Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

---

Mathematica Balkanica

New Series Vol.10, 1996, Fasc.4

A Probabilistic Validity Measure in Intuitionistic Propositional Logic[1]

Branislav Boričić†, Miodrag Rašković††

Presented by P. Kenderov

An extension of the Heyting propositional logic obtained by adding some probability measure axioms is considered. The corresponding Kripke—type and algebraic models are described.

1. Introduction

Our work on this subject was directly inspired by H. J. Keisler's treatment of probability quantifiers (see [2] and [3]). Instead of the first—order predicate logic and quantifiers (see also [4]), we consider propositional calculi with a family of propositional operators satisfying formally the same list of probability axioms, as those concerning quantifiers (see [1] or [5]). In this paper we present an extension of the Heyting propositional logic by probability operators. The main aim was to describe syntax and semantics enabling an adequate expression of the following statement: "probability of truthfulness of A is greater (less) or equal to r", denoted by $\pi_r A$ ($\pi^r A$, respectively). The next paragraph deals with our formal system. Afterwards we define the corresponding Kripke—type semantics and algebraic models followed by the completeness results. The paragraph concerning the Kripke models is written by the first (co)author and the treatment of algebraic semantics belongs to the second one.

[1]The results contained in this paper were presented at the Conference on Mathematical Logic held in Belgrade on March 25, 1993. The work on this paper was partly supported by the Ministry of Science and Technology of Serbia, grant number 0401A.

2. Formal system

The language of our system consists of usual symbols for logical connectives, propositional letters, parentheses and two types of *probability operators:* π^r and π_r, for each $r \in S$, where S is a finite subset of the real interval $[0,1]$, such that $0,1 \in S$. The set of formulae is the smallest set containing propositional letters and closed under formation rules: *if A and B are formulae, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \to B)$, $\pi^r A$ and $\pi_r A$ are formulae.*

Let $\mathcal{H}$ be the Heyting propositional calculus. Then by $\mathcal{H}_p$ we denote the logic having the following axioms:

(0) All formulae provable in $\mathcal{H}$.

(1) (monotonicity)

$$\pi^r A \to \pi^s A, \quad \textit{for } r \leq s$$

$$\pi_r A \to \pi_s A, \quad \textit{for } r \geq s$$

(2)

$$\pi^1 A$$

$$\pi_0 A$$

(3) (finite semiadditivity)

$$\pi^r A \wedge \pi^s B \to \pi^t (A \vee B), \quad \textit{where } t = \min(1, r+s)$$

$$\pi_r A \wedge \pi_s B \wedge \pi^0 (A \wedge B) \to \pi_t (A \vee B), \quad \textit{where } t = \min(1, r+s)$$

and inference rules:

$$\frac{A \quad A \to B}{B} \ (mp) \quad \textit{and} \quad \frac{A}{\pi_1 A} \ (\pi_1)$$

The axioms (1), (2) and (3), and the inference rule (π_1), concerning the probability operators are called *the probability measure conditions*.The derivation from hypotheses may be defined in a usual way: if Γ is a set of formulae, then we say that a formula A can be derived from the set of hypotheses Γ in $\mathcal{H}_p$ (denoted by $\Gamma \vdash_{\mathcal{H}_p} A$) *iff* there exists a finite sequence of formulae $B_1, ..., B_k$ such that $B_k = A$ and every formula of this sequence is a theorem of $\mathcal{H}_p$, belongs to Γ or is obtained by the *modus ponens* rule from the formulae preceding it in the sequence. It is easy to prove that the *Deduction Theorem* holds with respect to the derivability relation just defined.

3. Kripke type semantics

In this paragraph, we are going to define the Kripke type model corresponding to the logic $\mathcal{H}_p$ obtained as an extension of $\mathcal{H}$ by the probability operators satisfying the probability measure conditions. By an $\mathcal{H}_p$—*frame* we mean the structure $\langle W, R, (p^x)_{x\in W}, (p_x)_{x\in W}\rangle$, where

(i) $\langle W, R\rangle$ is an $\mathcal{H}$—frame, *and*

(ii) p^x and p_x are partial functions from the power set of W to $S \subset [0,1]$ such that the following conditions are satisfied:

monotonicity: for every $x, y \in W$ *and every measurable* $X \subseteq W$, *if* xRy, *then* $p^x(X) \geq p^y(X)$ *and* $p_x(X) \leq p_y(X)$;

subadditivity: for every $x \in W$ *and all measurable* $X, Y \subseteq W$,

$$p^x(X \cup Y) \leq \min(p^x(X) + p^x(Y), 1)$$

superadditivity: for every $x \in W$ *and all measurable* $X, Y \subseteq W$, *if* $p^x(X \cap Y) = 0$, *then*

$$p_x(X \cup Y) \geq \min(p_x(X) + p_x(Y), 1)$$

where the class of measurable sets is a σ*—algebra ower* W *and for any* $x \in W$, $p_x(W) = 1$.

Note that the properties considered above correspond to the probability measure conditions.

An $\mathcal{H}_p$—frame together with a *valuation*, i. e. a mapping from the set of propositional variables to the power set of W, such that for every propositional variable P and all $x, y \in W$, if xRy and $x \in v(P)$, then $y \in v(P)$, will be denoted by $\mathcal{M} = \langle W, R, (p^x)_{x\in W}, (p_x)_{x\in W}, v\rangle$ and called an $\mathcal{H}_p$—*model.*

Let $\mathcal{M} = \langle W, R, (p^x)_{x\in W}, (p_x)_{x\in W}, v\rangle$ be an $\mathcal{H}_p$—model. Then, for any $x \in W$ and any formula A, we define the relation $x \models A$, "A holds in the world

x in $\mathcal{M}$", by induction on the complexity of the formula A, as follows:

$$x \models P \ \ iff \ \ x \in v(P)$$

$$x \models B \wedge C \ \ iff \ \ x \models B \ \ and \ x \models C$$

$$x \models B \vee C \ \ iff \ \ x \models B \ \ or \ x \models C$$

$$x \models B \to C \ \ iff \ \ for \ all \ y \in W, \ if \ xRy \ and \ y \models B, \ then \ y \models C$$

$$x \models \neg B \ \ iff \ \ for \ all \ y \in W, \ if \ xRy, \ then \ not \ y \models B$$

$$x \models \pi^r B \ \ iff \ \ p^x\{y|y \models B\} \leq r$$

$$x \models \pi_r B \ \ iff \ \ p_x\{y|y \models B\} \geq r$$

where we suppose that, for every formula A, the set $\{y|y \models A\}$ is both p^x and p_x—measurable, for any $x \in W$. We say that A *holds* in a model $\mathcal{M}$, denoted by $\mathcal{M} \models A$, *iff* for any $x \in W$, $x \models A$. We simply write $\models A$ and say that A is $\mathcal{H}_p$—*valid iff* for every $\mathcal{H}_p$—model, $\mathcal{M} \models A$. In the sequel of this paper we shall use abbreviations $p^x(B)$ and $p_x(B)$ for $p^x\{y|y \models B\}$ and $p_x\{y|y \models B\}$, respectively.

Now, following a usual procedure, we are going to present the main steps of the proof that $\mathcal{H}_p$ is sound and complete with respect to the class of $\mathcal{H}_p$—models.

Intuitionistic Heredity Lemma. *In every $\mathcal{H}_p$—model, for any formula A and any $x, y \in W$, if $x \models A$ and xRy, then $y \models A$.*

Proof. By induction on the complexity of A. Let us cosider just the cases with the probability operators.

(1) If $A = \pi^r B$, xRy and $x \models A$, i. e. $x \models \pi^r B$, then by definition of $\models$, we have $p^x(B) \leq r$. But having in mind the monotonicity of $p^{\cdot}(X)$, for xRy, we have $p^x(B) \geq p^y(B)$, as wel l. Consequently: $p^y(B) \leq r$, i. e. $y \models A$.

(2) Similarly, for $A = \pi_r B$, xRy and $x \models A$, by definition of satisfiability relation and monotonicity of $p_{\cdot}(X)$, we have $r \leq p_x(B) \leq p_y(B)$, i. e. $y \models A$.$\dashv$

For any set of formulae Γ, we define its *deductive closure* $Cn(\Gamma) = \{A|\Gamma \vdash_{\mathcal{H}_p} A\}$. We say that the set of formulae Γ is a *saturated theory iff* Γ is $\mathcal{H}_p$—consistent (i. e. there is a formula A such that *not* $\Gamma \vdash_{\mathcal{H}_p} A$), deductively closed (i. e. $\Gamma = Cn(\Gamma)$) and has the disjunction property (i. e. for any formulae A and B, if $A \vee B \in \Gamma$, then $A \in \Gamma$ *or* $B \in \Gamma$*)*.

Lemma. *A maximal consistent extension of any set of formulae is a saturated theory.*

Proof. Let Γ be a consistent set of formulae such that *not* $\Gamma \vdash A$ and $S = \{\Pi | \Gamma \subseteq \Pi \text{ and not } \Pi \vdash A\}$. System S is a nonempty set of consistent extensions of Γ closed under unions of nonempty chains and, consequently, by Zorn's Lemma, S has a maximal element Δ. Δ is a consistent extension of Γ, obviously. We prove that Δ is deductively closed. Let us suppose that there is a formula B, such that $\Delta \vdash B$, but $B \notin \Delta$. Then, since Δ is a proper subset of $\Delta \cup \{B\}$ and Δ is a *maximal* consistent extension of Γ, we have that $\Delta \cup \{B\} \vdash A$, wherefrom $\Delta \vdash A$, which is a contradiction! We can also prove that Δ has the disjunction property. Let us suppose that there are the formulae B and C, such that $B \vee C \in \Delta$, $B \notin \Delta$ and $C \notin \Delta$. Then, since Δ is a maximal consistent extension of Γ, we have $\Delta \cup \{B\} \vdash A$ and $\Delta \cup \{C\} \vdash A$, wherefrom, by the deduction theorem, we infere $\Delta \vdash B \to A$ and $\Delta \vdash C \to A$, and, consequently, $\Delta \vdash B \vee C \to A$. But, since $B \vee C \in \Delta$, we can get a contradiction: $\Delta \vdash A$! It follows that Δ is a saturated theory extending the set Γ.$\dashv$ ∎

Consequence. *If not $\Gamma \vdash A$, then there exists a saturated theory Δ extending Γ, such that not $\Delta \vdash A$.*

The next usual step is to introduce the canonical $\mathcal{H}_p$—model. Let X^c be the class of all saturated theories with respect to $\mathcal{H}_p$, and for any $\Gamma, \Delta \in X^c$,

$$\Gamma R^c \Delta \quad iff(def) \quad \Gamma \subseteq \Delta$$

$$v^c(P) =_{def} \{\Gamma | P \in \Gamma\}$$

$$p^{c\Gamma}(A) = p^{c\Gamma}\{\Delta | \Delta \models A\} =_{def} \max\{t | t = \min\{s | \pi^s A \in \Delta\},\ \Gamma \subseteq \Delta\}$$

and

$$p^c_\Gamma(A) = p^c_\Gamma\{\Delta | \Delta \models A\} =_{def} \min\{t | t = \max\{s | \pi_s A \in \Delta\},\ \Gamma \subseteq \Delta\}$$

Then the structure $\langle X^c, R^c, (p^{c\Gamma})_{\Gamma \in X^c}, (p^c_\Gamma)_{\Gamma \in X^c}, v^c \rangle$ will be called *the canonical $\mathcal{H}_p$—model.* Hereafter, for the shortness, we shall drop the superscript c, when it is clear that we work in the canonical model.

Note that, according to the definition above, the measurable sets of the canonical structure have the form $\{\Delta | \Delta \models A\}$.

Lemma. *The canonical $\mathcal{H}_p$—model is an $\mathcal{H}_p$—model.*

Proof. First, we prove that $\langle X, R, (p^\Gamma)_\Gamma, (p_\Gamma)_\Gamma \rangle$ is an $\mathcal{H}_p$—frame. $\langle X, R \rangle$ is supposed to be an $\mathcal{H}$—frame. For each $\Gamma \in X$, both mappings $p^{\cdot}(A)$ and $p_{\cdot}(A)$ are monotone in the sense of our definition: if $p^\Gamma(A) = p^\Gamma\{\Pi | \Pi \models A\} = \max\{t | t = \min\{s | \pi^s A \in \Pi\}, \Gamma \subseteq \Pi\} = r_1$ and $p^\Delta(A) = p^\Delta\{\Pi | \Pi \models A\} = \max\{t | t = \min\{s | \pi^s A \in \Pi\}, \Delta \subseteq \Pi\} = r_2$, then, obvious ly, due to the condition $\Gamma \subseteq \Delta$, we have $r_1 \geq r_2$. Similarly, for $p_\Gamma(A) = p_\Gamma\{\Pi | \Pi \models A\} = \min\{t | t = \max\{s | \pi_s A \in \Pi\}, \Gamma \subseteq \Pi\} = r_1$, $p_\Delta(A) = p_\Delta\{\Pi | \Pi \models A\} = \min\{t | t = \max\{s | \pi_s A \in \Pi\}, \Delta \subseteq \Pi\} = r_2$ and $\Gamma \subseteq \Delta$, we have $r_1 \leq r_2$.

If $\Gamma \in X$, then, having in mind the probability measure conditions, i. e. finite semiadditivity axioms, as well as that $p\{y | y \models A \vee B\} = p(A \vee B) = p((A) \cup (B)) = p(\{y | y \models A\} \cup \{y | y \models B\})$ and $p\{y | y \models A \wedge B\} = p(A \wedge B) = p((A) \cap (B)) = p(\{y | y \models A\} \cap \{y | y \models B\})$, where p may be p^x or p_x, then $p^\Gamma(A \vee B) \leq p^\Gamma(A) + p^\Gamma(B)$ and, for $p^\Gamma(A \wedge B) = 0$, $p_\Gamma(A \vee B) \geq p_\Gamma(A) + p_\Gamma(B)$. $\dashv$ ■

Lemma. *In the canonical $\mathcal{H}_p$—model, for every formula A and every $\Gamma \in X$,*

$$\Gamma \models A \quad iff \quad A \in \Gamma$$

Proof. By induction on the complexity of A. We present only the part of proof related to the probability operators.

(1) If $A = \pi^r B$ and $\Gamma \models \pi^r B$, then, by definition, we have $\max\{t | t = \min\{s | \pi^s B \in \Pi\}, \Gamma \subseteq \Pi\} \leq r$ and, consequently, for $t = \min\{s | \pi^s B \in \Gamma\}$, $t \leq r$ and $\pi^t B \in \Gamma$, we conclude $\pi^r B \in \Gamma$.

Conversely, if $\pi^r B \in \Gamma$, then, *for each t*, if $t = \min\{s | \pi^s B \in \Pi\}$, where $\Gamma \subseteq \Pi$, we have $t \leq r$, and, consequently, $\max\{t | t = \min\{s | \pi^s B \in \Pi\}, \Gamma \subseteq \Pi\} \leq r$, i. e. $\Gamma \models \pi^r B$.

(2) If $A = \pi_r B$ and $\Gamma \models \pi_r B$, then, by definition $p_\Gamma\{\Pi | \Pi \models B\} \geq r$, i. e. $\min\{t | t = \max\{s | \pi_s B \in \Pi\}, \Gamma \subseteq \Pi\} \geq r$. In such a case, if $t = \max\{s | \pi_s B \in \Gamma\}$, we have $\pi_t B \in \Gamma$, $t \geq r$ and, consequently, $\pi_r B \in \Gamma$.

Conversely, for $\pi_r B \in \Gamma$, we can conclude that, *for each t*, if $t = \max\{s | \pi_s B \in \Pi\}$, where $\Gamma \subseteq \Pi$, then $t \geq r$, and, consequently, $\min\{t | t = \max\{s | \pi_s B \in \Pi\}, \Gamma \subseteq \Pi\} \geq r$, i. e. $\Gamma \models \pi_r B$. $\dashv$ ■

Completeness and Soundness Theorem. *For any formula A, A is provable in $\mathcal{H}_p$ iff A is $\mathcal{H}_p$—valid.*

Proof. The soundness part can be proved by induction on the length of the proof of A in $\mathcal{H}_p$, and we omit it.

■

On the other hand, the completeness part is an immediate consequence of the preceding lemmata. ⊣

4. Algebraic models

A pseudo—Boolean (or Heyting) probabilistic algebra (PBPA) is a structure $\langle B, \leq, (p^r)_{r\in S}, (p_r)_{r\in S}\rangle$, such that $\langle B, \leq\rangle$ is a partial ordering with a least element denoted by 0 and greatest element denoted by 1, and for any $x, y \in B$:

i) there is a least upper bound of x and y, denoted $x \cup y$;

ii) there is a greatest lower bound of x and y, denoted $x \cap y$;

iii) there is a pseudo–complement of x relative to y, denoted $x \Rightarrow y$ and defined as the largest $z \in B$ such that $x \cap z \leq y$ (The pseudo–complemen t $\sim x$ of x can be defined as $x \Rightarrow 0$.);

iv) p_r and p_r are unary operators, such that:

$p^r x \leq p^s x$, for $r \leq s$;

$p_r x \leq p_s x$, for $r \geq s$;

$1 \leq p^1 x$

$1 \leq p_0 x$

$p^r x \cap p^s y \leq p^t(x \cup y)$, where $t = \min(1, r+s)$;

$p_r x \cap p_s y \cap p^0(x \cap y) \leq p_t(x \cup y)$, where $t = \min(1, r+s)$;

$x \leq p_1 x$

A PBPA model is a mapping h from the set of all formulae to some PBPA, such that for any formulae A and B:

$$h(A \wedge B) = h(A) \cap h(B), \qquad h(\neg A) = \sim h(A),$$

$$h(A \vee B) = h(A) \cup h(B), \qquad h(\pi^r A) = p^r h(A),$$

$$h(A \to B) = h(A) \Rightarrow h(B) \quad \text{and} \quad h(\pi_r A) = p_r h(A)$$

h is a model of A, if $h(A) = 1$, and h is a model of a theory T when h is a model of each $A \in T$.

Alternatively, a model h can be taken to be a mapping from the propositional variables into PBPA and then extend h to arbitrary formulae according to the above conditions.

The following extended completeness theorem may be obtained.

Theorem . *A formula A is provable in a theory T iff every* PBPA model of T is a PBPA model of A.

Proof. We give a brief outline of the proof. The trivial part is to prove that $T \vdash A$ implies $T \models A$. The opposite direction is a Lindenbaum—Tarski algebra construction. First note that the relation "$A \to B$ *and* $B \to A$ are provable in T" is an equivalence relation. The function which maps each sentence into its own equivalence class is a PBPA model and each sentence of T is mapped into 1. If A is not provable in T, then A is not mapped into 1.$\dashv$

■

References

[1] B. R. Boričić. Validity measurement in some propositional logics, *Mathematicl Logic Quarterly*

[2] H. J. Keisler. Hyperfinite model theory, *Logic Colloquium '76*, (R. O. Gandy, J. M. E. Hyland eds.), North–Holland, Amsterdam, 1977, 5–110.

[3] H. J. Keisler. Probability quantifiers, *Model–Theoretic Logics* (J. Barwise and S. Feferman eds.), Perspectives in Mathematical Logic, Springer–Verlag, Berlin, 1985, 509—556.

[4] M. Rašković. A completeness theorem for an infinitary intuitionistic logic with both ordinary and probability quantifiers, *Publications de l'Institut Mathématique*, Nouvelle série tome 50 , **64**, 1991, 7—13.

[5] M. Rašković. Classical logic with some probability operators. *Publications de l'Institut Mathématique*, Nouvelle série tome 53,**67**, 1993, 1–3.

† *University of Belgrade*
Kamenička 6
11000 Beograd,
YUGOSLAVIA

Received 14.07.1994

†† *University of Kragujevac*
Radoja Domanovića 12
34000 Kragujevac,
YUGOSLAVIA