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On Dirichlet Averages

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In the present work we have established the correlation between Dirichlet averages and fractional derivatives of the function $x^n(\log x)^p$ and the Dirichlet series. Some particular cases are also mentioned.

1. Introduction

In a series of papers [1] Carlson has defined the Dirichlet averages of functions which denote a certain kind of integral averages with respect to a Dirichlet measure. Many of the most important special functions have been derived as Dirichlet averages of the function x^t, e^x, $e^{1/x}$ and $x^n \log x$.

In the present paper our aim is to establish the correlation between Dirichlet averages and fractional derivatives. We have considered the function

$$x^n(\log x)^p$$

which is the p-times derivative of x^n with respect to the degree of homogeneity n and also considered the Dirichlet series [cf. 4]. This Dirichlet series is a certain kind of double series.

We list some definitions used to investigate the results of the present article.

Standard simplex in R^n; $n \geq 1$:

We denote the standard simplex in R^n; $n \geq 1$: by

$$E = E_n = S(u_1, u_2, ..., u_n)\ u_1 \geq 0,\ \ u_2 \geq 0, ..., u_n \geq 0$$

and

$$\sum_{j=1}^{n} u_j \leq 1 \tag{1.1}$$

Dirichlet measure:

Let $b \in \mathbb{C}^k$, $k \geq 2$, and let $E = E_{k-1}$ be the standard simplex in R^{k-1}. The complex measure μ_b is defined on E:

$$d\mu_b(u) = \frac{1}{B(b)} \prod_{j=1}^{k-1} u_j^{b_j-1}(1 - \sum_{j=1}^{k-1} u_j)^{b_k-1} du_1 ... du_{k-1} \tag{1.2}$$

will be called as Dirichlet measure.
Here

$$B(b) = B(b_1, \ldots, b_k) = \frac{\Gamma(b_1)...\Gamma(b_k)}{\Gamma(b_1 + ... + b_k)}$$

and $\mathbb{C}_> = \{z \in \mathbb{C}\ z \neq 0,\ |ph\ z| < \pi/2\}$, open right half plane and $\mathbb{C}_> = \mathbb{C}_> \times ... \times \mathbb{C}_>$ k-times.

Dirichlet averages:

Let Ω be a complex set in $\mathbb{C}_>$, $z = (z_1, \ldots, z_k) \in \Omega^k$, $k \geq 2$ and $u.z$ be a complex linear combination of $z_1, ..., z_k$.

Let f be a measurable function on Ω, and μ_b be a Dirichlet measure on the standard simplex E in R^{k-1}. Define

$$F(b, z) = \int_E f(u.z) d\mu_b(u). \tag{1.3}$$

We shall call $F(b, z)$ the Dirichlet averages of f with variables $z = (z_1, \ldots, z_k)$ and the parameters $b = (b_1, \ldots, b_k)$.

Here

$$u.z = \sum_{j=1}^{k} u_j z_j \text{ and } u_k = 1 - \sum_{j=1}^{k-1} u_j \tag{1.4}$$

If $k = 1$, define $F(b, z) = f(z)$. Following are the Lemmas, which we need for our investigation.

Lemma 1. *(Dirichlet averages of x^n) Let $n \in N = \{1, 2, 3, ...\}$ and μ_b be a Dirichlet measure on the standard simplex E in R^{k-1}, $k \geq 2$. For every $z \in \mathbb{C}^k$ define*

$$R_n(b, z) = \int_E (u.z)^n d\mu_b(u); \tag{1.5}$$

called the Dirichlet averages of x^n.
Particularly when $k = 2$

$$R_n(\beta, \beta'; x, y) = \frac{\Gamma(\beta + \beta')}{\Gamma(\beta)\Gamma(\beta')} \int_0^1 [ux + (1-u)y]^n u^{\beta-1}(1-u)^{\beta'-1} du, \tag{1.6}$$

where β and β' have positive real part, x and y are independent and n is unrestricted.

Equation (1.6) is equivalent to

$$R_n(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}(y-x)^{1-\beta-\beta'}D_{y-x}^{-\beta'}x^n(y-x)^{\beta-1}, \tag{1.7}$$

where

$$D_z^{\alpha}F(z)=\frac{1}{\Gamma(-\alpha)}\int_0^z(z-t)^{\alpha-1}F(t)dt \tag{1.8}$$

called the Riemann-Liouville operator or Fractional Derivative of order α for $Re(\alpha)>0$.

Equation (1.7) can also be written as [cf. 3]

$$\begin{aligned} R_n(\beta,\beta';x,y) &= y^n{}_2F_1(-n,\beta',\beta+\beta';1-\tfrac{x}{y}) \\ &= x^n{}_2F_1(-n,\beta',\beta+\beta';1-\tfrac{y}{x}) \end{aligned} \tag{1.9}$$

Lemma 2. *The p^{th} derivative of the multivariate hypergeometric function R_n with respect to the degree of homogeneity n, can equivalently be thought of as a Dirichlet average of $\frac{\partial^p}{\partial n^p}(x^n)=x^n(\log x)^p$, denoted by $A_n^p(b,z)$.*

Then

$$A_n^p(b,z)=\frac{\partial^p}{\partial n^p}[R_n(b,z)] \tag{1.10}$$

where $R_n(b,z)$ is defined in from (1.5) to (1.8).

We shall often display the components of b and z by writing

$$A_n(b_1,\dots,b_k;z_1,\dots,z_k)$$

Lemma 3. *(Dirichlet average of e^x) Let μ_b be a Dirichlet measure on the standard simplex E in R^{k-1}, $k\geq 2$ defined in (1.2), then for every $z\in\mathbb{C}$ we have*

$$\begin{aligned} S(b,z) &= \textstyle\int_E e^{u.z}d\mu_n(u) \\ &= \textstyle\sum_{m=0}^{\infty}\frac{1}{m!}R_m(b,z) \end{aligned} \tag{1.11}$$

called the Dirichlet average of e^x.

Particularly when $k = 1$

$$S(b, z) = e^z.$$

Equation (1.11) may also be equivalent to:
For k=2

$$S(\beta, \beta'; x, y) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}(x-y)^{1-\beta-\beta'} e^y D_{x-y}^{-\beta'} e^{x-y}(x-y)^{\beta-1} \tag{1.12}$$

$$= e^y {}_1F_1[\beta,\ \beta+\beta'; x-y],\ (Re\beta, \beta' > 0)$$

Dirichlet Series:

Let $f(x)$ *be an entire function represented by the Dirichlet series [4]*

$$f(x) = \sum_{n=1}^{\infty} a_n \exp(x\lambda_n) \tag{1.13}$$

$x = \sigma + i\rho$, $\sqrt{-1} = i$, σ *and* ρ *are real.*
where the constants λ_n *satisfy the following condition:*

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots; \lambda_n \to \infty \text{ as } n \to \infty.$$

The Dirichlet series defined by (1.13) is a certain kind of double series as it contains the exponential terms.

2.Main results

In this section we shall show the equivalence of single Dirichlet average ($k = 2$) with the fractional derivative of the function $x^n(\log x)^p$ and of the Dirichlet series defined in (1.13).

Let $f(x)$ be defined by

$$f(x) = x^n(logx)^p \quad \text{for } n, p \in N,\ (n > p) \tag{2.1}$$

$$= 0 \qquad\qquad (n < p)$$

then for $k = 2$ (i.e. the single Dirichlet average of $f(x)$) is given by the equivalence relation

$$A_n^p(\beta, \beta'; x, y) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}(y-x)^{1-\beta-\beta'} D_{y-x}^{-\beta'}\left[x^n(\log x)^p.(y-x)^{\beta-1}\right] \tag{2.2}$$

where $Re(\beta)$, $Re(\beta') > 0$.

Let $f(x)$ be defined by the Dirichlet series (1.13); then for $Re(\beta)$, $Re(\beta') > 0$ and $\lambda_n > 0$ also $\{\lambda_n\}$ is an increasing sequence, then

$$T_n^\lambda(\beta,\beta';x,y) = \sum_{n=0}^{\infty} a_{n\,1}F_1[\beta,\ \beta+\beta';\lambda_n(x-y)] \tag{2.3}$$

which we shall call Dirichlet averages of the Dirichlet series.

Proof.

To establish the result (2.2), we use (1.6) and (1.10) to write, for $k = 2$,

$$A_n^p(\beta,\beta';x,y) = \frac{\partial^p}{\partial n^p}\left[\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\int_0^1[ux+(1-u)y]^n \times u^{\beta-1}(1-u)^{\beta'-1}du\right]$$

After differentiating partially the above equation p-times with respect to n and making use of (1.8) we arrive at the desired result.

As a particular case of the result, if we take $p = 1$ then, our result reduces to Gupta and Agarwal [3]. Also if we set $p = n$ in (2.2), without the loss of generality we get

$$\begin{aligned} A_n^n(\beta,\beta';x,y) &= A_n(\beta,\beta';x,y) \\ &= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}(y-x)^{1-\beta-\beta'}D_{y-x}^{-\beta'}\left[(x\log x)^n.(y-x)^{\beta-1}\right] \end{aligned} \tag{2.4}$$

Now to establish the second result (2.3) we use (1.11) to write for $k = 2$,

$$T_n^\lambda(\beta,\beta';x,y) = \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty}\frac{1}{m!}R_m(\beta,\beta';\lambda_m x,\lambda_m y).$$

With the aid of equation (1.6) we may write the above equation as

$$T_n^\lambda(\beta,\beta';x,y) = \sum_{n=0}^{\infty}\sum_{m=o}^{\infty}\frac{a_n}{m!}\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')} \times\lambda_n^m\int_0^1[ux+(1-u)y]^m u^{\beta-1}(1-u)^{\beta'-1}du.$$

By putting $u(x-y) = t$ under simple manipulation, and using (1.8), we may write

$$\sum_{n=0}^{\infty}\sum_{m=o}^{\infty}\frac{a_n}{m!}\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}(x-y)^{1-\beta-\beta'}D_{x-y}^{-\beta'}(\lambda_n x)^m(x-y)^{\beta-1}$$

Again using the definition of Dirichlet average given in Lemma 3 to write

$$T_n^\lambda(\beta,\beta';x,y)=\sum_{n=0}^{\infty} a_{n\,1}F_1[\beta;\ \beta+\beta';\lambda_n(x-y)]$$

thereby we arrive at the desired result.

Following particular cases might be of interest to extend the work.

Particular cases

Particular cases of (2.3):
(i) If $\beta'=\gamma-\beta$ and $y=0$, we have

$$T_n^\lambda(\beta,\gamma-\beta;x,0)=\sum_{n=0}^{\infty} a_{n\,1}F_1(\beta;\gamma;\lambda_n x)$$

where ${}_1F_1(a;b;x)$ is the confluent hypergeometric function [2].
(ii) If we set $\beta=-r$, $\beta'=1+\alpha+r$ and $y=0$ in (2.3), we obtain

$$\begin{aligned}T_n^\lambda(-r,1+\alpha+r;x,0)&=\textstyle\sum_{n=0}^{\infty} a_{n\,1}F_1(-r,1+\alpha;\lambda_n x)\\&=\textstyle\sum_{n=0}^{\infty} a_n\frac{L_r^\alpha(\lambda_n x)}{L_r^\alpha(0)},\end{aligned}$$

where L_r^α is the familiar Laguerre polynomial [cf. 2].
(iii) If we set $\beta=\nu+\frac{1}{2}$, $\beta'=\nu+\frac{1}{2}$, $x\equiv i\rho$ and $y\equiv -i\rho$ in (2.3) we have

$$\begin{aligned}\frac{(\frac{\lambda_n\rho}{2})^\nu}{\Gamma(\nu+1)}\quad T_n^\lambda(\nu+\tfrac{1}{2},\nu+\tfrac{1}{2};i\rho,-i\rho)&=\\=\textstyle\sum_{n=0}^{\infty}\frac{(\frac{\lambda_n\rho}{2})^\nu}{\Gamma(\nu+1)}a_{n\,1}F_1(\nu+\frac{1}{2};2\nu+1;2\lambda_n i\rho)&=\\=\textstyle\sum_{n=0}^{\infty} a_n J_\nu(2\lambda_n i\rho),\end{aligned}$$

where $J_\nu(x)$ is the well-known Bessel function of the first kind [2].

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