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Composition of Fractional Integration Operators Involving Multivariable H-Function

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In this paper the authors derive compositions of fractional integral operators associated with multivariable H-function and product of r-general class of polynomials introduced and studied recently by the authors and V.S. Kiryakova in the same journal. Special classes of these results yield compositions concerning fractional integration operators, associated with the ${}_pF_q$-function, Meijer's G-function, Fox's H-function and multivariable H-function.

1. Introduction

In a recent paper [22] R. K. Saxena, V. Kiryakova and O. P. Dave introduced two-generalized fractional integration operators in the following form:

$$I_{z_r}^{\eta,\alpha}\{f(x)\} = \xi x^{-\eta-\xi\alpha-1}\int_0^x t^{\eta}(x^{\xi}-t^{\xi})^{\alpha}$$

$$\times H_{p,q:p_1,q_1;\dots;p_r q_r}^{0,n:m_1,n_1;\dots;m_r n_r}\left[\begin{matrix} z_1U_1 \\ \cdot \\ \cdot \\ \cdot \\ z_rU_r\end{matrix}\;\middle|\;\begin{matrix}(a_j;\alpha'_j,\dots,\alpha_j^{(r)})_{1,p}: \\ (b_j;\beta'_j,\dots,\beta_j^{(r)})_{1,q}:\end{matrix}\right.$$

$$\left.\begin{matrix}(c'_j,\gamma'_j)_{1,p_1},\dots,(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ (d'_j,\delta'_j)_{1,q_1},\dots,(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix}\right]\prod_{i=1}^{r}\left\{S_{B_i}^{A_i}\left[y_i\left(\frac{t^{\xi}}{x^{\xi}}\right)^{e_i}\right.\right.$$

$$(1.1)\qquad .\left(1-\frac{t^{\xi}}{x^{\xi}}\right)^{\sigma_i}\ \Big]\Big\}\,\varphi\Big[\frac{t^{\xi}}{x^{\xi}}\Big]f(t)dt$$

$$R_{z_r}^{\delta,\alpha}\{f(x)\} = \xi x^{\delta} \int_x^{\infty} t^{-\delta-\xi\alpha-1}(t^{\xi} - x^{\xi})^{\alpha}$$

$$\times H_{p,q:p_1,q_1;\dots;p_rq_r}^{0,n:m_1,n_1;\dots;m_rn_r}\left[\begin{array}{c} z_1V_1 \\ \cdot \\ \cdot \\ \cdot \\ z_rV_r \end{array}\middle| \begin{array}{l} (a_j;\alpha'_j,\dots,\alpha_j^{(r)})_{1,p} : \\ (b_j;\beta'_j,\dots,\beta_j^{(r)})_{1,q} : \end{array}\right.$$

$$\left.\begin{array}{l}(c'_j,\gamma'_j)_{1,p_1},\dots,(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ (d'_j,\delta'_j)_{1,q_1},\dots,(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{array}\right] \prod_{i=1}^{r}\left\{S_{B_i}^{A_i}\left[y_i(\frac{x^{\xi}}{t^{\xi}})^{e_i}\right.\right.$$

$$.(1 - \frac{x^{\xi}}{t^{\xi}})^{\sigma_i}]\} \varphi[\frac{x^{\xi}}{t^{\xi}}]f(t)dt, \tag{1.2}$$

where U_i and V_i; $i = 1, 2, \dots, r$ respectively, represent the expressions:

$$U_i = (\frac{t^{\xi}}{x^{\xi}})^{M_i}(1 - \frac{t^{\xi}}{x^{\xi}})^{N_i};$$

$$V_i = (\frac{x^{\xi}}{t^{\xi}})^{M_i}(1 - \frac{x^{\xi}}{t^{\xi}})^{N_i};\ i = 1, \dots, r,$$

and ξ, M_i, N_i are positive numbers. The kernel $\varphi(\frac{t^{\xi}}{x^{\xi}})$ occurring in (1.1) and (1.2) is supposed to be a continuous function, such that the integrals make sense for a wide class of functions $f(x)$.

The operators (1.1) and (1.2) exist under the following sets of conditions:

(i) $1 \le p,\ q < \infty,\ p^{-1} + q^{-1} = 1$;

(ii) $Re[\eta + \xi \sum_{i=1}^{r}(M_i \frac{d_j^{(i)}}{\delta_j^{(i)}} + e_i g_i)] > -\frac{1}{q};\ j = 1, \dots, m_r$;

$Re[\alpha + \sum_{i=1}^{r}(N_i \frac{d_j^{(i)}}{\delta_j^{(i)}} + \sigma_i g_i)] > -\frac{1}{q\xi};\ j = 1, \dots, m_r$;

$Re[\delta + \xi \sum_{i=1}^{r}(M_i \frac{d_j^{(i)}}{\delta_j^{(i)}} + e_i g_i)] > -\frac{1}{p};\ j = 1, \dots, m_h$;

where $g_i = 0, 1, 2, \dots, [B_i/A_i]$, $i = 1, \dots, r$;

(iii) $\Omega_i = -\sum_{j=n+1}^{p}\alpha_j^{(i)} + \sum_{j=1}^{n_i}\gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i}\gamma_j^{(i)} - \sum_{j=1}^{q}\beta_j^{(i)}$
$+\sum_{j=1}^{m_i}\delta_j^{(i)} - \sum_{j=n_i+1}^{q_i}\delta_j^{(i)} > 0;\ |\arg z_i| < \frac{\pi}{2} - \Omega_i$;

$$i = 1, \dots, r; \tag{1.3}$$

(iv) $f(x) \in L_p(0, \infty)$.

The last condition ensures that $I_{z_r}^{\eta,\alpha}\{f(x)\}$ and $R_{z_r}^{\delta,\alpha}\{f(x)\}$ both exist and belong to $L_p(0, \infty)$.

The multivariable H-function introduced and studied by H. Srivastava and R. Panda [28] (also see [27], pp.251-253) appearing in (1.1) and (1.2) is defined as follows:

$$H[z_1,...,z_r] = H^{0,n:m_1,n_1;...;m_r n_r}_{p,q:p_1,q_1;...;p_r q_r}\left[\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\middle| \begin{array}{l} (a_j;\alpha'_j,...,\alpha^{(r)}_j)_{1,p}: \\ (b_j;\beta'_j,...,\beta^{(r)}_j)_{1,q}: \end{array}\right.$$

(1.4)

$$\left.\begin{array}{l}(c'_j,\gamma'_j)_{1,p_1},...,(c^{(r)}_j,\gamma^{(r)}_j)_{1,p_r} \\ (d'_j,\delta'_j)_{1,q_1},...,(d^{(r)}_j,\delta^{(r)}_j)_{1,q_r}\end{array}\right]$$

$$= \frac{1}{(2\pi w)^r}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)...\theta_r(s_r)\psi(s_1,\ldots,s_r) \times z_1^{s_1}...z_r^{s_r}\,ds_1...ds_r$$

where $w=\sqrt{-1}$;

$$\theta_i(s_i) = \frac{\prod_{j=1}^{m_i}\Gamma(d^{(i)}_j-\delta^{(i)}_j s_i)\prod_{j=1}^{n_i}\Gamma(1-c^{(i)}_j+\gamma^{(i)}_j s_i)}{\prod_{j=m_i+1}^{q_i}\Gamma(1-d^{(i)}_j+\delta^{(i)}_j s_i)\prod_{j=n_i+1}^{p_i}\Gamma(c^{(i)}_j-\gamma^{(i)}_j s_i)},$$

(1.5) $$i=1,...,r;$$

(1.6) $$\psi(s_1,\ldots,sk) = \frac{\prod_{j=1}^{n}\Gamma(1-a_j+\sum_{i=1}^{r}\alpha^{(i)}_j s_i)}{\prod_{j=n+1}^{p}\Gamma(a_j-\sum_{i=1}^{r}\alpha^{(i)}_j s_i)\prod_{j=1}^{q}\Gamma(1-b_j-\sum_{i=1}^{r}\beta^{(i)}_j s_i)}$$

All the Greek letters are assumed to be positive real numbers; the definition of the multivariable H-function is however meaningful even if some of these numbers are zero. For convergence conditions of this function the reader is referred to the original monograph [27].

The general class of polynomials $S^{\nu}_{\mu}(z)$ is introduced and studied by H. Srivastava 24,p.1,(1)], in the form:

(1.7) $$S^{\nu}_{\mu}(x) = \sum_{t=0}^{[\mu/nu]}\frac{(-\mu)_{\nu t}}{t!}A_{\mu,t}(x^t);\ \mu=0,1,2,...$$

where ν is an arbitrary positive integer and the coefficients $A_{\mu,t}(\mu,t\geq 0)$ are arbitrary real or complex constants.

If we set $\varphi(x)\equiv 1$, $M_i=0$, $e_i=0$ in (1.1) and (1.2), then it yields the following operators:

$$I^{*\eta,\alpha}_{z_r}\{f(x)\} = \xi x^{-\eta-\xi\alpha-1}\int_0^x t^{\eta}(x^{\xi}-t^{\xi})^{\alpha}$$

$$\times H^{0,n:m_1,n_1;...;m_r n_r}_{p,q:p_1,q_1;...;p_r q_r}\left[\begin{matrix} z_1X_1 \\ \cdot \\ \cdot \\ \cdot \\ z_rX_r \end{matrix} \middle| \begin{matrix} (a_j;\alpha'_j,...,\alpha^{(r)}_j)_{1,p}: \\ (b_j;\beta'_j,...,\beta^{(r)}_j)_{1,q}: \end{matrix}\right.$$

$$\left.\begin{matrix}(c'_j,\gamma'_j)_{1,p_1},...,(c^{(r)}_j,\gamma^{(r)}_j)_{1,p_r} \\ (d'_j,\delta'_j)_{1,q_1},...,(d^{(r)}_j,\delta^{(r)}_j)_{1,q_r}\end{matrix}\right]\prod_{i=1}^{r}\left\{S^{A_i}_{B_i}\left[y_i(1-\frac{t^\xi}{x^\xi})^{\sigma_i}\right]\right\}]f(t)dt \tag{1.8}$$

and

$$R^{*\delta,\beta}_{z_r}\{f(x)\} = \xi x^\delta \int_x^\infty t^{-\delta-\xi\beta-1}(t^\xi - x^\xi)^\beta$$

$$\times H^{0,N:M_1,N_1;...;M_r N_r}_{P,Q:P_1,Q_1;...;P_r Q_r}\left[\begin{matrix} Z_1Y_1 \\ \cdot \\ \cdot \\ \cdot \\ Z_rY_r \end{matrix} \middle| \begin{matrix} (h_j;\epsilon'_j,...,\epsilon^{(r)}_j)_{1,P}: \\ (k_j;\mu'_j,...,\mu^{(r)}_j)_{1,Q}: \end{matrix}\right.$$

$$\left.\begin{matrix}(e'_j,\theta'_j)_{1,P_1},...,(e^{(r)}_j,\theta^{(r)}_j)_{1,P_r} \\ (f'_j,\varphi'_j)_{1,Q_1},...,(f^{(r)}_j,\varphi^{(r)}_j)_{1,Q_r}\end{matrix}\right]\prod_{i=1}^{r}\left\{S^{C_i}_{D_i}\left[y'_i(1-\frac{x^\xi}{t^\xi})^{\rho_i}\right]\right\}]f(t)dt \tag{1.9}$$

where $X_i = (1-\frac{t^\xi}{x^\xi})^{\lambda_i}$ and $Y_i = (1-\frac{x^\xi}{t^\xi})^{\nu_i}$; $i = 1, 2, ..., r$

These operators exist under the following sets of conditions: (i) $1 \le p$, $q < \infty$, $p^{-1} + q^{-1} = 1$;

(ii) $Re(\eta) > -\frac{1}{q}$; $Re[\alpha + \sum_{i=1}^{r}(\lambda_i \frac{d^{(i)}_j}{\delta^{(i)}_j} + \sigma_i g_i)] > -\frac{1}{q\xi}$;

$Re[\beta + \sum_{i=1}^{r}(\nu_i \frac{f^{(i)}_{j'}}{\varphi^{(i)}_{j'}} + \rho_i l_i)] > -\frac{1}{q\xi}$;

$Re[\delta + \xi\sum_{i=1}^{r}(\nu_i \frac{f^{(i)}_{j'}}{\varphi^{(i)}_{j'}} + \rho_i l_i)] > -\frac{1}{p}$

where $g_i = 0, 1, 2, ..., [B_i/A_i]$, $l_i = 0, 1, 2, ..., [D_i/C_i]$;

$i = 1, ..., r$; $j = 1, ..., m_r$; $j' = 1, ..., M_r$;

(iii) $\Lambda_i = -\sum_{j=N+1}^{P}\epsilon^{(i)}_j + \sum_{j=1}^{N_i}\theta^{(i)}_j - \sum_{j=N_i+1}^{P_i}\theta^{(i)}_j$

$$-\sum_{j=1}^{Q}\mu^{(i)}_j + \sum_{j=1}^{M_i}\varphi^{(i)}_j - \sum_{j=M_i+1}^{Q_i}\varphi^{(i)}_j > 0 \tag{1.10}$$

the condition (1.3) is also satisfied, and

(iv) $f(x) \in L_p(0, \infty)$.

The object of this paper is to establish some compositions of the operators (1.8) and (1.9). The technique developed here is believed to be new.

The main results of this paper unify the earlier results by S.P. Goyal, R.M. Jain and Neelima Gaur [7], R.K. Saxena, Y. Singh and A. Ramawat [23], H.M. Srivastava and R.G. Buschman [25], M. Saigo [19], H.M. Srivastava, S.P. Goyal and R.M. Jain [26], V.S. Kiryakova [12,13], S. L. Kalla and V.S. Kiryakova [11], S. L. Kalla [9,10] and several others.

2. Auxiliary results

The following results are used in next sections [8, p.286, (3.197, 3)], [3, p.64, (23)], [4, p.201, (8)] and [3, p.62, (15)]:

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-\beta x)^{-\nu}dx = B(\lambda,\mu)\,{}_2F_1(\nu;\lambda;\lambda+\mu;\beta) \tag{2.1}$$

provided that $Re(\lambda) > 0$, $Re(\mu) > 0$, $|\beta| < 1$;

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z),\ |z| < 1; \tag{2.2}$$

$$\int_y^\infty x^{-\lambda}(x+\alpha)^{\nu}(x-y)^{\mu-1}dx$$

$$= y^{\mu+\nu-\lambda}B(\lambda,\lambda-\mu-\nu)(1+\alpha/y)^{\mu+\nu}\,{}_2F_1(\lambda,\mu;\lambda-\nu;-za/w), \tag{2.3}$$

provided that $0 < Re(\mu) < Re(\lambda-\mu)$, $|\alpha/y| < 1$,

$${}_2F_1(a,b;c;-z) = \frac{1}{2\pi w}\int_{c-w\infty}^{c+w\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} z^s ds \tag{2.4}$$

provided that $|\arg(-z)| < \pi$ and $q = \sqrt{-1}$.

3.Compositions of the operators defined by (1.8) and (1.9)

Theorem 1. *If* $f(x) \in L_p(0,\infty)$, $1 \leq p \leq 2$ *[or* $f(x) \in M_p(0,\infty)$, $p > 2$*],* $Re(\eta) > -\frac{1}{d}$; $Re[\alpha + \sum_{i=1}^{r}(\lambda_i \frac{d_j^{(i)}}{\delta_j^{(i)}} + \sigma_i g_i)] > -\frac{1}{g\xi}$, $g_i = 0, 1, 2, ..., [B_i/A_i]$, $i = 1, ..., r$;

$$Re[\beta + \sum_{i=1}^{r}(\lambda_i \frac{d_j^{(i)}}{\delta_j^{(i)}} + \sigma_i l_i)] > -\frac{1}{g\xi};\ l_i = 0, 1, 2, ..., [B_i/A_i],$$

$i = 1, ..., r$, $Re(\delta) > -\frac{1}{d}$; $Re[\alpha + \beta + \sum_{i=1}^{r}(\sigma_i(g_i + l_i) + \lambda_i \frac{d_j^{(i)}}{\delta_j^{(i)}})] > -2$, *then* $I_{z_r}^{*\eta,\alpha} I_{z_r}^{*\delta,\beta}\{f(x)\} \in L_p(0,\infty)$ *and the following result holds:*

$$I_{z_r}^{*\eta,\alpha} I_{z_r}^{*\delta,\beta}\{f(x)\} = \xi(x^\xi)^{-\alpha-\beta-1-\frac{\delta+1}{\xi}}$$
$$\times \prod_{i=1}^{r} \left\{ \sum_{g_i=0}^{(B_i/A_i)} \sum_{l_i=0}^{(D_i/C_i)} \frac{(-B_i)_{A_i g_i} (-B)_{A_i l_i} A_{B_i, g_i} A_{B_i, l_i} y_i^{(g_i+l_i)}}{g_i! l_i!} \right\}$$
$$\times (x^\xi)^{-\sum_{i=1}^{r} \sigma_i(g_i+l_i)} \int_0^x u^\delta (x^\xi - u^\xi)^{\alpha+\beta+1+\sum_{i=1}^{r} \sigma_i(g_i+l_i)}$$
$$\times H^{0,\ 2n+4:m_1 n_1;...:m_r n_r; m_1 n_1;...;m_r, n_r;0,1}_{2p+4, 2q+4:p_1 q_1;...:p_r q_r; p_1 q_1;...;p_r, q_r;0,1}$$

$$\times \left[\begin{array}{c|c} z_1(1-\frac{u^\xi}{x^\xi})^{\lambda_1} & \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r(1-\frac{u^\xi}{x^\xi})^{\lambda_r} & \\ z_1(1-\frac{u^\xi}{x^\xi})^{\lambda_1} & \chi_1 \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r(1-\frac{u^\xi}{x^\xi})^{\lambda_r} & \\ (\frac{u^\xi}{x^\xi} - 1) & \end{array} \right] f(u)du, \tag{3.1}$$

where

$$\chi_1 = \begin{array}{l} (a_j; \alpha'_j, ..., \alpha_j^{(r)}, \overbrace{0, ..., 0}^{(r+1)})_{1,p} : (a_j; \overbrace{0, ..., 0}^{(r)}, \alpha'_j, ..., \alpha_j^{(r)}, 0)_{1,p} : \\ (b_j; \beta'_j, ..., \beta_j^{(r)}, \underbrace{0, ..., 0}_{(r+1)})_{1,q} : (b_j; \underbrace{0, ..., 0}_{(r)}, \beta'_j, ..., \beta_j^{(r)}, 0)_{1,q} : \end{array}$$

$$(-\alpha-\textstyle\sum_{i=1}^r \sigma_i g_i; \lambda_1,\dots,\lambda_r, \overbrace{0,\dots,0}^{(r+1)}) : (-\beta-\sum_{i=1}^r \sigma_i l_i; \overbrace{0,\dots,0}^{(r)},$$
$$(-1-\alpha-\beta-\textstyle\sum_{i=1}^r \sigma_i(g_i+l_i); \lambda_1,\dots,\lambda_r,\lambda_1,\dots,\lambda_r,0) :$$

$$\lambda_1,\dots,\lambda_r,0) : (-\beta-\tfrac{(\delta-\eta)}{\xi}-\textstyle\sum_{i=1}^r \sigma_i l_i; \overbrace{0,\dots,0}^{(r)},\lambda_1,\dots,\lambda_r,1) :$$
$$(-1-\alpha-\beta-\textstyle\sum_{i=1}^r \sigma_i(g_i+l_i); \lambda_1,\dots,\lambda_r; \lambda_1,\dots,\lambda_r,1) :$$

$$: \ (-\alpha-\textstyle\sum_{i=1}^r \sigma_i g_i; \lambda_1,\dots,\lambda_r, \overbrace{0,\dots,0}^{(r)},1) \ : (c'_j,\gamma'_j)_{1,p_1};\dots;$$
$$: \qquad - \qquad : (d'_j,\delta'_j)_{1,q_1};\dots;$$

$$(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}; (c'_j,\gamma'_j)_{1,p_1};\dots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}; -$$
$$(d_j^{(r)},\delta_j^{(r)})_{1,q_r}; (d'_j,\delta'_j)_{1,q_1};\dots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r}; (0,1)$$

Proof. In view of definition (1.8), it follows that

$$\begin{aligned}(3.2)\quad & I_{z_r}^{*\eta\alpha} I_{z_r}^{*\delta\beta}\{f(x)\} = \xi x^{-\eta-\xi\alpha-1}\textstyle\int_0^x t^\eta (x^\xi-t^\xi)^\alpha \\ & \times H[z_1,X_1,\dots,z_rX_r]\textstyle\prod_{i=1}^r \left\{S_{B_i}^{A_i}\left[y_i(1-\frac{t^\xi}{x^\xi})^{\sigma_i}\right]\right\}\left(\xi t^{-\delta-\xi\beta-1}\right. \\ & \times \textstyle\int_0^t u^\delta (t^\xi-u^\xi)^\beta f(u) H\left[z_1(1-\frac{u^\xi}{t^\xi})^{\lambda_1},\dots,z_r(1-\frac{u^\xi}{t^\xi})^{\lambda_r}\right] \\ & \times \textstyle\prod_{i=1}^r \left\{S_{B_i}^{A_i}\left[y_i(1-\frac{u^\xi}{t^\xi})^{\sigma_i}\right]\right\} \left.du\right) dt\end{aligned}$$

If we interchange the order of integration, which is permissible under the conditions stated with the theorem, it is found that

$$(3.3)\qquad I_{z_r}^{*\eta\alpha} I_{z_r}^{*\delta\beta}\{f(x)\} = \xi^2 x^{-\eta-\xi\alpha-1}\int_0^x \Omega_1 u^\delta f(u)du,$$

where

$$\begin{aligned}(3.4)\quad \Omega & = \textstyle\int_u^x t^{\eta-\delta-\xi\beta-1}(x^\xi-t^\xi)^\alpha (t^\xi-u^\xi)^\beta H[z_1X_1,\dots,z_rX_r] \\ & \times \textstyle\prod_{i=1}^r \left\{S_{B_i}^{A_i}\left[y_i(1-\frac{t^\xi}{x^\xi})^{\sigma_i}\right]\right\} H\left[z_1(1-\frac{u^\xi}{t^\xi})^{\lambda_1},\dots,z_r(1-\frac{u^\xi}{t^\xi})^{\lambda_r}\right] \\ & \times \textstyle\prod_{i=1}^r \left\{S_{B_i}^{A_i}\left[y_i(1-\frac{u^\xi}{t^\xi})^{\sigma_i}\right]\right\} dt.\end{aligned}$$

If we write the series representation (1.7) for $\prod_{i=1}^r S_{\mu_i}^{\nu_i}(z_i)$ and Mellin-Barnes integral representation (1.4) for the multivariable H-functions appearing in (3.4), and interchange the order of integration, we will obtain Ω as

$$\begin{aligned}(3.5)\quad & \textstyle\prod_{i=1}^r \left\{\sum_{g_i=0}^{(B_i/A_i)}\sum_{l_i=0}^{(B_i/A_i)} \frac{(-B_i)_{A_i g_i}(-B_i)_{A_i l_i} A_{B_i,g_i} A_{B_i,l_i} y_i^{(g_i+l_i)}}{g_i! l_i!}\right\} \\ & \times \tfrac{1}{(2\pi w)^{2r}} \textstyle\int_{L_1}\cdots\int_{L_{2r}} \psi(s_1,\dots,s_r)\psi(s'_1,\dots,s'_r \\ & \times \textstyle\prod_{i=1}^r \left\{\theta_i(s_i)\theta_i(s'_i) z_i^{s_i+s'_i}\right\} \left(\int_u^x t^{\eta-\delta-\xi\beta-\xi\sum_{i=1}^r(\lambda_i s'_i+\sigma_i l_i)-1}\right. \\ & \times (x^\xi-t^\xi)^{\alpha+\sum_{i=1}^r(\lambda_i s_i+\sigma_i g_i)}(x^\xi)^{-\sum_{i=1}^r(\lambda_i s_i+\sigma_i g_i)} \\ & \times (t^\xi-u^\xi)^{\beta+\sum_{i=1}^r(\lambda_i s'_i+\sigma_i l_i)} \left.dt\right) ds_1\dots ds_r ds'_1\dots ds'_r.\end{aligned}$$

The t-integral can be evaluated fairly easily by the substitution

$$w^* = \frac{x^\xi - t^\xi}{x^\xi - u^\xi}.$$

The w^*-integral can be evaluated with the help of the integral (2.1) and then on using (2.4) the equation (3.5) transforms into the desired form (3.1), when we apply the definition (1.4) appropriately. ∎

The proof of (3.1) presented here in reasonable detail, can be applied mutatis mutandis in order to prove the composition formulas contained in next Theorem 2 and Theorem 3.

Theorem 2. *If* $f(x) \in L_p(0,\infty)$, $1 \le p \le 2$ *[or* $f(x) \in M_p(0,\infty)$, $p > 2]$, $p^{-1} + q^{-1} = 1$

$$Re[\alpha + \sum_{i=1}^{r}(\nu_i \frac{f_j^{(i)}}{\varphi_j^{(i)}} + \rho_i g_i)] > -\frac{1}{g\xi};$$

$$Re[\beta + \sum_{i=1}^{r}(\nu_i \frac{f_j^{(i)}}{\varphi_j^{(i)}} + \rho_i l_i)] > -\frac{1}{g\xi};$$

$$g_i, l_i = 0, 1, 2, ..., [D_i/C_i], \ i = 1, ..., r;$$

then $R_{z_r}^{\eta,\alpha} R_{z_r}^{\delta,\beta}\{f(x)\} \in L_p(0,\infty)$ *and the following result holds:*

$$R_{z_r}^{\eta,\alpha} R_{z_r}^{\delta,\beta}\{f(x)\} = \xi x^\eta \prod_{i=1}^r \Big\{\sum_{g_i=0}^{(D_i/C_i)} \sum_{l_i=0}^{(D_i/C_i)} \frac{(-D_i)_{C_i g_i}}{g_i!}$$
$$\times \frac{(-D_i)_{C_i l_i} A_{D_i, g_i} A_{D_i, l_i}}{l_i!} (y_i')^{(g_i + l_i)} \Big\}$$
$$\times \int_x^\infty (u^\xi)^{-\frac{(\eta+1)}{\xi} - \alpha - \beta - 1 - \sum_{i=1}^r \rho_i (g_i + l_i)}$$
$$\times (u^\xi - x^\xi)^{\alpha + \beta + \sum_{i=1}^r \rho_i (g_i + l_i) + 1} \cdot H^{0,\ 2N+4:M_1 N_1;...:M_r N_r;}_{2P+4, 2Q+2:P_1 Q_1;...:P_r Q_r;}$$

$$\text{(3.6)} \qquad \begin{matrix} M_1, N_1; ...; M_r N_r; 1, 0 \\ P_1, Q_1; ...; P_r Q_r; 0, 1 \end{matrix} \left[\begin{matrix} z_1(1 - \frac{x^\xi}{u^\xi})^{\nu_1} \\ \cdot \\ \cdot \\ \cdot \\ z_r(1 - \frac{x^\xi}{u^\xi})^{\nu_r} \\ z_1(1 - \frac{x^\xi}{u^\xi})^{\nu_1} \\ \cdot \\ \cdot \\ \cdot \\ z_r(1 - \frac{x^\xi}{u^\xi})^{\nu_r} \\ (\frac{x^\xi}{u^\xi} - 1) \end{matrix} \middle| \chi_2 \right] f(u) du,$$

where

$$\chi_2 = \begin{array}{l} (h_j;\epsilon'_j,...,\epsilon_j^{(r)},\overbrace{0,...,0}^{(r+1)})_{1,P} : (h_j;\overbrace{0,...,0}^{(r)},\epsilon'_j,...,\epsilon_j^{(r)},0)_{1,P} : \\ (k_j;\mu'_j,...,\mu_j^{(r)},\underbrace{0,...,0}_{(r+1)})_{1,Q} : (k_j;\underbrace{0,...,0}_{(r)},\mu'_j,...,\mu_j^{(r)},0)_{1,Q} : \end{array}$$

$$(-\beta-\textstyle\sum_{i=1}^r \rho_i l_i;\overbrace{0,...,0}^{(r+1)}) : (-\alpha-\sum_{i=1}^r \rho_i g_i;\nu_1,...,\nu_r,\overbrace{0,...,0}^{(r+1)}) :$$

$$(-1-\alpha-\beta-\textstyle\sum_{i=1}^r \rho_i(g_i+l_i);\nu_1,...,\nu_r,\nu_1,...,\nu_r,0) :$$

$$(-1-\alpha-\beta-\textstyle\sum_{i=1}^r \rho_i(g_i+l_i);$$

$$(\tfrac{\delta-\eta}{\xi}-\alpha-\textstyle\sum_{i=1}^r \rho_i g_i;\nu_1,...,\nu_r,\overbrace{0,...,0}^{(r)},1) :$$

$$\nu_1,...,\nu_r,\nu_1,...,\nu_4,1) :$$

$$: \;(-\beta-\textstyle\sum_{i=1}^r \rho_i l_i;\overbrace{0,...,0}^{(r)},\nu_1,...,\nu_r,1) \quad : (e'_j,\theta'_j)_{1,P_1};...;(e_j^{(r)},\theta_j^{(r)})_{1,P_r};$$

$$- \qquad : (f'_j,\varphi'_j)_{1,Q_1};...;(f_j^{(r)},\varphi_j^{(r)})_{1,Q_r};$$

$$(e'_j,\theta'_j)_{1,P_1};...;(e_j^{(r)},\theta_j^{(r)})_{1,P_r}-$$

$$(f'_j,\varphi'_j)_{1,Q_1};...;(f_j^{(r)},\varphi_j^{(r)})_{1,Q_r};(0,1)$$

4.Composition of mixed type

Theorem 3. *If* $f(x) \in L_p(0,\infty)$, $1 \le p \le 2$ *[or* $f(x) \in M_p(0,\infty)$, $p > 2$*]*, $p^{-1}+q^{-1}=1$, $Re(\eta) > -\frac{1}{q}$;

$$Re[\alpha+\sum_{i=1}^r(\lambda_i\frac{d_j^{(i)}}{\delta_j^{(i)}}+\sigma_i g_i)] > -\frac{1}{g\xi};$$

$$Re[\beta+\sum_{i=1}^r(\nu_i\frac{f_j^{(i)}}{\varphi_j^{(i)}}+\rho_i l_i)] > -\frac{1}{g\xi}; g_i, l_i = 0,1,2,...,[B_i/A_i],$$

$l_i = 0, 1, 2, ..., [D_i/C_i]$, $Re(\delta) > -\frac{1}{p}$ *then*

$$I^{*\eta,\alpha}_{z_r} R^{*\delta,\beta}_{Z_r}\{f(x)\} = R^{*\eta,\alpha}_{Z_r} I^{*\delta,\beta}_{z_r}\{f(x)\} \in L_p(0,\infty)$$

and the following result holds:

$$I^{*\eta,\alpha}_{z_r} R^{*\delta,\beta}_{Z_r}\{f(x)\} = R^{*\eta,\alpha}_{Z_r} I^{*\delta,\beta}_{z_r}\{f(x)\}$$

$$= \xi(x^{\xi})^{-\frac{\eta+1}{\xi}}\Gamma(\tfrac{\eta+\delta+1}{\xi})\prod_{i=1}^{r}\Big\{\sum_{g_i=0}^{(B_i/A_i)}\sum_{l_i=0}^{(D_i/C_i)}$$

$$\frac{(-B_i)_{A_i g_i}(-D_i)_{C_i l_i} A_{B_i,g_i} A_{D_i,l_i} y_i^{g_i}(y_i')^{l_i}}{g_i! l_i!}\Big\}$$

$$\times \int_0^x u^{\eta}(1-\tfrac{u^{\xi}}{x^{\xi}})^{\alpha+\beta+\sum_{i=1}^{r}(\sigma_i g_i+\rho_i l_i)+1}$$

$$\times H^{0,\ n+N+3\ :m_1 n_1;...:m_r n_r;M_1,N_1;...;M_r,N_r;1,0}_{p+p+3,q+Q+2:p_1 q_1;...:p_r q_r;P_1,Q_1;...P_r Q_r;0,1}$$

$$\left[\begin{array}{l|l} z_1(1-\frac{u^{\xi}}{x^{\xi}})^{\lambda_1} & \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r(1-\frac{u^{\xi}}{x^{\xi}})^{\lambda_r} & \\ z_1(1-\frac{u^{\xi}}{x^{\xi}})^{\lambda_1} & \chi_3 \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r(1-\frac{u^{\xi}}{x^{\xi}})^{\lambda_r} & \\ (\frac{u^{\xi}}{x^{\xi}}-1) & \end{array}\right] f(u)du + \xi(x^{\delta})\Gamma(\frac{\eta+\delta+1}{\xi})$$

$$\prod_{i=1}^{r}\Big\{\sum_{g_i=0}^{(B_i/A_i)}\sum_{l_i=0}^{(D_i/C_i)}\frac{(-B_i)_{A_i g_i}(-D_i)_{C_i l_i} A_{B_i,g_i} A_{D_i,l_i}}{g_i! l_i!}$$

$$\times y_i^{g_i}(y_i')^{l_i}\int_x^{\infty} u^{-\delta-1}(1-\tfrac{x^{\xi}}{u^{\xi}})^{\alpha+\beta+\sum_{i=1}^{r}(\sigma_i g_i+\rho_i l_i)+1}$$

$$\times H^{0,\ n+N+3\ :m_1 n_1;...:m_r n_r;M_1,N_1;...;M_r,N_r;1,0}_{p+p+3,q+Q+2:p_1 q_1;...:p_r q_r;P_1,Q_1;...P_r Q_r;0,1}$$

$$
(4.1)\qquad \left[\begin{array}{c|c|}
z_1(1-\frac{x^\xi}{u^\xi})^{\lambda_1} & \\
\cdot & \\
\cdot & \\
\cdot & \\
z_r(1-\frac{x^\xi}{u^\xi})^{\lambda_r} & \\
z_1(1-\frac{x^\xi}{u^\xi})^{\nu_1} & \chi_4 \\
\cdot & \\
\cdot & \\
\cdot & \\
z_r(1-\frac{x^\xi}{u^\xi})^{\nu_r} & \\
(\frac{x^\xi}{u^\xi}-1) &
\end{array}\right] f(u)du,
$$

where

$$
\chi_3 = \begin{array}{l}
(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\overbrace{0,\ldots,0}^{(r+1)})_{1,p} : (h_j;\overbrace{0,\ldots,0}^{(r)},\epsilon'_j,\ldots,\epsilon_j^{(r)},0)_{1,P} : \\
(b_j;\beta'_j,\ldots,\beta_j^{(r)},\underbrace{0,\ldots,0}_{(r+1)})_{1,q} : (k_j;\underbrace{0,\ldots,0}_{(r)},\mu'_j,\ldots,\mu_j^{(r)},0)_{1,Q} :
\end{array}
$$

$$
(-\alpha-\beta-\tfrac{\eta+\delta+1}{\xi}-\textstyle\sum_{i=1}^{r}(\sigma_i g_i+\rho_i l_i);\lambda_1,\ldots,\lambda_r,\nu_1,\ldots,\nu_r,0) :
$$

$$
(-\alpha-\beta-\tfrac{2(\eta+\delta+1)}{\xi}-\textstyle\sum_{i=1}^{r}(\sigma_i g_i+\rho_i l_i);\lambda_1,\ldots,\lambda_r,\nu_1,\ldots,\nu_r,0) :
$$

$$
(-\alpha-\beta-\tfrac{\eta+\delta+1}{\xi}-\textstyle\sum_{i=1}^{r}(\sigma_i g_i+\rho_i l_i);\lambda_1,\ldots,\lambda_r,\nu_1,\ldots,\nu_r,1) :
$$

$$
(-\beta-\tfrac{\eta+\delta+1}{\xi}-\textstyle\sum_{i=1}^{r}\rho_i l_i;\underbrace{0,\ldots,0}_{(r)},\nu_1,\ldots,\nu_r,1) :
$$

$$
\begin{array}{lll}
: & (-\beta-\sum_{i=1}^{r}\rho_i l_i;\overbrace{0,\ldots,0}^{(r)},\nu_1,\ldots,\nu_r,1) & : (c'_j,\gamma'_j)_{1,p_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}; \\
: & - & : (d'_j,\delta'_j)_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r};
\end{array}
$$

$$
(e'_j,\theta'_j)_{1,P_1};\ldots;(e_j^{(r)},\theta_j^{(r)})_{1,P_r}-
$$

$$
(f'_j,\varphi'_j)_{1,Q_1};\ldots;(f_j^{(r)},\varphi_j^{(r)})_{1,Q_r};(0,1)
$$

and

$$\chi_4 = \begin{matrix} (a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\overbrace{0,\ldots,0}^{(r+1)})_{1,p} : (h_j;\overbrace{0,\ldots,0}^{(r)},\epsilon'_j,\ldots,\epsilon_j^{(r)},0)_{1,P} : \\ (b_j;\beta'_j,\ldots,\beta_j^{(r)},\underbrace{0,\ldots,0}_{(r+1)})_{1,q} : (k_j;\underbrace{0,\ldots,0}_{(r)},\mu'_j,\ldots,\mu_j^{(r)},0)_{1,Q} : \end{matrix}$$

$$(-\alpha-\beta-\tfrac{\eta+\delta+1}{\xi}-\textstyle\sum_{i=1}^{r}(\sigma_i g_i+\rho_i l_i);\lambda_1,\ldots,\lambda_r,\nu_1,\ldots,\nu_r,0) :$$

$$(-\alpha-\beta-\tfrac{2(\eta+\delta+1)}{\xi}-\textstyle\sum_{i=1}^{r}(\sigma_i g_i+\rho_i l_i);\lambda_1,\ldots,\lambda_r,\nu_1,\ldots,\nu_r,0) :$$

$$(-\tfrac{\eta+\delta+1}{\xi}-\alpha-\beta-\textstyle\sum_{i=1}^{r}(\sigma_i g_i+\rho_i l_i);\lambda_1,\ldots,\lambda_r,\nu_1,\ldots,\nu_r,1) :$$

$$(-\tfrac{\eta+\delta+1}{\xi}-\alpha-\textstyle\sum_{i=1}^{r}(\sigma_i g_i);\lambda_1,\ldots,\lambda_r,\underbrace{0,\ldots,0}_{(r)},1) :$$

$$\begin{matrix} : \ (-\alpha-\sum_{i=1}^{r}\sigma_i g_i;\lambda_1,\ldots,\lambda_r,\overbrace{0,\ldots,0}^{(r)}, & :(c'_j,\gamma'_j)_{1,p_1};\ldots; \\ - & :(d'_j,\delta'_j)_{1,q_1};\ldots; \end{matrix}$$

$$;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r};(e'_j\theta'_j)_{1,P_1};\ldots;(e_j^{(r)},\theta_j^{(r)})_{1,P_r};-$$

$$;(d_j^{(r)},\delta_j^{(r)})_{1,q_r};(f'_j\varphi'_j)_{1,Q_1};\ldots;(f_j^{(r)},\varphi_j^{(r)})_{1QP_r};(0,1)$$

5.Special case

If we set $r=1$, $n=p=q=0=N=P=Q$ for the multivariable H-function and product of the general class of polynomials, (1.8) and (1.9) respectively reduce to the following operators:

$$E_z^{\eta,\alpha}\{f(x)\} = \xi x^{-\eta-\xi\alpha-1}\int_0^x t^{\eta}(x^{\xi}-t^{\xi})^{\alpha} H_{p,q}^{m,n}\left[z(1-\tfrac{t^{\xi}}{x^{\xi}})^{\lambda}\,\middle|\,\begin{matrix}(c_p,\gamma_p)\\(d_q,\delta_q)\end{matrix}\right] S_B^A[y(1-\tfrac{t^{\xi}}{x^{\xi}})^{\sigma}]f(t)dt \tag{5.1}$$

and

$$F_z^{\delta,\beta}\{f(x)\} = \xi x^{\delta}\int_0^x t^{-\delta-\xi\beta-1}(t^{\xi}-x^{\xi})^{\beta} H_{P,Q}^{M,N}\left[Z(1-\tfrac{x^{\xi}}{t^{\xi}})^{\nu}\,\middle|\,\begin{matrix}(e_p,\theta_p)\\(f_Q,\varphi_Q)\end{matrix}\right] S_D^C[y'(1-\tfrac{x^{\xi}}{t^{\xi}})^{\rho}]f(t)dt. \tag{5.2}$$

These operators exist under the following simplified set of conditions:
(i) $1 \le p,\ q < \infty,\ p^{-1} + q^{-1} = 1$;
(ii) $Re(\eta) > -\frac{1}{q}$; $Re(\alpha + \lambda\frac{d_j}{\delta_j} + \sigma g) > -\frac{1}{q\xi}$;
$Re(\beta + \nu\frac{f_{j'}}{\varphi_{j'}} + \rho(l)) > -\frac{1}{q\xi}$; $Re(\delta) > -\frac{1}{p}$;
where $g = 0, 1, 2, ..., [B/A]$; $l = 0, 1, 2, ..., [D/C]$;
$j = 1, ..., m$; $j' = 1, ..., M$

$$\text{(5.3)}\quad \text{(iii)}\begin{cases} \sum_{j=1}^{n}(\gamma_j) - \sum_{j=n+1}^{p}(\gamma_j) + \sum_{j=1}^{m}(\delta_j) - \sum_{j=m+1}^{q}(\delta_j) = \delta \ge 0; \\ |\arg z| < \frac{1}{2}\pi\delta, \sum_{j=1}^{q}(\delta_j) - \sum_{j=1}^{p}(\gamma_j) \ge 0 \end{cases}$$

and

$$\text{(5.4)}\quad \begin{cases} \sum_{j=1}^{N}(\theta_j) - \sum_{j=N+1}^{P}(\theta_j) + \sum_{j=1}^{M}(\varphi_j) - \sum_{j=M+1}^{Q}(\varphi_j) = \delta' \ge 0; \\ |\arg Z| < \frac{1}{2}\pi\delta', \sum_{j=1}^{Q}(\varphi_j) - \sum_{j=1}^{P}(\theta_j) \ge 0 \end{cases}$$

(iv) $f(x) \in L_p(0, \infty)$.

It is interesting to observe that the composition of these operators follow from the result given in the preceding sections by setting $r = 1$, $n = p = q = 0 = N = P = Q$.

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