Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica Balkanica

New Series Vol.10, 1996, Fasc.2-3

Arithmetic Functions Defined on Sets of Primes of Positive Density[1]

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Presented by P. Kenderov

Several asymptotic formulas are proved for arithmetic sums, which involve the largest prime factor of an integer and certain large additive functions. All the functions are defined on a set of primes having density δ ($0 < \delta < 1$) in the set of all primes.

1. Introduction

Let Q be a set of primes such that there exists some constant δ satisfying $0 < \delta < 1$ and

$$\pi(x,Q) := \sum_{p\le x,\ p\in Q} 1 = \delta \mathrm{Li}\, x + O\left(\frac{x}{\log^B x}\right). \tag{1.1}$$

Here and later p denotes primes, $\mathrm{Li}\, x = \int_2^x \frac{dt}{\log t}$, and B is a constant satisfying $B > 2$. It is possible to treat the case when one assumes only $B > 1$ in (1.1) (see R. W a r l i m o n t [14]), but as in [3] and [11] we find it sufficient to assume $B > 2$ in (1.1). In fact, the present work may be considered as a continuation of the first author's work [3] and the second author's [11]. All the relevant notation from these papers will be retained here. We define $P(n,Q)$ as

$$P(n,Q) = \begin{cases} \max\{p : p|n \wedge p \in Q\} & \text{if } (n,Q) > 1, \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

where $(n,Q) > 1$ (resp. $(n,Q) = 1$) means that n has a prime factor (resp. has no prime factor) from Q. Thus $P(n,Q)$ is the largest prime factor of n belonging

[1]Research of the first author supported by grants from NSERC of Canada and FCAR of Quebec. Research of the second author financed by the Mathematical Institute of Belgrade.

to Q, and analogously we define the k-th largest prime factor of n belonging to Q as

$$P_k(n,Q) = \begin{cases} P\left(\frac{n}{P_1(n,Q)\dots P_{k-1}(n,Q)}, Q\right) & \text{if } \Omega(n,Q) \geq k, \\ 0 & \text{otherwise,} \end{cases} \tag{1.3}$$

if $k \geq 2$, where $P_1(n,Q) \equiv P(n,Q)$ and

$$\Omega(n,Q) = \sum_{p^\alpha \| n,\ p \in Q} \alpha \tag{1.4}$$

is the total number of prime factors of n belonging to Q, while

$$\omega(n,Q) = \sum_{p|n,\ p \in Q} 1 \tag{1.5}$$

is the number of distinct prime factors of n belonging to Q. Here as usual $p^\alpha \| n$ means that p^α divides n, but $p^{\alpha+1}$ does not. The functions defined by (1.2)-(1.5) are the analogues of the classical functions

$$P(n) = \max\{p : p|n\}, \quad P_k(n) = \begin{cases} P\left(\frac{n}{P_1(n)\dots P_{k-1}(n)}\right) & \text{if } \Omega(n) \geq k, \\ 0 & \text{otherwise,} \end{cases}$$

if $k \geq 2$, and

$$\Omega(n) = \sum_{p^\alpha \| n} \alpha p, \quad \omega(n) = \sum_{p|n} 1.$$

Likewise in [11] we defined large additive functions

$$\beta(n,Q) = \sum_{p|n,\ p \in Q} p, \quad B(n,Q) = \sum_{p^\alpha \| n,\ p \in Q} \alpha p, \tag{1.6}$$

and $\beta(n,Q) = B(n,Q) = 0$ if $(n,Q) = 1$. The functions in (1.6) are the analogues of the large additive functions

$$\beta(n) = \sum_{p|n} p, \quad B(n) = \sum_{p^\alpha \| n} \alpha p,$$

for which there exists an extensive literature (e.g. see the monograph [4] and the papers [1], [5], [7], [8], [12], [15], where references to other works may be found).

In [3] the first author proved

$$\sum_{n\le x}{}' \frac{1}{P(n,Q)} = \left(\eta(Q) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^{\delta}}, \tag{1.7}$$

where $\eta(Q)$ is a positive constant depending on Q (i.e. δ) which may be written down in closed form. In general $\sum'_{n\le x} 1/f(n)$ denotes the sum over n not exceeding x for which $f(n)\neq 0$, so that

$$\sum_{n\le x}{}' \frac{1}{P(n,Q)} = \sum_{n\le x,\ (n,Q)>1} \frac{1}{P(n,Q)}.$$

Several results involving $\beta(n,Q)$ and $B(n,Q)$ were established by the second author [11]. For instance, it was proved that

$$\sum_{n\le x}\beta(n,Q) = \sum_{j\le B}\frac{\delta A_j x^2}{\log^j x} + O\left(\frac{x^2}{\log^B x}\right) \tag{1.8}$$

with explicitly given constants A_j,

(1.9)

$$\sum_{n\le x}(B(n,Q)-\beta(n,Q)) = \delta x\log\log x + E(Q) + x\sum_{j\le B}\frac{E_j(Q)}{\log^j x} + O\left(\frac{x}{\log^B x}\right),$$

$$\sum_{n\le x}{}' \frac{1}{B(n,Q)-\beta(n,Q)} = A(Q)x + O\left(x^{\frac{1}{2}}\log x\right) \quad (A(Q)>0), \tag{1.10}$$

and

$$\sum_{n\le x}{}' \frac{B(n,Q)}{\beta(n,Q)} = x + O\left(\frac{x\log\log x}{(\log x)^{\delta}}\right), \tag{1.11}$$

where the constants in (1.9) and (1.10) are effectively computable, and in view of (1.9) it is seen that (1.8) remains valid if $\beta(n,Q)$ is replaced by $B(n,Q)$. Moreover it was conjectured in [11] that

$$\sum_{n\le x}{}' \frac{1}{\beta(n,Q)} = \left(\eta_1(Q) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^{\delta}}, \tag{1.12}$$

and

$$\sum_{n\le x}{}' \frac{1}{B(n,Q)} = \left(\eta_2(Q) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^{\delta}}, \tag{1.13}$$

with $0 < \eta_2(Q) \le \eta_1(Q) \le \eta(Q)$, where $\eta(Q)$ is the constant appearing in (1.7). It is the aim of this paper to establish the asymptotic formulas (1.12) and (1.13), and to prove some other results involving the functions $\beta(n,Q)$, $B(n,Q)$ and $P_k(n,Q)$.

2. Statement of results

K. A l l a d i and P. E r d ő s [2] proved that, for any fixed $k \ge 2$,

$$\sum_{2\le n\le x} \frac{P_k(n)}{P(n)} = (1+o(1))a_k \frac{x}{(\log x)^{k-1}} \qquad (x\to\infty), \tag{2.1}$$

where the a_k's are effectively computable positive constants. Thus the asymptotic behaviour of the sum in (2.1) changes with k. However all the sums of $P_k(n,Q)/P(n,Q)$ are of the same order of magnitude, which shows a completely different behaviour. Our result is contained in

Theorem 1. *For any fixed integer $k \ge 2$ we have, with a suitable constant $C_k(Q) > 0$,*

$$\sum_{n\le x,\ (n,Q)>1} \frac{P_k(n,Q)}{P(n,Q)} = \left(C_k(Q) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^{\delta}}. \tag{2.2}$$

Actually it will transpire from the proof that

$$C_k(Q) = C(Q)\sum_{m=1}^{\infty}\frac{1}{m}\sum_{\substack{p_1\ge P(m)\\ p_1\in Q}} \ell_Q(p_1)\sum_{\substack{p_2\ge p_1\\ p_2\in Q}}\frac{1}{p_2}\cdots\sum_{\substack{p_{k-1}\ge p_{k-2}\\ p_{k-1}\in Q}}\frac{1}{p_{k-1}}\sum_{\substack{p_k\ge p_{k-1}\\ p_k\in Q}}\frac{1}{p_k^2}, \tag{2.3}$$

where $C(Q)$ is the constant appearing in Lemma 4, $\ell_Q(y)$ is given by (3.8), and $p_1, p_2, \dots, p_k$ denote primes. The multiple series in (2.3) is easily seen to be convergent by the prime number theorem, Lemma 6 and Lemma 2. The method of proof of Theorem 1 may be used to treat some other arithmetic sums, such as

$$\sum_{n\le x,\ \Omega(n,Q)\ge m} \frac{P_k(n)}{P_m(n)}, \tag{2.4}$$

where $k > m \ge 1$ are fixed integers. Also this method may be used to treat two sums related to the sum in (1.11). We shall prove

Theorem 2. *There exist constants* $0 < D_1(Q) < D_2(Q)$ *such that*

$$\sum_{n\le x,\ (n,Q)>1} \frac{\beta(n,Q)}{P(n,Q)} = x + \left(D_1(Q) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^{\delta}}. \tag{2.5}$$

and

$$\sum_{n\le x,\ (n,Q)>1} \frac{B(n,Q)}{P(n,Q)} = x + \left(D_2(Q) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^{\delta}}. \tag{2.6}$$

The explicit expressions for $D_1(Q)$ and $D_2(Q)$ will be given in the proof. The next result establishes the asymptotic formulas (1.12) and (1.13). This is

Theorem 3. *There exist constants* $0 < \eta_2(Q) < \eta_1(Q)$ *such that*

$$\sum_{n\le x,\ (n,Q)>1} \frac{1}{\beta(n,Q)} = \left(\eta_1(Q) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^{\delta}} \tag{2.7}$$

and

$$\sum_{n\le x,\ (n,Q)>1} \frac{1}{B(n,Q)} = \left(\eta_2(Q) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^{\delta}}. \tag{2.8}$$

The asymptotic formulas (2.7) and (2.8) display the difference in behaviour of $\beta(n)$ (resp. $B(n)$) and $\beta(n,Q)$ (resp. $B(n,Q)$), since it is known that

$$\sum_{2\le n\le x} \frac{1}{\beta(n)} = x\exp\left\{-(2\log x\log\log x)^{1/2} + O\left((\log x\log\log\log x)^{1/2}\right)\right\}. \tag{2.9}$$

The asymptotic formula (2.9), which remains true if $\beta(n)$ is replaced by $B(n)$ or $P(n)$, was proved in [10], and then sharpened in [12] and [8]. This is analogous to the difference in behaviour between the sum of reciprocals of $P(n)$ and $P(n,Q)$, as noted in [3] and [11]. The difference in behaviour between $P(n)$ and $P(n,Q)$ is also reflected in the asymptotic behaviour of two further arithmetic sums which contain the logarithms of these functions. The results are

Theorem 4. *There exists an effectively computable constant* B *such that*

$$\sum_{2\le n\le x} \frac{1}{n\log P(n)} = e^{\gamma}\log\log x + B + O\left(\frac{1}{\log x}\right), \tag{2.10}$$

where γ is Euler's constant.

Theorem 5. *There exists an effectively computable constant $F(Q) > 0$ such that*

$$\sum_{n\le x,\ (n,Q)>1} \frac{1}{n\log P(n,Q)} = \left(F(Q) + O\left(\frac{1}{\log\log x}\right)\right)\log^{1-\delta} x. \tag{2.11}$$

Theorem 4 sharpens a result of J. M. De K o n i n c k - R. S i t a r a m a c h a n d r a r a o [6], who obtained

$$\sum_{2\le n\le x} \frac{1}{n\log P(n)} = e^{\gamma}\log\log x + O(1);$$

their paper contains a discussion on earlier results on this problem. The bound for the error term in (2.10) is of the correct order of magnitude. This was kindly pointed out to us by G. Tenenbaum.

3. The necessary lemmas

This section is devoted to the lemmas needed in the sequel.

Lemma 1. *For $2 \le y \le x$ we have uniformly*

$$\psi(x,y) = \sum_{n\le x,\ P(n)\le y} 1 \ll x\exp\left(-\frac{\log x}{2\log y}\right), \tag{3.1}$$

while for $\exp\left((\log\log x)^{5/3+\varepsilon}\right) \le y \le x$ we have the asymptotic formula

$$\psi(x,y) = x\rho(u)\left\{1 + O\left(\frac{\log(u+2)}{\log y}\right)\right\}, \quad u = \frac{\log x}{\log y}, \tag{3.2}$$

where the error term is uniform. The Dickman - de Bruijn function $\rho(u)$ is the continuous solution of the equation $u\rho'(u) = -\rho(u-1)$ with the initial condition $\rho(u) = 1$ for $0 \le u \le 1$. It satisfies

$$\rho(u) = \exp\{-u(\log u + \log\log u + O(1))\}. \tag{3.3}$$

These are standard results on $\psi(x,y)$, to be found e.g. in G. Tenenbaum [13].

Lemma 2. *For $\xi > 1$ fixed,*

$$\sum_{n>x} \frac{1}{n(\log P(n))^{\xi}} \ll_{\xi} \frac{1}{\log^{\xi-1} x}.$$

Proof. By partial summation the above sum may be written as

$$\begin{aligned}\sum_{p} \frac{1}{p\log^{\xi} p} \sum_{\substack{m>\frac{x}{p} \\ P(m)\le p}} \frac{1}{m} &= \sum_{p} \frac{1}{p\log^{\xi} p}\left(\frac{\psi(\frac{x}{p},p)}{\frac{x}{p}} + \int_{x/p}^{\infty} \frac{\psi(t,p)}{t^2}\,dt\right) \\ &= \Sigma_1 + \Sigma_2,\end{aligned}$$

where in Σ_1 summation is over $p \le \sqrt{x}$, and in Σ_2 over $p > \sqrt{x}$. Then using (3.1) we obtain, after change of variable $\frac{\log t}{\log p} = u$, $\frac{\log x}{\log v} = y$,

$$\begin{aligned}\Sigma_1 &\ll \sum_{p\le\sqrt{x}} \frac{1}{p\log^{\xi} p}\left(e^{-\frac{\log x}{2\log p}} + \int_{x/p}^{\infty} e^{-\frac{\log t}{2\log p}}\frac{dt}{t}\right) \\ &\ll \sum_{p\le\sqrt{x}} \frac{1}{p\log^{\xi-1} p} e^{-\frac{\log x}{2\log p}} = \int_2^{\sqrt{x}} \frac{1}{v\log^{\xi} v} e^{-\frac{\log x}{2\log v}}\,dv + O_{\xi}\left(\frac{1}{\log^{\xi-1} x}\right) \\ &= \frac{1}{\log^{\xi-1} x}\left(\int_2^{\log x/\log 2} y^{\xi-2}e^{-y/2}\,dy + O_{\xi}(1)\right) \ll_{\xi} \frac{1}{\log^{\xi-1} x},\end{aligned}$$

where we used the prime number theorem in the form

$$\pi(x) = \sum_{p\le x} 1 = \operatorname{Li} x + \Delta(x), \quad \Delta(x) = O\left(xe^{-\sqrt{\log x}}\right). \tag{3.4}$$

The trivial estimate $\psi(x,y) \le x$ gives

$$\begin{aligned}\Sigma_2 &= \sum_{p>\sqrt{x}} \frac{1}{p\log^{\xi} p}\left(\frac{\psi(\frac{x}{p},p)}{\frac{x}{p}} + \int_{x/p}^{\infty} \frac{\psi(t,p)}{t^2}\,dt\right) \\ &\ll_{\xi} \frac{1}{\log^{\xi} x} + \sum_{p>\sqrt{x}} \frac{1}{p\log^{\xi} p}\left(\int_{x/p}^{p} \frac{dt}{t} + \int_{p}^{\infty} \frac{\psi(t,p)}{t^2}\,dt\right) \\ &\ll_{\xi} \frac{1}{\log^{\xi-1} x} + \sum_{p>\sqrt{x}} \frac{1}{p\log^{\xi} p}\int_{p}^{\infty} \frac{\psi(t,p)}{t^2}\,dt.\end{aligned}$$

Finally by Lemma 1 the last sum is

$$\ll \sum_{p>\sqrt{x}} \frac{1}{p\log^{\xi} p} \int_p^{\infty} e^{-\frac{\log t}{2\log p}} \frac{dt}{t} \ll \sum_{p>\sqrt{x}} \frac{1}{p\log^{\xi-1} p} \ll_{\xi} \frac{1}{\log^{\xi-1} x}.$$

■

R e m a r k. By elaborating the method of proof given above, it can be shown that there exists $\kappa(\xi) > 0$ such that

$$\sum_{n>x} \frac{1}{n(\log P(n))^{\xi}} = \frac{\kappa(\xi)}{\log^{\xi-1} x} + O\left(\frac{(\log\log x)^{2\xi}}{\log^{\xi} x}\right).$$

Lemma 3. *If $p(n)$ is the smallest prime factor of an integer $n \geq 2$, then uniformly for $2 \leq y \leq x$*

$$\sum_{n\leq x,\ p(n)>y} 1 \ll \frac{x}{\log y}. \tag{3.5}$$

This is a well-known sieve bound. For a thorough discussion of estimates for the sum in (3.5) the reader is referred to G. Tenenbaum [13].

Lemma 4. *There is a positive constant $C(Q)$ such that*

$$\sum_{n\leq x,\ (n,Q)=1} 1 = \left(C(Q) + O\left(\frac{1}{\log\log x}\right)\right) \frac{x}{(\log x)^{\delta}}. \tag{3.6}$$

This is Lemma 5 of J. M. De K o n i n c k [3], and follows from the work of D. A. G o l d s t o n - K. S. M c C u r l e y [9].

Lemma 5. *If $2 \leq y \leq e^{\log^{\alpha} x}$ for some $0 < \alpha < 1$, then uniformly*

$$\sum_{n\leq x,\ (n,Q)>1,\ p(n)>y} 1 = \left(C(Q) + O\left(\frac{1}{\log\log x}\right)\right) \ell_Q(y) \frac{x}{(\log x)^{\delta}}, \tag{3.7}$$

where $C(Q)$ is the constant appearing in (3.6), and

$$\ell_Q(y) = \prod_{p\leq y,\ p\nmid Q} \left(1 - \frac{1}{p}\right). \tag{3.8}$$

This follows, in the special case when $y \le e^{\log^a x}$, from Lemma 7 of [3].

Lemma 6. *If $\ell_Q(y)$ is given by (3.8), then there is a positive constant ν_Q such that*

$$\ell_Q(y) = \left(\nu_Q + O\left(\frac{1}{\log y}\right)\right)\log^{\delta-1} y. \tag{3.9}$$

This is Lemma 8(i) of [3]. One has

$$\begin{aligned}
\log \ell_Q(y) &= \textstyle\sum_{p\le y,\ p\notin Q} \log\left(1-\frac{1}{p}\right)\\
&= -\textstyle\sum_{p\le y,\ p\notin Q}\frac{1}{p} - \sum_{m\ge 2}\frac{1}{m}\sum_{p\le y,\ p\notin Q}\frac{1}{p^m}\\
&= \textstyle\sum_{p\le y,\ p\in Q}\frac{1}{p} - \sum_{p\le y}\frac{1}{p} - \sum_{m\ge 2,\ p\in Q}\frac{1}{mp^m} + O\left(\frac{1}{y}\right)\\
&= \textstyle\int_{3/2}^{y}\frac{d\pi(t,Q)}{t} - \log\log y + D(Q) + O\left(\frac{1}{\log y}\right)
\end{aligned} \tag{3.10}$$

with some constant $D(Q)$. If we use (1.1) and integration by parts to evaluate the above integral, then we obtain (3.9) by exponentiating (3.10).

4. Proof of Theorem 1 and Theorem 2

We pass now to the proof of Theorem 1. A detailed proof will be given only for the case $k = 2$, and it will be indicated how to treat the general case, which is merely technically more complicated than the case $k = 2$. In evaluating

$$S(x) := \sum_{n\le x,\ (n,Q)>1} \frac{P_2(n,Q)}{P(n,Q)}$$

first note that the integers n for which $P(n,Q) = P(n)$ contribute $\ll x/\log x$, which follows from (2.1). If $P(n,Q) < P(n)$ and n contains at least two prime factors $p, q \in Q$, then n may be uniquely written as

$$n = mpqr,\ P(m) \le p \le q;\ p,q \in Q,\ (r,Q) = 1,\ p(r) > p,\ P(r) = P(n) > q. \tag{4.1}$$

Hence we have

$$S(x) = \sum_{\substack{mpqr\le x,\ P(m)\le p\le q;\ p,q\in Q\\ (r,Q)=1,\ p(r)>p,\ P(r)>q}} \frac{p}{q} + O\left(\frac{x}{\log x}\right). \tag{4.2}$$

Trivially in (4.2) we need to consider only p and q for which

$$\frac{p}{q} \geq \frac{1}{\log x}, \tag{4.3}$$

and we shall show now that we may consider only those m for which

$$m \leq e^{(\log x)^\alpha} \qquad (\delta < \alpha < 1). \tag{4.4}$$

Namely from Lemma 3 and Lemma 2 (with $\xi = 2$) we have

$$\begin{aligned}\sum_{\substack{mpqr\leq x,\ P(m)\leq p\leq q;\ p,q\in Q\\ (n,Q)=1,\ p(r)>p,\ P(r)>q,\ m\geq e^{\log^\alpha x}}} \frac{p}{q} &\leq \sum_{e^{(\log x)^\alpha}\leq m\leq x} \sum_{P(m)\leq p\leq q} \frac{p}{q} \sum_{r\leq \frac{x}{mpq},\ p(r)>p} 1\\ &\ll x \sum_{e^{(\log x)^\alpha}\leq m\leq x} \frac{1}{m} \sum_{p\geq P(m)} \frac{1}{\log p} \sum_{q\geq p} \frac{1}{q^2}\\ &\ll x \sum_{m\geq e^{(\log x)^\alpha}} \frac{1}{m(\log P(m))^2} \ll \frac{x}{(\log x)^\alpha},\end{aligned}$$

which is absorbed by the error term in Theorem 1 if $\delta < \alpha < 1$. Likewise we may suppose that

$$p \leq e^{(\log x)^\alpha} \qquad (\delta < \alpha < 1), \tag{4.5}$$

since by the preceding argument we obtain

$$\sum_{\substack{mpqr\leq x,\ P(m)\leq p\leq q;\ p,q\in Q\\ (r,Q)=1,\ p(r)>p,\ P(r)>q\\ m\leq e^{(\log x)^\alpha},\ p>e^{(\log x)^\alpha}}} \frac{p}{q} \ll x \sum_{m\leq e^{(\log x)^\alpha}} \frac{1}{m} \sum_{p>e^{(\log x)^\alpha}} \frac{1}{p\log^2 p} \ll \frac{x}{(\log x)^\alpha}.$$

In the portion of the sum in (4.2) for which the conditions (4.3)-(4.5) hold we have

$$mpq \leq mp^2 \log x \leq e^{3(\log x)^\alpha} \log x,$$

hence for these m, p and q we have

$$\frac{1}{\log\left(\frac{x}{mpq}\right)} = \frac{1}{\log x}\left(1 + O\left(\frac{\log(mpq)}{\log x}\right)\right) = \frac{1}{\log x}\left(1 + O\left((\log x)^{\alpha-1}\right)\right). \tag{4.6}$$

Further the condition $P(r) > q$ may be omitted, since by using Lemma 1 we obtain

$$\sum_{m\le e^{(\log x)^\alpha}} \sum_{P(m)\le p\le e^{(\log x)^\alpha}} p \sum_{p\le q\le p\log x} \frac{1}{q} \sum_{r\le \frac{x}{mpq},\ P(r)\le q} 1$$

(4.7)

$$\ll x \sum_{m\le e^{(\log x)^\alpha}} \frac{1}{m} \sum_{P(m)\le p\le e^{(\log x)^\alpha}} \sum_{p\le q\le p\log x} \frac{1}{q^2} e^{-\frac{\log(x/mpq)}{2\log q}},$$

and by (4.6)

$$\frac{\log(x/mpq)}{\log q} \ge \frac{1}{2}\frac{\log x}{\log q} \ge \frac{1}{2}\frac{\log x}{\log p + \log\log x} \ge \frac{1}{3}(\log x)^{1-\alpha}.$$

Hence the contribution of the left-hand side of (4.7) is

$$\ll x \exp\left(-\frac{1}{6}(\log x)^{1-\alpha}\right) \sum_{m\le e^{(\log x)^\alpha}} \frac{1}{m} \sum_{P(m)\le p\le e^{(\log x)^\alpha}} \frac{1}{p\log p} \ll$$

$$\ll x \exp\left(-\frac{1}{10}(\log x)^{1-\alpha}\right),$$

which is negligible. Finally by using Lemma 5 and (4.6) we obtain

$$\begin{aligned} S(x) = {} & \sum_{m\le e^{(\log x)^\alpha}} \sum_{\substack{P(m)\le p\le e^{(\log x)^\alpha} \\ p\in Q}} p \sum_{\substack{p\le q\le p\log x \\ p\in Q}} \frac{1}{q} \sum_{\substack{r\le x/(mpq) \\ (r,Q)=1,\ P(r)>p}} 1 \\ + {} & O\left(\frac{x}{(\log x)^\alpha}\right) \\ (4.8) \quad = {} & \left(C(Q)+O\left(\frac{1}{\log\log x}\right)\right) \frac{x}{(\log x)^\delta} \sum_{m\le e^{(\log x)^\alpha}} \frac{1}{m} \\ & \sum_{\substack{P(m)\le p\le e^{(\log x)^\alpha} \\ p\in Q}} \ell_Q(p) \times \times \sum_{\substack{p\le q\le p\log x \\ q\in Q}} \frac{1}{q^2} + O\left(\frac{x}{(\log x)^\alpha}\right) \\ = {} & \left(C(Q)+O\left(\frac{1}{\log\log x}\right)\right) E(Q) \frac{x}{(\log x)^\delta}, \end{aligned}$$

where

$$E(Q) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\substack{p\ge P(m) \\ p\in Q}} \ell_Q(p) \sum_{\substack{q\ge p \\ q\in Q}} \frac{1}{q^2}.$$

This follows since, by using Lemma 2 and Lemma 6, we obtain,

$$\begin{aligned} & \sum_{m\le e^{(\log x)^\alpha}} \frac{1}{m} \sum_{\substack{P(m)\le p\le e^{(\log x)^\alpha} \\ p\in Q}} \ell_Q(p) \sum_{\substack{p\le q\le p\log x \\ q\in Q}} \frac{1}{q^2} \\ = {} & \sum_{m\le e^{(\log x)^\alpha}} \frac{1}{m} \sum_{\substack{P(m)\le p\le e^{(\log x)^\alpha} \\ p\in Q}} \ell_Q(p) \sum_{\substack{q\ge p \\ q\in Q}} \frac{1}{q^2} + O\left(\frac{1}{\log x}\right) \\ = {} & \sum_{m\le e^{(\log x)^\alpha}} \frac{1}{m} \sum_{\substack{p\ge P(m) \\ p\in Q}} \ell_Q(p) \sum_{\substack{q\ge p \\ q\in Q}} \frac{1}{q^2} + \end{aligned}$$

$$+O\left(\sum_{m\le e^{\log^\alpha x}}\frac{1}{m}\sum_{p>e^{\log^\alpha x}}\frac{1}{p(\log p)^{2-\delta}}\right)+O\left(\frac{1}{\log x}\right)$$

$$= E(Q)+O\left(\sum_{m>e^{(\log x)^\alpha}}\frac{1}{m}\sum_{p\ge P(m)}\frac{1}{(\log p)^{1-\delta}}\sum_{q\ge p}\frac{1}{q^2}\right)+O\left((\log x)^{\alpha(\delta-1)}\right)$$

$$= E(Q)+O\left(\sum_{m>e^{(\log x)^\alpha}}\frac{1}{m(\log P(m))^{2-\delta}}\right)+O\left((\log x)^{\alpha(\delta-1)}\right)$$

$$= E(Q)+O\left((\log x)^{\alpha(\delta-1)}\right).$$

Theorem 1 for $k=2$ follows then from (4.8) with $C_2(Q)=C(Q)E(Q)$.

To treat the general sum

$$S_k(x):=\sum_{n\le x,\ (n,Q)>1}\frac{P_k(n,Q)}{P(n,Q)}\qquad (k\ge 3,\ \Omega(n,Q)\ge k)$$

one proceeds similarly as in the case $k=2$. Again we may suppose that $P(n,Q)<P(n)$ in view of (2.1). If n has at least k prime factors from Q (otherwise $P_k(n,Q)=0$ by definition) then we may write n uniquely as

$$n=mp_1p_2\dots p_kr,\ P(m)\le p_1\le\dots\le p_k;\ p_1\in Q,\dots,p_k\in Q;$$

$$(r,Q)=1,\ p(r)>p_1, P(r)>p_k. \tag{4.9}$$

Furthermore we may suppose that

$$\frac{p_1}{p_k}\ge\frac{1}{\log x},\quad m\le e^{\log^\alpha x},\quad p_1\le e^{\log^\alpha x}\qquad(\delta<\alpha<1), \tag{4.10}$$

using the same arguments that were used in the case $k=2$. Likewise the condition $P(r)>p_k$ may be discarded, and from (4.9) and (4.10) we shall obtain

$$\begin{aligned}S_k(x)=\ & O\left(\frac{x}{(\log x)^\alpha}\right)\\ &+\sum_{m\le e^{\log^\alpha x}}\sum_{\substack{P(m)\le p_1\le e^{\log^\alpha x}\\ p_1\in Q}}p_1\sum_{\substack{p_1\le p_2\le p_1\log x\\ p_2\in Q}}\dots\\ &\dots\sum_{\substack{p_{k-1}\le p_k\le p_1\log x\\ p_k\in Q}}\frac{1}{p_k}\sum_{\substack{r\le x/(mp_1\dots p_k)\\ (r,Q)=1,\ P(r)>p_1}}1.\end{aligned}$$

To evaluate the innermost sum we apply Lemma 5 with x replaced by $x/(mp_1\dots p_k)$ and $y=p_1$, which is possible in view of (4.10) and $p_1\le p_2\le\dots\le p_k$. The

ensuing estimations are performed as in the case $k = 2$. Theorem 1, with $C_k(Q)$ given by (2.3), readily follows.

We turn now to Theorem 2. We shall prove only (2.6), since the proof of (2.5) is quite similar. If $U(x)$ is the sum appearing in (2.6), then

$$U(x) = \sum_{n\le x,\ (n,Q)>1} 1 + \sum_{n\le x,\ (n,Q)>1} \frac{B(n,Q)-P(n,Q)}{P(n,Q)} = U_1(x) + U_2(x),$$

say. By using Lemma 4 we immediately obtain

$$U_1(x) = [x] - \sum_{n\le x,\ (n,Q)=1} 1 = x - \left(C(Q) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^{\delta}},$$

so it remains to evaluate $U_2(x)$. From (2.1) we have

$$\sum_{\substack{n\le x,\ (n,Q)>1\\ P(n,Q)=P(n)}} \frac{B(n,Q)-P(n,Q)}{P(n,Q)} \le \sum_{2\le n\le x,\ (n,Q)>1} \frac{P_2(n,Q)+\Omega(n,Q)P_3(n,Q)}{P(n)} \ll \frac{x}{\log x},$$

since for $n \ge 2$ and $k \ge 1$

$$P_k(n,Q) \le P_k(n), \quad \Omega(n,Q) \le \Omega(n) \le \frac{\log n}{\log 2}.$$

For the remaining n counted by $U_2(x)$ the decomposition (4.1) holds, since the sum is non-zero only if n has at least two prime factors from Q. Thus we have

$$U_2(x) = \sum_{\substack{mpqr\le x\\ P(m)\le p\le q;\ p,q\in Q\\ (r,Q)=1,\ P(r)>p,\ P(r)>q}} \frac{p+B(m,Q)}{q} + O\left(\frac{x}{\log x}\right), \tag{4.11}$$

since

$$B(n,Q) - P(n,Q) = B(mpq,Q) - q = p + B(m,Q)$$

by the additivity of $B(n,Q)$ (see (1.6)). We may suppose that the condition (4.3) holds, since

$$\sum_{\substack{mpqr\le x,\ p,q\in Q\\ P(m)\le p\le q,\ p/q\le 1/\log x\\ (r,Q)=1,\ P(r)>p,\ P(r)>q}} \frac{p+B(m,Q)}{q} \le \sum_{\substack{mpqr\le x,\ p,q\in Q\\ P(m)\le p\le q,\ p/q\le 1/\log x\\ (r,Q)=1,\ P(r)>p,\ P(r)>q}} \frac{p(1+\Omega(m))}{q} \le \frac{1}{\log x}\sum_{n\le x}\Omega(n) \ll \frac{x\log\log x}{\log x}.$$

We may also assume that (4.4) holds. Namely we have, for any fixed $c > 1$, that the contribution of the sum in (4.11) for which $m > e^{\log^\alpha x}$ is

$$\begin{aligned}
&\ll \sum_{e^{\log^\alpha x} < m \le x} \Omega(m) \sum_{p \ge P(m)} p \sum_{q \ge p} \frac{1}{q} \sum_{r \le \frac{x}{mpq},\ (r,Q)=1,\ p(r)>p} 1 \\
&\ll x \sum_{e^{\log^\alpha x} < m \le x} \frac{\Omega(m)}{m} \sum_{p \ge P(m)} \frac{1}{\log p} \sum_{q \ge p} \frac{1}{q^2} \ll x \sum_{e^{\log^\alpha x} < m \le x} \frac{\Omega(m)}{m \log^2 P(m)} \\
&= x \sum_{e^{\log^\alpha x} < m \le x} \frac{\Omega(m)}{m^{1/c}} \cdot \frac{1}{m^{(c-1)/c} \log^2 P(m)} \\
&\ll x \left(\sum_{m \le x} \frac{\Omega^c(m)}{m} \right)^{1/c} \left(\sum_{m > e^{\log^\alpha x}} \frac{1}{m (\log P(m))^{\frac{2c}{c-1}}} \right)^{\frac{c-1}{c}} \\
&\ll x (\log x)^{\frac{1-\alpha-\alpha c}{c}} \log\log x.
\end{aligned}$$

Here we used Lemma 3, Hölder's inequality, Lemma 2 with $\xi = \frac{2c}{c-1}$, and the elementary estimate

$$\sum_{n \le x} \Omega^c(n) \ll_c x (\log\log x)^c$$

if c is an integer. Since $\alpha < \delta < 1$ and

$$\lim_{c \to \infty} \frac{1 - \alpha - \alpha c}{c} = -\alpha,$$

it follows that the above contribution is certainly $\ll x (\log^\delta x \log\log x)^{-1}$ if c is a sufficiently large integer. Similarly we may assume that (4.5) holds, since the contribution of $p > e^{\log^\alpha x}$ is

$$\ll x \sum_{m \le e^{\log^\alpha x}} \frac{\Omega(m)}{m} \sum_{p \ge e^{\log^\alpha x}} \frac{1}{p \log^2 p} \ll x \log^\alpha x \log\log x \cdot (\log x)^{-2\alpha} = \frac{x \log\log x}{(\log x)^\alpha},$$

and that the condition $P(r) > q$ may be omitted. Thus following the method of proof of Theorem 1 we obtain, by using Lemma 5,

$$\begin{aligned}
U_2(x) &= \sum_{m \le e^{\log^\alpha x}} \sum_{\substack{P(m) \le p \le e^{\log^\alpha x} \\ p \in Q}} (p + B(m,Q)) \sum_{\substack{p \le q \le p \log x \\ q \in Q}} \frac{1}{q} \sum_{\substack{r \le x/(mpq) \\ (r,Q)=1,\ p(r)>q}} 1 \\
&\quad + O\left(\frac{x}{(\log x)^\delta \log\log x} \right)
\end{aligned}$$

$$= \frac{C(Q)x}{(\log x)^{\delta}} \sum_{m \le e^{\log^{\alpha} x}} \sum_{\substack{P(m) \le p \le e^{\log^{\alpha} x} \\ p \in Q}} (p + B(m,Q)) \ell_Q(p) \sum_{\substack{p \le q \le p \log x \\ q \in Q}} \frac{1}{q^2}$$

$$+ \; O\left(\frac{x}{(\log x)^{\delta} \log\log x}\right)$$

$$= \left(D_2'(Q) + O\left(\frac{1}{\log\log x}\right)\right) \frac{x}{(\log x)^{\delta}},$$

where

$$D_2'(Q) = C(Q) \sum_{m=1}^{\infty} \sum_{p \ge P(m),\ p \in Q} (p + B(m,Q)) \ell_Q(p) \sum_{q \ge p,\ q \in Q} \frac{1}{q^2}. \tag{4.12}$$

Hence we obtain

$$U(x) = x + \left(D_2'(Q) - C(Q) + O\left(\frac{1}{\log\log x}\right)\right) \frac{x}{(\log x)^{\delta}},$$

which proves (2.6) with $D_2(Q) = D_2'(Q) - C(Q)$. In proving (2.5) we shall encounter $\omega(m)$ instead of $\Omega(m)$, which is harmless since $\omega(m) \le \Omega(m)$. The only change is that, as $\beta(n,Q)$ counts the sum of distinct prime factors of n which belong to Q, in the analogue of (4.11) we shall suppose that $P(m) < p < q$, as the cases when $P(m) = p$, $p = q$ will make a negligible contribution. Hence the constant analogous to $D_2'(Q)$ of (4.12) will be

$$D_1'(Q) = C(Q) \sum_{m=1}^{\infty} \sum_{p > P(m),\ p \in Q} (p + \beta(m,Q)) \ell_Q(p) \sum_{q > p,\ q \in Q} \frac{1}{q^2},$$

which will clearly give $0 < D_1(Q) < D_2(Q)$ in Theorem 2. The essential reason why the method of proof of Theorem 1 could be extended to yield Theorem 2 is that one encounters $\Omega(m)$ at various places in the estimations (coming from $B(m,Q) \le \Omega(m) P(m,Q)$). Since $\Omega(m)$ has average and normal order equal to $\log\log m$, all the estimates are only affected by this factor which is small and therefore harmless.

5. Proof of Theorem 3

We shall prove (2.7) only, since the proof of (2.8) is similar. We have

$$\begin{aligned} \sum_{n \le x,\ (n,Q)>1} \frac{1}{\beta(n,Q)} &= \sum_{\substack{n \le x,\ (n,Q)>1 \\ P(n,Q) < P(n)}} \frac{1}{\beta(n,Q)} + \sum_{\substack{n \le x,\ (n,Q)>1 \\ P(n,Q) = P(n)}} \frac{1}{\beta(n,Q)} \\ &= \sum_{\substack{n \le x,\ (n,Q)>1 \\ P(n,Q) < P(n)}} \frac{1}{\beta(n,Q)} + O\left(\frac{x}{\log x}\right) \\ &= T(x) + O\left(\frac{x}{\log x}\right), \end{aligned} \tag{5.1}$$

say, since (2.9) holds with $P(n)$ in place of $\beta(n)$. If n is counted by $T(x)$, then n can be written uniquely as

$$n = mr,\ (r,Q) = 1,\ p(r) > P(m) \in Q. \tag{5.2}$$

Therefore we have

$$T(x) = \sum_{m\le x,\ P(m)\in Q} \frac{1}{\beta(m,Q)} \sum_{r\le x/m,\ (r,Q)=1,\ p(r)>P(m)} 1 = T_1(x) + T_2(x), \tag{5.3}$$

say, where in $T_1(x)$ we have $m \le e^{\log^\alpha x}$ $(\delta < \alpha < 1)$, and in $T_2(x)$ we have $e^{\log^\alpha x} < m \le x$. By using Lemma 3, Lemma 2 and

$$\beta(m,Q) \ge P(m) \ge \log P(m)$$

if $P(m) \in Q$, we obtain that

$$T_2(x) \ll x \sum_{m>e^{\log^\alpha x},\ P(m)\in Q} \frac{1}{m\beta(m,Q)\log P(m)}$$

$$\ll x \sum_{m>e^{\log^\alpha x}} \frac{1}{m\log^2 P(m)} \ll \frac{x}{(\log x)^\alpha}. \tag{5.4}$$

Now applying Lemma 5 we have

$$\begin{aligned} T_1(x) &= \textstyle\sum_{m\le e^{\log^\alpha x},\ P(m)\in Q} \frac{1}{\beta(m,Q)} \sum_{r\le x/m,\ (r,Q)=1,\ p(r)>P(m)} 1 \\ &= \textstyle\frac{C(Q)x}{(\log x)^\delta}\left(1 + O\left(\frac{1}{\log\log x}\right)\right) \sum_{m\le e^{\log^\alpha x},\ P(m)\in Q} \frac{\ell_Q(P(m))}{m\beta(m,Q)} \qquad (5.5) \\ &= \textstyle\left\{C(Q)\sum_{m=2,\ P(m)\in Q}^{\infty} \frac{\ell_Q(P(m))}{m\beta(m,Q)} + O\left(\frac{1}{\log\log x}\right)\right\} \frac{x}{(\log x)^\delta} \end{aligned}$$

by repeating the argument used in the estimation of $T_1(x)$. Thus we obtain from (5.1), (5.3), (5.4) and (5.5) that (2.7) holds with

$$\eta_1(Q) = C(Q) \sum_{m=2,\ P(m)\in Q}^{\infty} \frac{\ell_Q(P(m))}{m\beta(m,Q)}.$$

Likewise we obtain (2.8) with

$$\eta_2(Q) = C(Q) \sum_{m=2,\ P(m)\in Q}^{\infty} \frac{\ell_Q(P(m))}{mB(m,Q)},$$

and $0 < \eta_2(Q) < \eta_1(Q)$ holds in view of (1.10).

6. Proof of Theorem 4 and Theorem 5

If we can establish, for $x \geq 2$,

$$\sum_{2\leq n\leq x} \frac{1}{\log P(n)} = \frac{e^\gamma x}{\log x} + R(x), \qquad R(x) = O\left(\frac{x}{\log^2 x}\right), \tag{6.1}$$

then by partial summation (6.1) readily implies (2.10) with

$$B = \int_2^\infty R(t)\frac{dt}{t^2} - e^\gamma \log\log 2.$$

To prove (6.1) note that

$$\begin{aligned}\sum_{2\leq n\leq x} \frac{1}{\log P(n)} &= \sum_{p\leq x} \frac{1}{\log p}\psi(\frac{x}{p}, p)\\ &= \left(\sum_{p\leq L} + \sum_{L<p\leq\sqrt{x}} + \sum_{\sqrt{x}<p\leq x}\right)\frac{1}{\log p}\psi(\frac{x}{p}, p) = S_1 + S_2 + S_3,\end{aligned}$$

say, where

$$L := \exp\left(\frac{\log x}{(\log\log x)^2}\right).$$

From (3.1) of Lemma 1 we have

$$S_1 \ll x\sum_{p\leq L}\frac{1}{p\log p}e^{-\frac{\log x}{2\log p}} \leq xe^{-\frac{1}{2}(\log\log x)^2}\sum_p \frac{1}{p\log p} \ll \frac{x}{\log^2 x}, \tag{6.2}$$

since $\sum_p 1/(p\log p)$ converges. In the range $L < p \leq \sqrt{x}$ in S_2 we may use the asymptotic formula (3.2) to evaluate $\psi(\frac{x}{p}, p)$. We obtain

$$S_2 = x\sum_{L<p\leq\sqrt{x}}\frac{1}{p\log p}\rho\left(\frac{\log x}{\log p} - 1\right) +$$

$$+O\left(x\sum_{L<p\leq\sqrt{x}}\frac{\log\left(\frac{\log x}{\log p}+1\right)}{p\log^2 p}\rho\left(\frac{\log x}{\log p} - 1\right)\right). \tag{6.3}$$

By using (3.3) and (3.4) it is seen that the contribution of the O-term in (6.3) is

$$\begin{aligned}
&\ll x \sum_{L<p\le\sqrt{x}} \frac{\log\left(\frac{\log x}{\log p}+1\right)}{p\log^2 p} e^{-\left(\frac{\log x}{\log p}-1\right)\log\left(\frac{\log x}{\log p}-1\right)} \\
&= x\int_{L+0}^{\sqrt{x}} \frac{\log\left(\frac{\log x}{\log t}+1\right)}{t\log^2 t} e^{-\left(\frac{\log x}{\log t}-1\right)\log\left(\frac{\log x}{\log t}-1\right)} d\pi(t) \\
&= x\int_{L}^{\sqrt{x}} \frac{\log\left(\frac{\log x}{\log t}+1\right)}{t\log^3 t} e^{-\left(\frac{\log x}{\log t}-1\right)\log\left(\frac{\log x}{\log t}-1\right)} dt + O\left(\frac{x}{\log^2 x}\right) \\
&= \frac{x}{\log^2 x}\int_2^{(\log\log x)^2} u\log(u+1)e^{-(u-1)\log(u-1)}du + O\left(\frac{x}{\log^2 x}\right) \ll \frac{x}{\log^2 x},
\end{aligned}$$

after the substitution $\frac{\log x}{\log t} = u$. Similarly, substituting $\frac{\log x}{\log t} - 1 = v$, the main term in (6.3) equals

$$\begin{aligned}
x\int_{L+0}^{\sqrt{x}} \frac{1}{t\log t}\rho\left(\frac{\log x}{\log t}-1\right) d\pi(t) &= x\int_L^{\sqrt{x}} \rho\left(\frac{\log x}{\log t}-1\right)\frac{dt}{t\log^2 t} \\
&\quad -x\int_{L+0}^{\sqrt{x}} \Delta(t)\, d\left(\frac{1}{t\log t}\rho\left(\frac{\log x}{\log t}-1\right)\right) + \\
&\quad + O\left(\frac{x}{\log^2 x}\right) \\
&= \frac{x}{\log x}\int_1^{(\log\log x)^2-1} \rho(v)\,dv + O\left(\frac{x}{\log^2 x}\right) \\
&= \frac{(e^\gamma-1)x}{\log x} + O\left(\frac{x}{\log^2 x}\right)
\end{aligned}$$

Here we used (3.4), $\rho(u) = 1$ for $0 \le u \le 1$, $\rho'(u) = -\frac{\rho(u-1)}{u} \ll e^{-u}$ and

$$\int_0^\infty \rho(v)\,dv = e^\gamma. \tag{6.4}$$

For a proof of the well-known relation (6.4), see e.g. G. T e n e n b a u m [13]. Incidentally (6.4) follows in an elementary way if we compare our proof of Theorem 3 with the elementary derivation of Theorem 1.2 of J. M. De K o n i n c k - R. S i t a r a m a c h a n d r a r a o [6]. Hence we obtain

$$S_1 + S_2 = \frac{(e^\gamma-1)x}{\log x} + O\left(\frac{x}{\log^2 x}\right). \tag{6.5}$$

Lastly we have, since $\psi(x,y) = [x]$ for $y \geq x$,

$$S_3 = \sum_{\sqrt{x}<p\leq x} \frac{1}{\log p}\psi(\frac{x}{p},p) = \sum_{\sqrt{x}<p\leq x} \frac{1}{\log p}\left[\frac{x}{p}\right] = \sum_{m\leq\sqrt{x}} \sum_{\sqrt{x}<p\leq x/m} \frac{1}{\log p}.$$

From the prime number theorem we obtain

$$\sum_{p\leq y} \frac{1}{\log p} = \frac{y}{\log^2 y} + O\left(\frac{y}{\log^3 y}\right),$$

which yields

$$\begin{aligned} S_3 &= \textstyle\sum_{m\leq\sqrt{x}} \left(\frac{\frac{x}{m}}{\log^2(\frac{x}{m})} + O\left(\frac{\frac{x}{m}}{\log^3(\frac{x}{m})}\right) + O\left(\frac{\sqrt{x}}{\log^2 x}\right)\right) \\ &= \textstyle\sum_{m\leq\sqrt{x}} \frac{\frac{x}{m}}{\log^2(\frac{x}{m})} + O\left(\frac{x}{\log^2 x}\right) = x\int_1^{\sqrt{x}} \frac{dt}{t\log^2(\frac{x}{t})} + O\left(\frac{x}{\log^2 x}\right) \\ &= x\int_{\sqrt{x}}^{x} \frac{du}{u\log^2 u} + O\left(\frac{x}{\log^2 x}\right) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right). \end{aligned} \tag{6.6}$$

The asymptotic formula (6.1) follows then from (6.5) and (6.6).

To prove Theorem 4 we shall prove

$$\sum_{n\leq x,\ (n,Q)>1} \frac{1}{\log P(n,Q)} = \left(D(Q) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^\delta}, \tag{6.7}$$

where

$$D(Q) = C(Q) \sum_{m=2,\ P(m)\in Q}^{\infty} \frac{\ell_Q(P(m))}{m\log P(m)}, \tag{6.8}$$

and $C(Q)$, $\ell_Q(y)$ are as in Lemma 5. By partial summation Theorem 4 follows from (6.7) with $F(Q) = D(Q)/(1-\delta)$. Let $\delta < \alpha < 1$. Then we have

$$\begin{aligned} \sum_{n\leq x,\ (n,Q)>1} \frac{1}{\log P(n,Q)} &= \sum_{\substack{n\leq x,\ (n,Q)>1 \\ P(n,Q)<P(n) \\ P(n,Q)\leq e^{\log^\alpha x}}} \frac{1}{\log P(n,Q)} + O\left(\frac{x}{(\log x)^\alpha}\right) \\ &= \Sigma_0 + O\left(\frac{x}{(\log x)^\alpha}\right), \end{aligned}$$

say. Here we used the bound

$$\sum_{\substack{n\leq x,\ (n,Q)>1 \\ P(n,Q)=P(n)}} \frac{1}{\log P(n,Q)} \leq \sum_{2\leq n\leq x} \frac{1}{\log P(n)} \ll \frac{x}{\log x},$$

which is a trivial consequence of (6.1). If n is counted by $\sum_0$, then n can be uniquely written as

$$n = mr,\ P(m) \in Q,\ (r, Q) = 1,\ p(r) > P(m),$$

since $P(m) = P(n, Q) < P(n) = p(r)$. Thus

$$\Sigma_0 = \sum_{\substack{m \le x,\ P(m) \in Q \\ P(m) \le e^{\log^\alpha x}}} \frac{1}{\log P(m)} \sum_{\substack{r \le x/m,\ (r,Q)=1 \\ p(r) > P(m)}} 1 = \Sigma_1 + \Sigma_2,$$

say, where in Σ_1 we have $m \le K := e^{\log^\beta x}$ $(\alpha < \beta < 1)$, and in Σ_2 we have $K < m \le x$. By (3.1) of Lemma 1 we obtain

$$\begin{aligned}
\Sigma_2 &\le x \sum_{\substack{K < m \le x \\ P(m) \le e^{\log^\alpha x}}} \frac{1}{m \log P(m)} = x \sum_{p \le e^{\log^\alpha x}} \frac{1}{p \log p} \sum_{\substack{\frac{K}{p} < n \le \frac{x}{p} \\ P(n) \le p}} \frac{1}{n} \\
&= x \sum_{p \le e^{\log^\alpha x}} \frac{1}{p \log p} \left(\frac{\psi(t,p)}{t} \Big|_{\frac{K}{p}}^{\frac{x}{p}} + \int_{\frac{K}{p}}^{\frac{x}{p}} \psi(t,p) \frac{dt}{t^2} \right) \\
&\ll x \sum_{p \le e^{\log^\alpha x}} \frac{1}{p \log p} \left(e^{-\frac{\log K}{2 \log p}} + \int_{\frac{K}{p}}^{\frac{x}{p}} e^{-\frac{\log t}{2 \log p}} \frac{dt}{t} \right) \\
&\ll x \sum_{p \le e^{\log^\alpha x}} \frac{1}{p} e^{-\frac{\log K}{2 \log p}} \ll x e^{-\frac{1}{2} \log^{\beta - \alpha} x} \sum_{p \le e^{\log^\alpha x}} \frac{1}{p} \ll \frac{x}{(\log x)^\delta \log\log x},
\end{aligned}$$

since $\beta > \alpha$ and $\sum_{p \le x} \frac{1}{p} \ll \log\log x$. In Σ_1 we evaluate the inner sum by applying Lemma 5 to obtain

$$\begin{aligned}
S_1 &= \left(C(Q) + O\left(\frac{1}{\log\log x} \right) \right) \frac{x}{(\log x)^\delta} \sum_{\substack{m \le e^{\log^\beta x},\ P(m) \in Q \\ P(m) \le e^{\log^\alpha x}}} \frac{\ell_Q(P(m))}{m \log P(m)} \\
&= \left(D(Q) + O\left(\frac{1}{\log\log x} \right) \right) \frac{x}{(\log x)^\delta},
\end{aligned}$$

where $D(Q)$ is given by (6.8). This proves (6.7), but to justify the last equality above we proceed as follows. From Lemma 6 and Lemma 2 we obtain

$$\sum_{m > X,\ P(m) \in Q} \frac{\ell_Q(P(m))}{m \log P(m)} \ll \sum_{m > X} \frac{1}{m \log^{2-\delta} P(m)} \ll \frac{1}{\log^{1-\delta} X}. \tag{6.9}$$

Hence setting

$$Y = \exp\left((\log\log x)^{\frac{1}{1-\delta}}\right)$$

and using (6.9) we have

$$\begin{aligned}
&\sum_{\substack{m\le e^{\log^{\beta} x},\ P(m)\in Q\\ P(m)\le e^{\log^{\alpha} x}}} \frac{\ell_Q(P(m))}{m\log P(m)}\\
&= \sum_{\substack{m\le Y,\ P(m)\in Q\\ P(m)\le e^{\log^{\alpha} x}}} \frac{\ell_Q(P(m))}{m\log P(m)} + O\left(\frac{1}{\log^{1-\delta} Y}\right)\\
&= \sum_{m\le Y,\ P(m)\in Q} \frac{\ell_Q(P(m))}{m\log P(m)} + O\left(\frac{1}{\log\log x}\right)\\
&= \sum_{m=2,\ P(m)\in Q}^{\infty} \frac{\ell_Q(P(m))}{m\log P(m)} + O\left(\frac{1}{\log\log x}\right),
\end{aligned}$$

since $P(m) > e^{\log^{\alpha} x}$ is impossible if $m \le Y$. This completes the proof of Theorem 4.

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Received 13.07.94