-i} + h_0\gamma(h)\Big\}, \quad j = 0, 1, ..., N_0 - k.$$

where $\gamma(h) = \gamma_*(h) + \gamma^*(h)$. It follows from Lemma 3 that there is $\overline{\lambda_h} : \{0, 1, ..., N_0\} \to R_+$ such that $\overline{\lambda_h}^{(j)} = \gamma_0(h)$ for $j = 0, 1, ..., k-1$ and

$$\overline{\lambda}^{(j+k)} \geq \sum_{i=1}^k \alpha_i \overline{\lambda}^{(j+k-i)} + h_0 \sum_{i=1}^k |\beta_i| \overline{\lambda}^{(j+k-i)} h_0\gamma(h), \quad j = 0, 1, ..., N_0 - k,$$

and $\lambda_h^{(j+1)} \geq \lambda_h^{(j)}$ for $0 \leq j \leq N_0 - 1$. Furthermore $\lambda^* : I_e \to R_+$ exists such that $|\overline{\lambda_h}^{(j)}|$ for $0 \leq j \leq N_0$ and $\lim_{h\to 0} \lambda^*(h) = 0$.

Finally (33) follows by virtue of Assumption H_5. ■

R e m a r k 4. We wish to remark that the results of the present paper can be extended to weakly coupled systems of differential-functional equations. We omit the details.

8. A numerical example

Let $E = [0,1] \times [0,1] \times [0,1]$, $D = \{0\} \times [0, 0.5] \times [-0.5, 0]$ and

$$\partial_0 E = ([0,1] \times [0,1.5] \times [-0.5,1]) \setminus ([0,1] \times [0,1) \times (0,1]).$$

We consider the following mixed problem

(37)

$$\begin{aligned} D_x z(x,y) &= (1 + xy_2)[D_{y_1} z(x,y) + \sin D_{y_1} z(x,y)] - \\ &-(1 + xy_1)[D_{y_2} z(x,y) - \cos D_{y_2} z(x,y)] - 2\int_D z(x, y+s)ds + \\ &+[\tfrac{1}{2} + 2x(1+x^2)^{-1}] z(x,y) + f_0(x,y), \\ z(0,y) &= \tfrac{1}{2}(y_1^2 + y_2^2) \quad \text{for } y = (y_1, y_2) \in [0, 1.5] \times [-0.5, 1], \\ z(x,y) &= \tfrac{1}{2}(y_1^2 + y_2^2)(1 + x^2) \quad \text{for } (x,y) \in \partial_0 E, \end{aligned}$$

where

$$\begin{aligned} f_0(x,y) = &-(1+xy_2)\sin(y_1 + y_1 x^2) - \\ &-(1+xy_1)\cos(y_2 + y_2 x^2) - (1+x^2)[\tfrac{7}{8}(y_1 - y_2) - \tfrac{1}{24}]. \end{aligned}$$

Let us consider the difference equation

(38)

$$\begin{aligned} z^{(m_0+1,m')} = Az^{(m)} + h_0 \Big\{ &(1 + x^{(m_0)} y_2^{(m_2)})(\delta_1 z^{(m)} + \sin \delta_1 z^{(m)}) - \\ -(1 + x^{(m_0)} y_1^{(m_1)})&(\delta_2 z^{(m)} - \cos \delta_2 z^{(m)}) - 2\int_D T_h(x^{(m_0)}, y^{(m')} + s) ds + \\ + [\tfrac{1}{2} + 2x^{(m_0)}&(1 + x^{(m_0)} x^{(m_0)})^{-1} z^{(m)} + f_0^{(m)} \Big\}. \end{aligned}$$

$$m_0 = 0, 1, \dots, N_0 - 1, \; m_1 = 0, 1, \dots, M_1 - 1, \; m_2 = 1, 2, \dots, M_2,$$

with initial-boundary conditions

$$\begin{aligned} z^{(0,m)} &= \tfrac{1}{2}[(y_1^{(m_1)})^2 + (y_2^{(m_2)})^2] \quad \text{for } 0 \le m_1 \le N_1, \; -N_2 \le m_2 \le M_2, \\ z^{(m)} &= \tfrac{1}{2}[(y_1^{(m_1)})^2 + (y_2^{(m_2)})^2](1 + (x^{(m_0)})^2) \quad \text{on } \partial_0 E[h], \end{aligned}$$

where $N_0 h_0 = 1$, $M_1 h_1 = 1$, $N_1 h_1 = 1.5$, $-N_2 h_2 = -0.5$, $M_2 h_2 = 1$ and T_h is defined by (18), (19) with $n = 2$ and

$$\begin{aligned} &Az^{(m)} = \\ &= \tfrac{1}{4}[z^{(m_0,m_1+1,m_2)} + z^{(m_0,m_1-1,m_2)} + z^{(m_0,m_1,m_2+1)} + z^{(m_0,m_1,m_2-1)}], \\ &\delta_1 z^{(m)} = (2h_1)^{-1}[z^{(m_0,m_1+1,m_2)} - z^{(m_0,m_1-1,m_2)}], \\ &\delta_2 z^{(m)} = (2h_2)^{-1}[z^{(m_0,m_1,m_2+1)} - z^{(m_0,m_1,m_2-1)}] \end{aligned} \tag{39}$$

for $m_0 = 0, 1, \dots, N_0 - 1,\ 1 \le m_1 \le M_1 - 1,\ 1 \le m_2 \le M_2 - 1,$

and

(40) $$Az^{(m)} = z^{(m)},\ \delta_1 z^{(m)} = h_1^{-1}[z^{(m_0, m_1+1, m_2)} - z^{(m)}],$$

$$\delta_2 z^{(m)} = h_2^{-1}[z^{(m)} - z^{(m_0, m_1, m_2-1)}],\ m_0 = 0, 1, \dots, N_0 - 1,$$

$$\text{if } m_1 = 0,\quad 1 \le m_2 \le M_2 \text{ or } m_2 = M_2,\ 1 \le m_2 \le M_1 - 1.$$

Note that method (38)-(40) satisfies condition (5) for $h_1 \ge 10h_0$, $h_2 \ge 10h_0$. Let u and v_h be solutions of (37) and (38)-(40) respectively.

Let $Q_h^{(m)} = v_h^{(m)} - u^{(m)}$, $m_0 = 0, 1, \dots, N_0$, $m_1 = 0, 1, \dots, M_1$, $m_2 = 0, 1, \dots, M_2$.

Suppose that $N_0 = 400$, $M_1 = M_2 = 40$, $h_0 = 0.0025$, $h_1 = h_2 = 0.025$. The values $Q_h^{(400, m_1, m_2)}$ are listed in the Table below.

Table of errors for $x = 1$

	$y_1 = 0.125$	$y_1 = 0.375$	$y_1 = 0.625$	$y_1 = 0.875$
$y_2 = 0.125$	2.080×10^{-2}	1.788×10^{-2}	1.554×10^{-2}	1.163×10^{-2}
$y_2 = 0.375$	5.211×10^{-2}	4.541×10^{-2}	3.836×10^{-2}	2.115×10^{-2}
$y_2 = 0.625$	7.396×10^{-2}	6.358×10^{-2}	4.956×10^{-2}	2.216×10^{-2}
$y_2 = 0.875$	8.604×10^{-2}	7.095×10^{-2}	5.081×10^{-2}	2.051×10^{-2}

During the computation a computer IBM AT was used.

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