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Integral Mamifolds of Singulary Perturbed Systems of Impulsive Differential Equations Defined on Tori

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Sufficient conditions are obtained for the existence of integral manifolds of singulary perturbed systems of impulsive differential equations defined on tori, and some of their properties are investigated.

1. Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and let T^m be an m-dimensional torus, i.e.

$$T^m = \{\varphi = (\varphi_1, \ldots, \varphi_m) : 0 \leq \varphi_i \leq 2\pi, i = 1, 2, \ldots, m\}$$

Consider the system of impulsive differential equations

$$\frac{d\varphi}{dt} = a(\varphi),$$

$$\mu \frac{dx}{dt} = A(\varphi)x, \ \varphi \overline{\in} \Gamma, \tag{1}$$

$$\Delta x = I(\varphi, \mu), \ \varphi \in \Gamma,$$

where $\mu > 0$ is a small parameter, $\varphi = (\varphi_1, \ldots, \varphi_m)$, $x = (x_1, \ldots, x_n)$ are respectively m - and n - dimensional vectors, $A(\varphi) = (a_{ij}(\varphi))$ is an $n \times n$ - dimensional matrix function, $a_{ij} : \mathbb{R}^m \to \mathbb{R}, a : \mathbb{R}^m \to \mathbb{R}^m, I : \mathbb{R}^m \times M \to \mathbb{R}^n, M \subset (0, \infty)$, Γ is an $(m-1)$ - dimensional manifold, $t \in \mathbb{R}$. Systems of the form (1) are characterized in the following way:

Between two successive meetings of the phase point with the set Γ the motion proceeds along one of the trajectories of the dynamical system $\frac{d\varphi}{dt} = a(\varphi), \mu\frac{dx}{dt} = A(\varphi)x$. At the moments of a meeting of the phase point with the set Γ the point $(\varphi(t), x(t))$ is momentarily transferred to the position $(\varphi(t), x(t) + I(\varphi(t), \mu))$.

The impulsive differential equations are adequate mathematical models of many evolutionary processes wich are subject to short-time effects during their evolution. They are successfully used for mathematical simulation in science and technology [1],[3]. It is know that the theory of these equations is considerably richer than the theory of the ordinary differential equations.

In the recent years the qualitative theory of the impulsive differential equations is developing intensively, while in the field of the integral manifolds the results obtained are relatively few.

In the present paper sufficient conditions are obtained for existence of integral manifolds of singularly perturbed systems of impulsive differential equations defined on tori, and some of their properties are investigated.

2. Preliminary notes. Statement of the problem

We shall denote by $< x, y >= x_1y_1 + \ldots + x_ny_n$ the scalar product of the vectors $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and by $\| x \| = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}}$ the norm of x.

Let the $(m-1)$ - dimensional manifold Γ be a subset of T^m, which is defined by the equation $\Phi(\varphi) = 0$, where $\Phi(\varphi)$ is a scalar function of the variable $\varphi = (\varphi_1, \ldots, \varphi_m)$, continuous and periodic with period 2π with respect to each of the variables $\varphi_i, i = 1, \ldots, m$, i.e. if $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, \ldots, 0), \ldots, e_m = (0, 0, \ldots, 1)$, then $\Phi(\varphi + 2\pi e_i) = \Phi(\varphi)$ for any $i = 1, \ldots, m$.

By $\varphi = \varphi(t, \varphi_0)$ we denote the solution of the Cauchy problem

$$\frac{d\varphi}{dt} = a(\varphi), \ \varphi(0) = \varphi_0$$

Then the system of equations

$$\Phi(\varphi) = 0$$

$$\varphi = \varphi(t, \varphi_0)$$

is equivalent to the equation.

$$\Phi(\varphi(t, \varphi_0)) = 0 \tag{2}$$

Henceforth we assume that the function $\Phi(\varphi)$ is chosen so that (2) has solutions of the form $t = t_i(\varphi_0), i \in \mathbb{Z} : t_i(\varphi_0) < t_{i+1}(\varphi_0)$ and

$$\lim_{i\to\pm\infty} t_i(\varphi_0) = \pm\infty,$$

$$\overline{\lim_{w\to\infty}} \frac{i(t, t+w)}{w} = p < \infty$$

uniformly with respect to $t \in \mathbb{R}$, where by $i(a, b)$ we have denoted the number of the points $t_i(\varphi_0)$ in the interval (a, b).

Here and whenever no misunderstandings arise, we write φ instead of φ_0. We shall note that if (2) has no solutions, the system (1) is not subject to impulse effect and becomes an ordinary dynamical system.

In the present paper we investigate the problem of existence of integral manifilds of the systems

$$\begin{aligned} \frac{d\varphi}{dt} = a(\varphi), \quad \mu\frac{dx}{dt} &= A(\varphi)x + f(\varphi, \mu), \quad \varphi\overline{\in}\Gamma, \\ \Delta x &= I(\varphi, \mu), \qquad \varphi \in \Gamma \end{aligned} \tag{3}$$

and

$$\begin{aligned} \frac{d\varphi}{dt} = a(\varphi), \quad \mu\frac{dx}{dt} &= A(\varphi)x + f(\varphi, x, \mu), \quad \varphi\overline{\in}\Gamma, \\ \Delta x &= I(\varphi, x, \mu), \qquad \varphi \in \Gamma \end{aligned} \tag{4}$$

The functions $f(\varphi, \mu)$ and $I(\varphi, \mu)$ map the set $\mathbb{R}^m \times M$ into $\mathbb{R}^n$, and the functions $f(\varphi, x, \mu)$ and $I(\varphi, x, \mu)$ map the set $\mathbb{R}^m \times \mathbb{R}^n \times M$ into $\mathbb{R}^n$, where $M \subset (0, \infty)$, and Γ is the manifold defined above.

By $\varphi(t) = \varphi(t, \varphi_0)$, $x(t) = x(t; x_0)$ we denote the respective solutions of (3) and (4) with initial conditions $\varphi(0) = \varphi_0, x(0) = x_0$.

Definition 1. *An arbitrary manifold J in the phase space of system (3) (or(4)) is said to be an* **integral manifold** *if for any solution $(\varphi(t), x(t))$ for which $(\varphi(t_0), x(t_0)) \in J$, it follows that $(\varphi(t), x(t)) \in J$ for $t \geq t_0$.*

In the paper we shall consider integral manifolds J defined as the graph of a function $x = U(\varphi)$ which is periodic with period 2π with respect to each of the components φ_i, $i = 1, \ldots, m$, and the function $U(\varphi(t))$ is piecewise continuous with discontinuities of the first kind of the points $t_i(\varphi)$, i.e.

$$J = \{(\varphi, x) : x = u(\varphi), \varphi \in T^m, x \in \mathbb{R}^n\} \tag{5}$$

Introduce the following conditions:

H1. The functions $f(\varphi, \mu)$ and $I(\varphi, \mu)$ are continuous for any $(\varphi, \mu) \in \mathbb{R}^m \times M$ and periodic with period 2π with respect to each of the variables

$$\varphi_i,\ i = 1, \ldots, m.$$

H2. The matrix functions $A(\varphi)$ and $A'(\varphi)$, where

$$A'(\varphi) = \frac{dA}{d\varphi} a(\varphi)$$

are continuous for $\varphi \in \mathbb{R}^m$ and periodic with period 2π with respect to each of the variables φ_i, $i = 1, \ldots, m$.

H3. The redl parts of the eigenvalues $\lambda_k(\varphi)$, $k = 1, 2, \ldots, n$, of the matrix $A(\varphi)$ satisgy the inequalities
Re $\lambda_k(\varphi) \leq -\alpha$, Re $\lambda_s(\varphi) \geq \beta$, $k = 1, 2 \ldots, r$, $s = r+1, \ldots, n$,
where α and β are positive constats.

H4. The function $a : \mathbb{R}^m \to \mathbb{R}^m$ is continuous for $\varphi \in \mathbb{R}^m$, periodic with period 2π with respect to each of the variables φ_i, $i = 1, \ldots, m$.

H5. The function $a(\varphi)$ is Lipschitz continuous a constant κ, i.e.

$$\| a(\varphi) - a(\overline{\varphi}) \| \leq \kappa \| \varphi - \overline{\varphi} \|, \qquad \varphi, \overline{\varphi} \in T^m.$$

3. Main results

Lemma 1. *Let a positive constant Θ exist such that for the solutions $t_i(\varphi), \varphi_0 \in T^m$ of equation (2) the inequality*

$$t_{i+1}(\varphi_0) - t_i(\varphi_0) \geq \Theta \tag{6}$$

holds for $i = \mathbb{Z}$. Then the following relation is valid

$$t_i(\varphi(-t, \varphi_0)) - t_i(\varphi_0) = t \tag{7}$$

for $\varphi_0 \in T^m$, $t \in \mathbb{R}$, $i \in \mathbb{Z}$.

Proof. Let $t_i(\varphi_0)$ be a solution of equation (2). Then $\Phi(\varphi(t_i(\varphi_0), \varphi_0)) = 0$ for $\varphi \in T^m$. From $\varphi(t, \varphi_0) \in T^m$ for all $t \in \mathbb{R}$ it follows that $\varphi(-t, \varphi_0) \in T^m$ for $t \in \mathbb{R}$. From the group property of the solution of the autonomous system $\frac{d\varphi}{dt} = a(\varphi)$ we otain

$$\Phi(\varphi(t_i(\varphi(-t, \varphi_0)) - t, \varphi_0)) = 0 \tag{8}$$

From (8) it folows that there exists $j \in \mathbb{Z}$ such that

$$t_j(\varphi_0) = t_i(\varphi(-t, \varphi_0)) - t$$

Obviously, for $t = 0$ from (6) we obtain that $i = j$ which implies (7). Consider the system

$$\mu \frac{dx}{dt} = A(\varphi(t, \varphi))x \tag{9}$$

where $\varphi(t, r)$ is an arbitrary solution of the system $\frac{d\varphi}{dt} = a(\varphi), \varphi \in T^m$ ■

Lemma 2. *Let conditions H2 - H5 hold. Then there exists a positive constant μ_0 such that for $\mu < \mu_0$ system (9) has a fundamental matrix $W(t, \varphi)$ such that the matrix*

$$G(t, s, \varphi) = \begin{cases} W(t, \varphi) P W^{-1}(s, \varphi), & t > s, \\ W(t, \varphi)(P - E_n) W^{-1}(s, \varphi), & s > t, \end{cases}$$

satisfies the inequality

$$\| G(t, s, \varphi) \| \leq N e^{-\frac{\gamma}{\mu}|t-s|}, t, s \in \mathbb{R} \tag{10}$$

where $\gamma > 0$, $N > 0$, $P = diag[E_r, 0]$, E_n, E_r are unit matrices of order respectively n and r.

P r o o f. This lemma follows directly from Theorem 3 in [2]. Indeed in the system (9) we set $t = \mu\tau$, we obtain

$$\frac{dx}{d\tau} = A(\varphi(\mu\tau, \varphi))\, x.$$

From conditions H1 and H2 it follows that A and A' are continuous as functions of t, hence they are continuous as functions of τ as well. On the other hand, from H3 it follows that, $\exists\, c_1 > 0,\ c_2 > 0$:

$$\| A(\varphi(\mu\tau, \varphi)) \| < c_1$$

$$\| \frac{d}{d\tau} A(\varphi(\mu\tau, \varphi)) \| \leq \| \frac{dA}{d\varphi} \cdot \frac{d\varphi}{d\tau} \| = \mu \| A'(\varphi(t, \varphi)) \| \leq \mu c_2 \tag{11}$$

From (11) it follows that for small values of the parameter μ all conditions of Theorem 3 in [2] are satisfied. Hence the fundamental matrix W of the system

$$\frac{dx}{d\tau} = A(\varphi(\mu\tau, \varphi))x,$$

exists and satisfies the inequalities:

$$(12) \qquad \| W(\mu\tau,\varphi)PW_{-1}(\mu s,\varphi) \| \leq Ke^{-(\alpha-\varepsilon_1)(\tau-s)}, \qquad \tau > s,$$

$$(13) \qquad \| W(\mu\tau,\varphi)W_{-1}(\mu s,\varphi) \| \leq Le^{-(\beta-\varepsilon_1)(\tau-s)}, \qquad \tau < s,$$

where $0 < \varepsilon_1 < \min(\alpha,\beta)$, and the positive constants K and L depend only on c_2, $\alpha+\beta$, ε_1. If we set $N = \max(K,L)$, $\tau = \frac{t}{\mu}$, $\gamma = \min(\alpha-\varepsilon,\beta-\varepsilon)$, then from (12) and (13) there follows (10). ■

Lemma 3. *Let $W(t,\varphi)$ be a fundamental matrix of system (9). Then the following relations are valid:*

$$(14) \qquad \mu\frac{dG(t,s,\varphi)}{dt} = A(\varphi(t))G(t,s,\varphi)$$

$$(15) \qquad G(t,t-0,\varphi) - G(t,t+0,\varphi) = E_n \qquad \textit{for } t \neq t_i(\varphi)$$

$$(16) \qquad G(t+0,t,\varphi) - G(t-0,t,\varphi) = E_n \qquad \textit{for } t = t_i(\varphi)$$

$$(17) \qquad G(t,s,\varphi(\tau,\varphi)) = G(t+\tau,s+\tau,\varphi)$$

$$(18) \qquad G(t,s,\varphi+2\pi e_i) = G(t,s,\varphi)$$

where $e_i = (0,\ldots,0,1,0,\ldots,0)$ for $i = 1,\ldots,m$.

P r o o f. Properties (14)-(16) are immediately verified, and (18) follows from the periodicity of the functions $a(\varphi), A(\varphi) >$ For the proof of (17) in

$$(19) \qquad \frac{dG(t,s,\varphi)}{dt} = A(\varphi(t))G(t,s,\varphi).$$

we replace φ by $\varphi(\tau)$. Then from the group property of teh solutions of the autonomous system $\frac{d\varphi}{dt} = a(\varphi)$, i.e.

$$\varphi(t,\varphi(\sigma)) = \varphi(t+\sigma,\varphi).$$

we obtain

$$(20) \qquad \frac{dG(t,s,\varphi(\sigma))}{dt} = A(\varphi(t+\sigma,\varphi))G(t,s,\varphi(\sigma))$$

From (19) and (20) there follows (17). ■

Theorem 1. *Let the following conditions hold:*
1. Conditions H1 - H5 are met.
2. The moments of impulse effect $t_i(\varphi)$ are such that

$$t_{i+1}(\varphi) - t_i(\varphi) \geq \theta,$$

where θ is a positive constant, $\varphi \in T^m$.

Then there exists $\mu_1(\mu_1 \leq \mu_0)$ such that for each $\mu \in (0, \mu_1]$ system (3) has an integral manifold of the form (5) defined as a graph of the function $x = U(\varphi, \mu)$ where $U(\varphi + 2\pi e_i, \mu) = U(\varphi, \mu)$, $i = 1, \dots, m$. Moreover, there exists a constant C independent of $f(\varphi, \mu)$ and $I(\varphi, \mu)$ such that the following inewuality is valid

$$\| U(\varphi, \mu) \| \leq C \max\{\max_{\varphi \in T^m} \| f \| \max_{\varphi \in T^m} \| I \|\} \tag{21}$$

Proof. From condition H6 it follows that system (1) is equivalent to system (9). Then each bounded solution of the system (3) depending on φ and μ as on parameters is given by the formula

$$x(t, \varphi, \mu) = \frac{1}{\mu} \int_{-\infty}^{\infty} G(t, s, \varphi) f(\varphi(s, \varphi), \mu) ds +$$

$$+ \sum_{i=-\infty}^{\infty} G(t, t_i, \varphi) I(\varphi(t_i), \mu) \tag{22}$$

In (22) we set $x = U(\varphi(t), \mu)$ and replace φ by $\varphi(-t)$. Then from Lemma 1, Lemma 2, Lemma 3 we obtain that

$$U(\varphi, \mu) = \frac{1}{\mu} \int_{-\infty}^{\infty} G(0, s, \varphi) f(\varphi(s, \varphi), \mu) ds +$$

$$+ \sum_{i=-\infty}^{\infty} G(0, t_i, \varphi) I(\varphi(t_i, \varphi), \mu). \tag{23}$$

From (10) it follows that for each $\mu \leq \mu_0$ the integral and the sum are uniformly convergent, and from (14) and (15) it follows that

$$\mu \frac{dU(\varphi(t), \mu)}{dt} = A(\varphi(t) U(\varphi(t), \mu) + f(\varphi(t), \mu), \quad t \neq t_i,$$

and for $t = t_i$ from (16) it follows that

$$\Delta U = U(\varphi(t_i + 0), \mu) - U(\varphi(t_i - 0), \mu) = I(\varphi(t_i - 0), \mu)$$

Hence (23) defines an integral manifold J of the form (5) . From the representstion of $U(\varphi, \mu)$ and (17) it follows that $U(\varphi, \mu)$ is periodic with period 2π with respect to the variable φ.

On the other hand,

$$\| U(\varphi, \mu) \| \leq \frac{1}{\mu} \int_{-\infty}^{\infty} \| G(0, s, \varphi) \| \, \| f(\varphi(s, \varphi), \mu) \| \, ds +$$

$$(24) \qquad + \sum_{i=-\infty}^{\infty} \| G(0, t_i, \varphi) \| \times \| I(\varphi(t_i, \varphi), \mu) \| \leq \frac{2K}{\gamma} \mid f \mid + \frac{2K}{1 - e^{-\frac{r\theta}{\mu_0}}} \mid I \mid$$

If in (24) we set $C = \max(\frac{2K}{\gamma}, \frac{2K}{1-e^{-\frac{\gamma\theta}{\mu_0}}})$, we obtain (21). ■

Theorem 2. *Let the following conditions hold:*

1. Conditions H2-H5 are satisfied.

2. The functions $f(\varphi, x, \mu)$ and $I\varphi, x, \mu)$ are continuous, periodic with period 2π with respect to each of the vaiables φ_i, $i = 1, \ldots, m$ for each $(x, \mu) \in \mathbb{R}^m \times M$ and

$$\max_{\varphi \in T^m, \mu > 0} \| f(\varphi, 0, \mu) \| + \max_{\varphi \in T^m, \mu > 0} \| I(\varphi, 0, \mu) \| = M^*$$

where M^* is a positive constant.

3. The Lipschitz condition is valid

$$\| f(\varphi, x, \mu) - f(\varphi, \overline{x}, \mu) \| + \| I(\varphi, x, \mu) - I(\varphi, \overline{x}, \mu) \| \leq \lambda \| x - \overline{x} \|,$$

where $\lambda = \lambda(\mu) \to 0$ as $\mu \to 0, x, \overline{x} \in \mathbb{R}^n$.

4. The moments of impulse effect $t_i(\varphi)$ are such that

$$t_{i+1}(\varphi) - t_i(\varphi) \geq \Theta > 0.$$

Then there existx $\mu_1 (0 < \mu_1 < \mu_0)$ such that for each $\mu \in (0, \mu_1]$ the system (4) has an integral manifold of the form (5).

Proof. Consider the sequence of functions $U^{(k)}(\varphi, \mu)$, where

$$U^{(0)} \equiv 0,$$

$$U^{(k)}(\varphi,\mu)=\frac{1}{\mu}\int_{-\infty}^{\infty}G(0,s,\varphi)f(\varphi(s).U^{(k-1)}(\varphi(s),\mu),\mu)ds+$$

(25) $$+\sum_{i=-\infty}^{\infty}G(0,t_i,\varphi)I(\varphi(t_i),U^{(k-1)}(\varphi(t_i),\mu),\mu)$$

Then

$$\| U^{(1)}(\varphi,\mu)\|\leq\frac{1}{\mu}\int_{-\infty}^{\infty}Ne^{-\frac{\gamma}{\mu}|s|}\| f(\varphi(s),0,\mu)\| ds+$$

(26) $$+\sum_{i=-\infty}^{\infty}Ne^{-\frac{\gamma}{\mu}|t_i|}\| I_i(\varphi(t_i),0,\mu)\|\leq\frac{2N}{\gamma}(1+\frac{\gamma}{1-e^{-\frac{r\theta}{\mu}}})M^*$$

Introduce the notation

(27) $$a=\frac{2N}{\gamma}(1+\frac{\gamma}{1-e^{-\frac{\gamma\theta}{\mu}}})M^*,\ b=\frac{\lambda a}{M^*}$$

Then there exists $\mu(\mu_1<\mu_0)$ such that for $\mu\in(0,\mu_1]$ we have $\frac{\lambda a}{m}<1$, i.e. the following inequality is valid

$$\| U^{(k)}(\varphi,\mu)\|\leq\| U^{(k)}-U^{(1)}\|+\| U^{(1)}\|\leq\frac{a}{1-b}.$$

From Lemma 2 it follows that

$$\| U^{(k+1)}(\varphi,\mu)-U^{(k)}(\varphi,\mu)\|\leq$$

$$\frac{1}{\mu}\int_{-\infty}^{\infty}Ne^{-\frac{\gamma}{\mu}|s|}\lambda\| U^{(k)}(\varphi(s),\mu)-U^{(k-1)}(\varphi(s),\mu)\| ds+$$

$$+\sum_{i=-\infty}^{\infty}Ne^{-\frac{\gamma}{\mu}|t_i|}\lambda\| U^{(k)}(\varphi(t_i),\mu)-U^{(k-1)}(\varphi(t_i),\mu)\|\leq$$

(28) $$\leq\frac{2N\lambda}{\gamma}(1+\frac{\gamma}{1-e^{-\frac{\gamma\theta}{\mu}}})\max_{\varphi\in T^m,\mu>0}\| U^{(k)}-U^{(k-1)}\|$$

Then from (27) and (28) there follows a uniform convergence of the sequence (25) to the function $U(\varphi,\mu)$, i.e. $\lim_{k\to\infty}U^{(k)}(\varphi,\mu)=U(\varphi,\mu)$ which proves the theorem. ■

Consider the system

$$\frac{d\varphi}{dt} = a(\varphi)$$

$$x(t) = G(t, t_0, \varphi)\eta + \frac{1}{\mu}\int_{t_0}^{\infty} G(t, s, \varphi) f(\varphi(s), x(s), \mu) ds +$$

$$+ \sum_{t_0 < t_i} G(t, t_i, \varphi) I(\varphi(t_i, x(t_i), \mu) \tag{29}$$

where η is an arbitrary vector, $t_0 \in \mathbb{R}$.

Lemma 4. *Let the following conditions hold:*

1. Conditions H2 - H5 are satisfied.

2. The functions $f(\varphi, x, \mu)$ and $I(\varphi, x, \mu)$ are continuous, periodic with period 2π with respect to each of the variables φ_i, $i = 1, \ldots, m$ for each $(x, \mu) \in \mathbb{R}^n \times M$ and

$$\max_{\varphi \in T^m, \mu > 0} \| f(\varphi, 0, \mu) \| + \max_{\varphi \in T^m, \mu} \| I(\varphi, 0, \mu) \| = M^*$$

where M^* is a constant.

3. The Lipschitz condition is valid

$$\| f(\varphi, x, \mu) - f(\varphi, \bar{x}, \mu) \| + \| I(\varphi, x, \mu) - I(\varphi, \bar{x}, \mu) \| \leq \lambda \| x - \bar{x} \|$$

where $\lambda = \lambda(\mu) \to 0$ as $\mu \to 0, x, \bar{x} \in \mathbb{R}^n$.

4. The moments of impulse effect $t_i(\varphi)$ are such that

$$t_{i+1}(\varphi) - t_i(\varphi) \geq \theta,$$

where θ is a positive constant.

Then there exists $\mu_2 (\mu_2 \leq \mu_1)$ such that for $\mu \leq \mu_2$ and for each η such that $\| \eta \| < \sigma_0$, where $\sigma_0 > 0$ has a unique bounded solution.

Proof. We construct the sequence

$$x^{(k)}(t) = G(t, t_0, \varphi)\eta +$$

$$\frac{1}{\mu}\int_{t_0}^{\infty} G(t, s, \varphi) f(\varphi(s), x^{(k_1)}(s), \mu) ds +$$

$$+ \sum_{t_0 < t_i} G(t, t_i, \varphi) I(\varphi(t_i), x^{(k-1)}(t_i), \mu) \tag{30}$$

where $x^{(0)} \equiv 0$. From (30) we obtain

$$\| x^{(1)} \| \leq Ne^{-\frac{\gamma}{\mu}(t-t_0)} \| \eta \| + \frac{2N}{\gamma} m[1 + \frac{\gamma}{1 - e^{-\frac{\gamma\theta}{\mu_0}}}]$$

Introduce the notation

$$c = \frac{2N\lambda}{\gamma}(1 + \frac{\gamma}{1 - e^{-\frac{\gamma\theta}{\mu_0}}})$$

$$d = Ne^{-\frac{\gamma}{\mu}(t-t_0)} \| \eta \| + \frac{cM^*}{\lambda}$$

Then there exists $\mu_3(\mu_3 \leq \mu_2)$ such that for $\mu \in (0, \mu_3]$ we have $c < 1$, i.e. the following inequality is valid

$$\| x^{(k)} \| \leq \| x^{(k)} - x^{(1)} \| + \| x^{(1)} \leq \frac{d}{1-c} \tag{31}$$

On the other hand,

$$\| x^{(k+1)} - x^{(k)} \| \leq \frac{1}{\mu} \int_{t_0}^{\infty} Ne^{-\frac{\gamma}{\mu}|t-s|} \lambda \| x^{(k)} - x^{(k-1)} \| ds +$$

$$+ \sum_{t_0 < t_i} Ne^{-\frac{\gamma}{\mu}|t-t_i|} \lambda \| x^{(k)} - x^{(k-1)} \| \leq C \mid x^{(k)} - x^{(k-1)}. \tag{32}$$

Then for $\mu \in (0, \mu_3]$ from (31) and (32) it follows that the sequence (30) is uniformly convergemt to a limit $\Psi(t, \varphi, \eta)$, i.e.

$$\lim_{k \to \infty} x^{(k)}(t) = \Psi(t, \varphi, \eta). \tag{33}$$

From (33) it follows that $(\varphi(t), \Psi(t, \varphi, \eta))$ is a solution of the system (29), and from (31) it follows that this solution is bounded.

■

R e m a r k 1. If $(\varphi(t), x(t))$ is a solution of (29), then a straightforward vertication yields that it is a solution of system (4) too. On the other hand, each solution of system (4) for which $\| x_0 \| < \sigma_0$, is a solution of the system (29) for $t \geq t_0$.

Consequently, those solution of (1) for which $\| x_0 \| < \sigma_0$, admit a parametric representation

$$\varphi(t), x(t) = \Psi(t, \varphi, \eta), \qquad \mid \eta \mid < \sigma_0. \tag{34}$$

Teorema 3. *Let the following conditions hold:*

1. Conditions (H) are satisfied.

2. The functions $f(\varphi, x, \mu)$ and $I(\varphi, x, \mu)$ are continuous periodic with period 2π with respect to the variables φ_i, $i = 1, \dots, m$ for each $(x, \mu) \in \mathbb{R}^n \times M$ and

$$\max_{\varphi \in T^m, \mu > 0} \| f(\varphi, 0, \mu) \| + \max_{\varphi \in T^m, \mu > 0} \| I(\varphi, 0, \mu) \| = M^*, \; M^* > 0.$$

3. The lipschitz condition is valid with respect to the variable x

$$\| f(\varphi, x, \mu) - f(\varphi, \overline{x}, \mu) \| + \| I(\varphi, x, \mu) - I(\varphi, \overline{x}, \mu) \| \leq \lambda \| x - \overline{x} \|$$

where $\lambda = \lambda(\mu) \to 0$ as $\mu \to 0$; $x, \overline{x} \in \mathbb{R}^n$.

4. The moments of impulse effect $t_i(\varphi)$ are such that

$$t_{i+1}(\varphi) - t_i(\varphi) \geq \Theta,$$

where Θ is a positive constant.

Then there exists $\mu_4 (\mu_4 \leq \mu_3)$ such that for each $\mu \in (0, \mu_4)$ in a σ_0 - neighbourhood of the point $x = 0$ there exists an r -dimensional manifold J_r of initial values with the property:

If $x(t_0) \in J_r$, then the inequality

$$| x(t) - u(\varphi(t), \mu) | \leq N_1 e^{-\frac{\gamma}{\mu}(t - t_0)} | x(t_0) - u(\varphi(t_0), \mu) |, \tag{35}$$

is valid, where N_1 is a positive constant.

P r o o f. Since the conditions for existence of integral manifolds of the system (4) and the conditions for existence of bounded solutions of the system (29) for $\| x_0 \| < \sigma_0$ are equivalent, then from Lemma 4 it follows that all solutions lying on the integral manifold J of the system (4) are solutions of the system (29). From (34) it follows that for each solution of (4) lying on J there exists a number $\overline{\eta}$ such that

$$\varphi(t), x = u(\varphi(t), \mu) = \Psi(t, \varphi, \overline{\eta})$$

i.e.

$$\begin{aligned} U(\varphi(t), \mu) = {} & G(t, t_0, \varphi)\overline{\eta} + \\ & + \frac{1}{\mu} \int_{t_0}^{\infty} G(t, s, \varphi) f(\varphi(s), U(\varphi(s), \mu) ds + \\ & + \sum_{t_0 < t_i} G(t, t_i, \varphi) I(\varphi(t_i), U(\varphi(t_i), \mu), \mu). \end{aligned} \tag{36}$$

We set in (36) $\overline{\eta} = U(\varphi(t_0), \mu)$. From (29) and Lemma 4 we obtain

$$x(t) - u(\varphi(t), \mu) = G(t, t_0, \varphi)(x_0 - u(\varphi(t_0), \mu)) +$$

$$+\frac{1}{\mu}\int_{t_0}^{\infty} G(t, s, \varphi)(f(\varphi(s), x(s), \mu) - f(\varphi(s), u(\varphi(s), \mu), \mu))ds$$

$$+\sum_{t_0<t_i} G(t, t_i, \varphi)(I(\varphi(t_i), x(t_i), \mu) - I(\varphi(t_i), U(\varphi(t_i)), \mu).$$

Then from Lemma 2 it follows that

$$\| x(t) - U(\varphi(t), \mu) \| \left[1 - \frac{2\lambda N}{\gamma}(1 + \frac{\gamma}{1 - e^{-\frac{\gamma\theta}{\mu_0}}})\right] \leq$$

$$\leq N e^{-\frac{\gamma}{\mu}(t-t_0)} \| x_0 - U(\varphi(t_0), \mu) \| .$$

From Lemma 4 it follows that there exists $\mu_4 \leq \mu_3$ such that for $\mu \in (0, \mu_4]$

$$1 - \frac{2\lambda N}{\gamma}(1 + \frac{\gamma}{1 - e^{-\frac{\gamma\theta}{\mu_0}}}) > 0,$$

i.e.

$$| x(t) - U(\varphi(t), \mu) | \leq N_1 e^{-\frac{\gamma}{\mu}(t-t_0)} | x(t_0) - U(\varphi(t_0), \mu) |,$$

where

$$N_1 = N\Big(1 - \frac{2\lambda N}{\gamma}(1 + \frac{\gamma}{1 - e^{-\frac{\gamma\theta}{\mu_0}}})\Big)^{-1}$$

By J_r we denote the set of points x in a bounded neighbourhood of the point $x = 0$ for which $x = \Psi(t_0, \varphi_0, \eta)$. Since each bounded solution of (4) for which $\| x_0 \| < \sigma_0$ satisfies the relation

$$x(t) = \Psi(t, \varphi(t), \eta), \; \| x_0 \| < \sigma_0$$

then for $t = t_0$ we obtain that for each bounded solution of (4) for which $\| x_0 \| < \sigma_0$ we have $x_0 \in J_r$.

■

R e m a r k 2. We shall note that if $n = r$, then J_r coincides with the whole σ_0 -neighbourhood of the point $x = 0$. In this case the integral manifold J is exponentiallt stable in the sense that if $\| x_0 \| < \sigma_0$, then (35) is fulfilled for each $t > t_0$.

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