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On a Class of Nelson–Type Examples for Higher Order Partial Differential Operators

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1. Introduction

The example of E. N e l s o n [1] was first to exhibit two symmetric operators A, B commuting algebraically (i.e. $AB = BA$) on the domain of essential self–adjointness, and noncommuting in strong sense (i.e. the projection valued [PV] measure of $\bar{A}$ does not commute with the PV measure of $\bar{B}$). Here a class of Nelson–type examples is extended from the case of second order elliptic partial differential operators with constant coefficients [2] to higher order operators with constant coefficients. The main result is contained in the Theorem proved in Section 2 giving necessary conditions for strong commutativity of A and B in terms of the geometry of boundaries of n–dimensional Eucledian domains. With help of the Theorem one can construct a variety of Nelson–type examples part of which are related to notions and statements of quantum theory of open systems. We return to this point at the end of Section 3.

2. Notations and statement of results

We proceed to formulate the Theorem's assumptions: [Bou], [Coe], [Dom] concerning the n–dimensional domain boundary, the coefficients and the Hilbert space domains of operators correspondingly.

Let Ω be a bounded domain in R^n with piecewise boundary $\partial\Omega$, which has no points in the interior of the closure $\bar{\Omega}$ of Ω. Let ω and ϑ, $\vartheta \subseteq \omega$ are two pieces of $\partial\Omega$ i.e. open connected $n-1$ dimensional subsets of the smooth part of $\partial\Omega$. Let the distance between ϑ and $\partial\Omega \setminus \omega$ is nonzero. One has

[Bou]: ω is supposed to be given by the equation $f(x) = 0$, $x = (x_1, x_2, \ldots, x_n)$, which can be solved with respect to some x_i where f is a smooth (infinitely differentiable for any x_i, $1 \le i \le n$) function.

Denote the partial derivative $\partial^{|I|}/\partial x_{i_1}, \partial x_{i_2}, \ldots \partial x_{i_q}$ by ∂_x^I, $I = \{i_1, i_2, \ldots, i_q\}$, $|I| = q$. With R an integer and $C^\infty(\bar{\Omega})$ – the set of infinetely differentiable functions on $\bar{\Omega}$, let $D^\infty(\bar{\Omega})$ be the following set of functions $u_k(x)$, $x \in \bar{\Omega}$:

$$
\begin{aligned}
D^\infty(\bar{\Omega}) &= \Big\{ u_k \in C^\infty(\bar{\Omega}) : u_k(x) = \partial_x u_k(x) = \partial_x^2 u_k(x) \\
&= \ldots \partial_x^{R-1} u_k(x) = 0,\ x \in \partial\bar{\Omega} \Big\}.
\end{aligned} \tag{0.1}
$$

Let $u_k \in D^\infty(\bar{\Omega})$ and A, B be the formally self–adjoint elliptic differential expressions with coefficients α_i, β_i, a_I, b_I in $C^\infty(\bar{\Omega})$:

$$
Au_k = \sum_{i=1}^{n} \alpha_i(x)\partial^{2R} u_k/\partial^{2R} x_i + \sum_{|I| \le 2R-1} a_I(x)\partial_x^I u_k, \tag{0.2}
$$

$$
Au_k = \sum_{i=1}^{n} \beta_i(x)\partial^{2R} u_k/\partial^{2R} x_i + \sum_{|I| \le 2R-1} b_I(x)\partial_x^I u_k, \tag{0.3}
$$

[Coe]: Suppose $\alpha_i(x)|\omega$, $\beta_i(x)|\omega$, $a_i(x)|\omega$, $b_i(x)|\omega$ are $C^\infty(\omega)$–functions and $\beta_i(x) > 0$, $x \in \omega$, $i = 1, 2, \ldots, n$. Let the set of coefficients ratios $r_i(x) = \alpha_i(x)/\beta_i(x)$, $i = 1, 2, \ldots, n$ is decomposed into subsets of equal on θ functions:

$$
\begin{aligned}
& r_{i_1}(x) = r_{i_2}(x) = \ldots = r_{i_\sigma}(x); \\
& r_{j_1}(x) = r_{j_2}(x) = \ldots = r_{j_\psi}(x); \\
& \ldots\ldots\ldots\ldots\ldots \\
& r_{k_1}(x) = r_{k_2}(x) = \ldots = r_{k_\pi}(x), \qquad x \in \theta.
\end{aligned}
$$

It is required that any open subset θ' of θ contains points $x \in \theta'$ such that the values of $r_i(x)$ at these points are diferent for $r_i(x)$ taken from different coefficient subsets, i.e. $r_{i_1}(x) \ne r_{j_1}(x) \ne \ldots \ne r_{k_1}(x)$, $x \in \theta'$.

For later use one needs to consider also the corresponding decomposition of coordinates x into subsets:

$$x_{i_1}(x), x_{i_2}(x), \dots x_{i_\sigma}(x); x_{j_1}(x), x_{j_2}(x), \dots x_{j_\psi}(x); \dots, x_{k_1}(x), x_{k_2}(x), \dots x_{k_\pi}(x).$$

Let the self–adjoint extensions $\hat{A}, \hat{B}$ of A and B exist in $L_2(\Omega)$ with the following domain condition:

[Dom]: *(a)* $D(\hat{A}) = D(\hat{B})$ and *(b)* $u(x) \in D(\hat{A}) \cap C^\infty(\Omega)$ implies $u(x) = 0$ if $x \in \partial\Omega$.

One has the following

Theorem. *Suppose [Bou], [Coe] and [Dom] are verified and $\hat{A}$ and $\hat{B}$ commute in the strong sense. Then on θ the equation $f(x) = 0$ of the boundary $\partial\Omega$ depends only on one coordinate subset from the corresponding decomposition of the coordinates into subsets.*

R e m a r k s. Formally self–adjoint elliptic diferential expressions A, B satisfying Dirichlet conditions (1) and [Dom] *(b)*, for smooth $\partial\Omega$ are discussed e.g. in [3] – Theorems 23 and 25 of Chapter XIV, 6 with $D(A) = H_0^R \cap H^{2R}$. Here H^R is the space of distributions together with R–th derivatives in $L_2(\Omega)$, H_0^R is the closure in H^R of $C^\infty(\Omega)$–functions with compact support. A special case of the Theorem with $R = 1$ and constant coefficients of A, B is sudied in [2].

In [Coe] it is sufficient to assume for fixed $x \in \omega$ more generally that $\beta_i(x) \neq 0$ and the *sign* of $\beta_i(x)$ is independent of i, instead of simply $\beta_i(x) > 0$.

3. Proof of the necessary conditions of strong commutativity

In this section we give a proof of the Theorem in several steps, starting from 1) to 6).

1) Consider a neighbourhood V of ω, $V \in R^N$, such that $V \cap \partial\Omega = \omega$. Introduce in V new coordinates $\xi_i = \xi_i(x_1, x_2, \dots, x_n)$, $i = 1, 2, \dots, n$, with $\xi_i \in C^\infty(\Omega)$ and so that ω is given by the equation $\xi_1 = f(x) = 0$.

Denote by ∂^I a derivative of order $|I|$ with respect to ξ and set: $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, $\xi' = (\xi_2, \xi_3, \dots, \xi_n)$. In the new coordinates one has

$$(2') \qquad Au_k = \textstyle\sum_{|I| \leq 2R} \hat{a}_I \partial^I u_k,$$

$$(3') \qquad Bu_k = \textstyle\sum_{|I| \leq 2R} \hat{b}_I \partial^I u_k,$$

where $\hat{a}_I$, $\hat{b}_I$ are functions of the coefficients $\alpha_i, \beta_i, a_I, b_I$ and of $\partial_x^I \xi_i$, $|I| \leq 2R$, and do not depend of u_k or $\partial^I u_k$. Then for the highest order derivative coefficients i.e. when $|I| = 2R$ one has:

$$\hat{b}_I = \hat{b}_{(i_1, i_2, \ldots, i_{2n})} = \sum_{j=1}^{n} \beta_j(x) \frac{\partial \xi_{i_1}}{\partial x_j} \frac{\partial \xi_{i_2}}{\partial x_j} \cdots \frac{\partial \xi_{i_{2n}}}{\partial x_j}. \tag{0.4}$$

Similar expressions hold for $\hat{a}_I$ if α_j is substituted in (4) instead of β_j.

2). In this step Au_k, Bu_k as given by (2′), (3′), are written at any point of Ω as bilinear forms $< . >$ of two kinds of sequences G_k, and Ξ, Ξ_α : $G_k = \{G_{k1}, G_{k2}, \ldots G_{k\delta}\}$, $\Xi = \{\Xi_1, \Xi_2, \ldots \Xi_\delta\}$, $\Xi_\alpha = \{\Xi_{\alpha 1}, \Xi_{\alpha 2}, \ldots \Xi_{\alpha\delta}\}$. For fixed u_k and $I = \{i_1, i_2, \ldots, i_q\}$, $q \leq 2R$ the derivatives $\partial^I u_k$ in (2′) and (3′) are written in the form of the sequences G_k with elements G_{ki}, indexed by i. Similarly, the coefficients $\hat{a}_I$ (coresp. $\hat{b}_I$), for $I = \{i_1, i_2, \ldots, i_q\}$, $q \leq 2R$ in (2′), corresp. (3′) are written in the form of the sequences Ξ_α (corresp. Ξ). The indexing inside the sequences G_k, Ξ_α and Ξ should be such as to permit to rewrite (2′) and (3′) correspondingly in the forms:

$$Au_k = < G_k, \Xi_\alpha > = \sum_{i=1}^{\delta} G_{ki} \Xi_{\alpha i}, \tag{0.5}$$

$$Bu_k = < G_k, \Xi > = \sum_{i=1}^{\delta} G_{ki} \Xi_i. \tag{5′}$$

In the following we need to fix by convention the indeces only of those elements of G_k, Ξ_α and Ξ which are used effectively later in the proof. For $i = \delta$ denote

$$\begin{aligned} \Xi_\delta &= \hat{b}_{(1,1,\ldots,1)} = \\ &= \textstyle\sum_{i=1}^{\delta} \beta_i(x) (\partial \xi_1 / \partial x_i)^{2R}, \\ G_{k\delta} &= \partial^{2R} u_k / d\xi_1^{2R} \end{aligned} \tag{5″}$$

Set $\partial_1^s = \partial^s / \partial \xi_1^s$ and let ∂_2^J denotes any derivative with respect to ξ' of order $|J|$, $J = (j_1, j_2, \ldots, j_r)$. Furthermore with the notations

(0.6) $$\Gamma_k^0 := u_k, \quad \Gamma_k^1 := \partial_1^s u_k, \; s = 0, 1, 2, \ldots, 2R,$$

we separate the dependence on ξ_1 and on $\xi' = (\xi_2, \xi_3, \ldots, \xi_n)$. So $\partial^I u_k$, $|I| \leq 2R$, can be represented in the form $\partial_2^J \Gamma_k^s$. Denote for fixed s by $\left[\partial_2^J \Gamma_k^s\right]$ a subsecuence of G_k consisting of all $\partial_2^J \Gamma_k^s$ such that $|J| = 0, 1, 2, \ldots, 2R - s$. Then G_k is partitioned into subsequences $[\ldots]$:

(0.7) $$G_k = \left\{ \left[\partial_2^J \Gamma_k^0\right], \left[\partial_2^J \Gamma_k^1\right], \left[\partial_2^J \Gamma_k^2\right], \ldots, \left[\partial_2^J \Gamma_k^{2R-1}\right], \Gamma_k^{2R}, \right\},$$

where $\Gamma_k^{2R} = G_{k\delta}$. Inside the s–th subsequence $[\ldots]_s$, assign to $\partial_2^0 \Gamma_k^s$ the smallest index. Next choose $[\ldots]_s$, arbitrarily some single variable $\xi_i \in \xi'$ and continue indexing different $\partial_2^J \Gamma_k^s$ within $[\ldots]_s$ increasingly as the order of derivatives of ξ_i increases, otherwise the ordering inside $[\ldots]_s$ is arbitrarily fixed.

Finally, define $\bar{G}_k = \{G_{k1}, G_{k2}, \ldots, G_{k\delta-1}, 0\}$ after substituting the end element $G_{k\delta}$ in G_k by zero. Denote by $\rho + 1$ the index corresponding to $\Gamma_k^R \equiv \partial_2^0 \Gamma_k^R$ as an element in $\bar{G}_k$ and set: $\Delta = \delta - (\rho + 1)$.

3). This step consists of substeps 3a), 3b) and 3c) and constructs the functions $v_k \in C^\infty(\bar{\Omega})$ and $\Phi_k^s \in C^\infty(\omega)$, $k = 1, 2, \ldots, \Delta$. In 3a) the construction of Φ_k^s is done for $s = R, R+1, \ldots, 2R-1$, while in 3b) – for $s = 2R, 2R+1, \ldots, 3R-1$. If $s = 0, 1, 2, \ldots, R-1$, one sets $\Phi_k^s = 0$. Finally, in 3c) v_k is constructed. Before defining Φ_k^s one first orders Φ_k^s and its derivatives for different s as a sequence with δ elements denoted by F_k^0. One defines F_k^0, if instead of Γ_k^s, $s = 0, 1, 2, \ldots, 2R-1$ in $\bar{G}_k$ one writes Φ_k^s and instead of $G_{k\delta}$ one puts 0, i.e.:

(0.8) $$F_k^0 = \left\{ \left[\partial_2^J \Phi_k^0\right], \left[\partial_2^J \Phi_k^1\right], \left[\partial_2^J \Phi_k^2\right], \ldots, \left[\partial_2^J \Phi_k^{R-1}\right], F_k, 0 \right\}.$$

Here F_k denotes a subsequence with Δ elements defined as follows:

(0.9) $$F_k = \left\{ \left[\partial_2^J \Phi_k^R\right], \left[\partial_2^J \Phi_k^{R+1}\right], \ldots, \left[\partial_2^J \Phi_k^{2R-1}\right] \right\}.$$

Denote by F the $\Delta \times \Delta$ matrix

$$F \equiv \left\| \begin{array}{c} F_1 \\ F_2 \\ \vdots \\ F_\Delta \end{array} \right\| .$$

If $R \le s \le 2R-1$, then the functions $\Phi_k^s \equiv \Phi_k^s(\xi')$ should satisfy the following equality (10):

$$(0.10) \qquad det\, F \neq 0,\ \xi' \in \omega.$$

The restriction on θ of the functions v_k should satisfy the following equalities (11), (12):

$$(0.11) \qquad \partial^I v_k \mid \theta = 0, \quad \mid I \mid = 1, 2, \ldots, R-1$$

$$(0.12) \qquad \partial_1^s v_k \mid \theta = \Phi_k^s, \quad s = R, R+1, \ldots, 3R-1.$$

3a). To construct Φ_k^s, $R \le s \le 2R-1$, $1 \le k \le \Delta$, satisfying (9) and (10) fix some row F_k in F, i.e. fix some sequence (9), with matrix elements F_{km}, $1 \le k, m \le \Delta$. Suppose $F_{kk} = \partial_2^J \Phi_k^{\bar{s}}$ for some $s = \bar{s}$ and some J inside the subsequence $\left[\partial_2^J \Phi_k^{\bar{s}}\right]$. Choose $\Phi_k^{\bar{s}}$ as a polynomial of single variable, namely in the variable $\xi_i \in \xi'$ whose derivatives are increasingly ordered inside $\left[\partial_2^J \Phi_k^{\bar{s}}\right]$. The degree of this polynomial should be so chosen, that $\partial_2^J \Phi_k^{\bar{s}} = const = F_{kk} \neq 0$. If $s < \bar{s}$, choose Φ_k^s as an arbitrary $C^\infty(\omega)$ function. If $s > \bar{s}$, choose $\Phi_k^s = 0$. The above procedure is repeated for any k, $1 \le k \le \Delta$. Then $F_{km} = 0$ for $m > k$ and $\det F \neq 0$.

3b). With help of Φ_k^s, $0 \le s \le 2R-1$, one constructs by recurrence formulas (14) functions Φ_k^s, $s = 2R, 2R+1, \ldots, 3R-1$, $1 \le k \le \Delta$. For this purpose one first orders Φ_k^s in the form of sequences F_k^μ, and defines the expressions D_k^μ. Denote by F_k^μ, $\mu = 0, 1, 2, \ldots, R-1$ a sequence with δ elements, defined recurrently as follows. If $\mu = 0$, F_k^0 is defined by (8). For fixed $\mu \ge 1$, F_k^μ is defined by formally writing Φ_k^{s+1} instead of Φ_k^s in places where Φ_k^s enters the sequence $F_k^{\mu-1}$. Hence

$$F_k^\mu = \left\{ \left[\partial_2^J \Phi_k^\mu\right], \left[\partial_2^J \Phi_k^{\mu+1}\right], \ldots, \left[\partial_2^J \Phi_k^{\mu+2R-1}\right] \right\}.$$

Define the sequence $\bar{\Xi} = \{\Xi_1, \Xi_2, \ldots, \Xi_{\delta-1}, 0\}$ by substituting the end element Ξ_δ of Ξ by zero.

Denote by $\left(\partial_1^i \bar{\Xi}\right) \mid \omega$, $i = 1, 2, \ldots, \mu$, a sequence whose elements are obtained as restrictions on ω of ∂_1^i–derivatives of the elements with the same indexes in the sequence $\bar{\Xi}$. Using the binomial coefficients $\begin{pmatrix} \mu \\ i \end{pmatrix}$, $\mu \geq 1$, the convention $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$ and the bilinear forms $< \cdot >$ of sequences, we denote by D_k^μ, $\mu = 0, 1, 2, \ldots, R-1$, the following sum:

$$\begin{aligned} D_k^\mu &= < F_k^\mu, \bar{\Xi} \mid \omega > + \begin{pmatrix} \mu \\ 1 \end{pmatrix} < F_k^{\mu-1}, \left(\partial_1 \bar{\Xi}\right) \mid \omega > + \\ &+ \begin{pmatrix} \mu \\ 2 \end{pmatrix} < F_k^{\mu-2}, \left(\partial_1^2 \bar{\Xi}\right) \mid \omega > + \ldots + \begin{pmatrix} \mu \\ \mu \end{pmatrix} < F_k^0, \left(\partial_1^\mu \bar{\Xi}\right) \mid \omega > . \end{aligned} \tag{0.13}$$

The functions $\Phi_k^{2R+\mu}$, $\mu = 0, 1, 2, \ldots, R-1$ are constructed by the following recurrence relations:

$$\begin{aligned} \Phi_k^{2R+\mu} &= -\left(\Xi_\delta \mid \omega\right)^{-1} \Big[D_\mu + \begin{pmatrix} \mu \\ 1 \end{pmatrix} \Phi_k^{2R-1+\mu} \left(\partial_1 \Xi_\delta\right) \mid \omega + \\ &+ \begin{pmatrix} \mu \\ 2 \end{pmatrix} \Phi_k^{2R-2+\mu} \left(\partial_1^2 \Xi_\delta\right) \mid \omega + \ldots + \\ &+ \begin{pmatrix} \mu \\ \mu \end{pmatrix} \Phi_k^{2R} \left(\partial_1^\mu \Xi_\delta\right) \mid \omega \Big] . \end{aligned} \tag{0.14}$$

Due to the nonzero Jacobian of the coordinate change the derivatives $\partial \xi_1 / \partial x_i$ can not be zero simultaneously. Furthermore, $\beta_i(x) \neq 0$ on ω (cf. [Coe]). Hence as seen by (5″) one can devide in (14) by $\Xi_\delta(x) \mid \omega \neq 0$.

3c). Finally, one constructs v_k, $k = 1, 2, \ldots, \Delta$. Let θ_1 be a piece of $\partial\Omega$ such that $\theta \subset \theta_1 \subset \omega$ and the distances between θ and $\omega \setminus \theta$ and also between θ_1 and $\partial\Omega \setminus \omega$ are nonzero numbers. Denote by S and S_1 two open subsets of Ω such that $S \subset S_1 \subset V$ and $\omega \cap S = \theta$, $\omega \cap S_1 = \theta_1$. Fix any function $\tau(\xi) \in C^\infty(V)$ with $supp\,\tau(\xi) = \overline{S}_1, \tau(\xi) = 1$ if $\xi \in \overline{S}$. Define for $\xi \in \Omega$ the functions $v_k(\xi)$ as follows:

$$(0.15) \quad v_k(\xi) = \begin{cases} \tau(\xi) \sum_{N=0}^{2R-1} [1/(R+N)!] \Phi_k^{R+N}(\xi') \xi_1^{R+N}, & \text{if} \quad \xi \in V \cap \Omega \\ 0 & \text{if} \quad \xi \in V \setminus \Omega \end{cases}$$

If $\xi \in S$ one has

$$(0.16) \qquad \partial_1^h \partial_2^J v_k(\xi) = \sum_{N=0}^{2R-1} [1/(R+N)!] \partial_2^J \Phi_k^{R+N}(\xi') \partial_1^h \xi_1^{R+N}.$$

Since ω is defined by $\xi_1 = 0$, then (15) and (16) imply (11) and (12). Hence $v_k \in D^\infty\left(\overline{\Omega}\right)$. In the sequel we set always $u_k = v_k$.

4). This step verifies $v_k \in D\left(\left[\hat{B}\right]^2\right)$. Since $D\left(\left[\hat{B}\right]^2\right) = \left\{g \in D\left(\hat{B}\right) : \hat{B}g \in D\left(\hat{B}\right)\right\}$ and $v_k \subset D^\infty\left(\overline{\Omega}\right) \subset D\left(\hat{B}\right)$ it is sufficient to check that $Bv_k \in D^\infty\left(\overline{\Omega}\right)$, i.e.:

$$(0.17) \qquad \left(\partial_1^\mu \partial_2^J B v_k\right) \mid \theta = 0, \quad \mu + |J| = 0, 1, 2, \ldots, R-1.$$

To verify (17) one first derives relations (22) and (23) used to rewrite the expression (5′) for Bv_k in a suitable form. With the notation (6) and (7) one rewrites (12) for $s = \mu + \nu$ as follows:

$$(0.18) \, (\partial_1^\mu \Gamma_k^\nu) \mid \theta = \Phi_k^{\nu+\mu}, \quad \nu = R, R+1, \ldots, 2R, \quad \mu = 0, 1, 2, \ldots, R-1.$$

With help of (18) one compares the definition (7) of G_k and of F_k^μ (cf. step 3a). Due to (18) it is clear that a ∂_1^μ–derivative of each element of the sequence $\overline{G}_k$

restricted on θ coincides with the element having the same index in the sequence F_k^μ or shortly:

$$\left(\partial_1^\mu \overline{G}_k\right) \mid \theta = F_k^\mu, \quad \mu = 0,1,2,\ldots,R-1. \tag{0.19}$$

With the notations $\overline{G}_k$ and $\overline{\Xi}$ one rewrites $(5')$ on θ as:

$$Bv_k \mid \theta =< \overline{G}_k, \overline{\Xi} >\mid \theta + \left(\Gamma_k^{2R}\Xi_\delta\right) \mid \theta. \tag{0.20}$$

One has for ∂_1^μ–derivatives of (20) on θ:

$$\partial_1^\mu Bv_k \mid \theta = \left(\partial_1^\mu < \overline{G}_k, \overline{\Xi} >\right) \mid \theta + \partial_1^\mu \left(\Gamma_k^{2R}\Xi_\delta\right) \mid \theta. \tag{0.21}$$

With help of Leibnitz' rule the first and second member on the right hand side of (21) are written in the forms given by (22) and (23) correspondingly:

$$\begin{aligned} \left(\partial_1^\mu < \overline{G}_k, \overline{\Xi} >\right) \mid \theta &= < \partial_1^\mu \overline{G}_k, \overline{\Xi} >\mid \theta + \\ &+ \binom{\mu}{1} < \partial_1^{\mu-1}\overline{G}_k, \partial_1 \overline{\Xi} >\mid \theta + \\ &+ \binom{\mu}{2} < \partial_1^{\mu-2}\overline{G}_k, \partial_1^2 \overline{\Xi} >\mid \theta + \ldots \\ &+ \binom{\mu}{\mu} < \overline{G}_k, \partial_1^\mu \overline{\Xi} >\mid \theta. \end{aligned} \tag{0.22}$$

$$\begin{aligned} \left[\partial_1^\mu \left(\Gamma_k^{2R}\Xi_\delta\right)\right] \mid \theta &= \left[\left(\partial_1^\mu \Gamma_k^{2R}\right)\Xi_\delta\right] \mid \theta + \\ &+ \binom{\mu}{1} \left[\left(\partial_1^{\mu-1}\Gamma_k^{2R}\right)\partial_1 \Xi_\delta\right] \mid \theta + \end{aligned}$$

(0.23)

$$+ \binom{\mu}{2} \left[\left(\partial_1^{\mu-2} \Gamma_k^{2R} \right) \partial_1^2 \Xi_\delta \right] \mid \theta +$$

$$+ \quad \ldots + \binom{\mu}{\mu} \left(\Gamma_k^{2R}, \partial_1^{\mu} \Xi_\delta \right) \mid \theta .$$

Now we have all the tools to check (17). Indeed, if $\mu = |J| = 0$ then the l.h.s. of (17) takes the form (20). Sustitute $G_k \mid \theta = F_k^0$ (cf. (19)) and $\Gamma_k^{2R} \mid \theta = \Phi_k^{2R}$ (cf. (18)) in (20). Then (14) is equivelent for $\mu = 0$ to the r.h.s. of (20) when it is set equal to zero. Hence the l.h.s. of (20) equals zero i.e. (17) is satisfied if $\mu = |J| = 0$.

Let $|J| = 0, \quad 1 \leq \mu \leq R-1$ and check that (21) equals zero. This is seen by comparison of the r.h.s. of (21) with (14) as follows. Rewrite (14) restricted to θ with zero to the left of the equality:

$$\begin{aligned} 0 &= \Phi_k^{2R+\mu} + (\Xi_\delta \mid \theta)^{-1} \left[D_\mu + \binom{\mu}{1} \Phi_k^{2R-1+\mu} (\partial_1 \Xi_\delta) \mid \theta + \right. \\ &\left. + \quad \ldots + \binom{\mu}{\mu} \Phi_k^{2R} (\partial_1^{\mu} \Xi_\delta) \mid \theta \right] . \end{aligned} \tag{14'}$$

It is seen that D_k^{μ} as defined by (13) coinsides with the r.h.s. of (22) after a substitution in (22) according to (19). All other (besides D_k^{μ}) members of (14′) equal r.h.s. of (23) after a substitution in (23) according to (18) with $\nu = 2R$. Therefore the sum of (22) and (23) (being equal to (21) and simultaneously equal to the r.h.s. of (11)) is zero.

Hence (17) is proved for any μ and $|J| = 0$, i.e. $(\partial_1^{\mu} B v_k) \mid \theta = 0$. A differentiation ∂_2^{J} where $\mu + |J| \leq R - 1$ of the above equality proves (17) in its general form.

5). This step checks relation (27) used as a starting point in step 6) to get the final result. Here one uses the following lemma.

Lemma *[2] Let $\hat{A}$ and $\hat{B}$ be self-adjoint and strongly commuting operators. Then the equality $D\left(\hat{A}\right) = D\left(\hat{B}\right)$ implies $D\left[\left(\hat{A}\right)^2\right] = D\left[\left(\hat{B}\right)^2\right]$.*

The above Lemma and step 4) imply $v_k \in D\left[\left(\hat{A}\right)^2\right]$. Hence $\hat{A}v_k \in D\left(\hat{A}\right)$ and the condition b) of [Dom] imply

$$(Av_k) \mid \theta = 0. \tag{0.24}$$

Consider also (17) for $\mu = |J| = 0$:

$$(Bv_k) \mid \theta = 0. \tag{24'}$$

Substitute zeros in places of the first ρ (of totaly δ) elements of Ξ (corresp. of Ξ_α) and denote the new sequences by Ξ^0 (corresp. Ξ^0_α).

Then one rewrites (24) and (24′) in terms of bilinear forms $< \cdot >$:

$$< G_k, \Xi^0_\alpha > \mid \theta = 0, \tag{0.25}$$

$$< G_k, \Xi^0_\alpha > \mid \theta = 0, \tag{25'}$$

According to (11) the first $\rho = \delta - (\Delta + 1)$ elements of $G_k \mid \theta$ are zero. Let $L(x)$ be a $\Delta + 1$ dimensional linear space associated with any $x \in \theta$. Then clearly the restrictions $G_k \mid \theta, \Xi^0_\alpha \mid \theta, \Xi^0 \mid \theta$ of G_k, Ξ^0_α, Ξ^0 belong to $L(x)$. Recall that $G_k \mid \theta$ is differ from $\overline{G}_k \mid \theta = F^0_k$ only by the last element $G_{k\delta} \mid \theta = \Phi^{2R}_k$. Then the notations (7), (8), (9) for G_k, F^0_k, F_k permit to rewrite (10) as

$$rank \left\| \begin{matrix} G_1 \\ G_2 \\ \vdots \\ G_\Delta \end{matrix} \right\| = \Delta, \tag{0.26}$$

Denote by $\chi(x)$ a function on θ with values in R^1. The orthogonality conditions (25) and (25′) of the Δ–dimensional subspace spanned by $G_k \mid \theta$, $k = 1, 2, \ldots, \Delta$, both to $\Xi^0_\alpha \mid \theta$ and to $\Xi^0 \mid \theta$ imply

$$\Xi^0_\alpha(x) = \chi(x)\Xi^0(x), \quad x \in \theta. \tag{0.27}$$

6). This step verifies the dependence of the equation of θ (39) only on one subset of the arguments. The general ideas of the proof in this step are similar to the relevant part of [2], although the arbitrariness of the power $2R$ and the nonconstant coefficients require modifications. Consider the components:

$$\hat{a}_{(1,1,\ldots,i)} = \sum_{j=1}^{n} \alpha_k \left(\frac{\partial\xi_1}{\partial x_j}\right)^{2R-1} \frac{\partial\xi_i}{\partial x_j} \quad \text{of } \Xi^0_\alpha \quad \text{and}$$

$$\hat{b}_{(1,1,\ldots,i)} = \sum_{j=1}^{n} \beta_k \left(\frac{\partial\xi_1}{\partial x_j}\right)^{2R-1} \frac{\partial\xi_i}{\partial x_j} \quad \text{of } \Xi^0 \quad i = 1,2,\ldots,n.$$

With the notations

$$(0.28) \qquad P_j(x) = [\alpha_j(x) - \chi(x)\beta_j(x)] \left(\frac{\partial\xi_1}{\partial x_j}\right)^{2R-1}, \quad j = 1,2,\ldots,n,$$

one rewrites (27) for the components $\hat{a}_{(1,1,\ldots,i)}$ and $\hat{b}_{(1,1,\ldots,i)}$ in the form:

$$(0.29) \qquad \left(\sum_{j=1}^{n} P_j(x)\frac{\partial\xi_i}{\partial x_j}\right) \mid \theta \quad i = 1,2,\ldots,n.$$

Denote by $\overline{x}$ the set $\{x_1, x_2, \ldots, x_{i_1-1}, x_{i_1+1}, \ldots, x_n\}$. In virtue of [Bou] one can solve equation $\xi_1 = 0$ with respect to some x_{i_1} in the form:

$$(0.30) \qquad \xi_1 = x_{i_1} - \zeta(\overline{x}) = 0.$$

Consequently for $\xi_1 = 0$ (i.e. for $x \in \theta$) x_{i_1} depends on $\overline{x}$. To emphasize this dependence one writes x on θ as $\Pi(\overline{x})$:

$$x = \{x_1, x_2, \ldots, x_{i-1}, \zeta(\overline{x}), x_{i+1}, \ldots, x_n\} = \Pi(\overline{x}).$$

Denote for $h = 2,3,\ldots,n$, $\eta_h(x) \equiv \xi_h(\Pi(\overline{x}))$. Due to (30) for $j \neq i_1$ one has

$$(0.31) \qquad \frac{\partial\eta_h}{\partial x_j} = \frac{\partial\xi_h}{\partial x_j} + \frac{\partial\xi_h}{\partial x_{i_1}}\frac{\partial\zeta}{\partial x_j}.$$

A substitution of $\partial\xi_h/\partial x_j$ from (31) in (29) gives for $h = 2.3.\ldots,n$:

$$(0.32)\ \sum_{\substack{j=1\\ j\neq i_1}}^{n} P_j(\Pi(\overline{x}))\frac{\partial\eta_h}{\partial x_j} - \frac{\partial\eta_h}{\partial x_{i_1}}\left[\sum_{\substack{j=1\\ j\neq i_1}}^{n} P_j(\Pi(\overline{x}))\frac{\partial\zeta}{\partial x_j} - P_{i_1}(\Pi(\overline{x}))\right] = 0.$$

Equality (29) for $i = 1$ due to $\partial\xi_1/\partial x_{i_1} = 1$ (cf. (30)) is written as:

$$(0.33)\qquad P_{i_1}(\Pi(\overline{x})) - \sum_{\substack{j=1\\ j\neq i_1}}^{n} P_j(\Pi(\overline{x}))\frac{\partial\zeta}{\partial x_j} = 0.$$

Using (33) one simplifies (32) as follows:

$$(0.34)\qquad \sum_{\substack{j=1\\ j\neq i_1}}^{n} P_j(\Pi(\overline{x}))\frac{\partial\eta_h}{\partial x_j} = 0, \quad h = 2,3,\ldots,n.$$

Remark that $\eta_2,\ldots,\eta_n$ are independent, which is implied by the nonzero Jacobian of the coordinate change. One comsiders them as $n-1$ independent solutions of (34) and gets:

$$(0.35)\qquad P_j(\Pi(\overline{x})) = 0, \quad j \neq i_1.$$

Furthermore (35) and (33) give:

$$(0.36)\qquad P_{i_1}(\Pi(\overline{x})) = 0.$$

Since $\partial\xi_1/\partial x_{i_1} \neq 0$ then (36) and (28) imply $\alpha_j(x) - \chi(x)\beta_j(x) = 0$. Due to [Coe] one has:

$$(0.37)\qquad r_{i_1}(x) = r_{i_2}(x) = \ldots = r_{i_\sigma}(x) = \chi(x), \quad x \in \theta.$$

If $j \neq i_q$, $q = 1, 2, \ldots, \sigma$, then (37) and [Coe] imply for any open subset θ' of θ existence of points $x \in \theta'$ such that

$$(0.38)\qquad r_j(x) - \chi(x) \neq 0.$$

Then at such points (28) and (35) imply $\partial\xi_1/\partial x_j = 0, \quad j \neq i_q$.

One shows that the last equality holds everywhere on θ. Assume on the contrary that there are points $\hat{x} \in \theta$, such that $r_j(\hat{x}) - \chi(\hat{x}) = 0$ for $j \neq i_q$. Then (28) and (35) offer two possibilities: $\partial\xi_1/\partial\hat{x}_j = 0$ and $\partial\xi_1/\partial\hat{x}_j \neq 0$. Continuitly properties extend the last inequality on a neighborhood $V(\hat{x}_j)$ of $\hat{x}_j$. Then (28) and (35) imply together with (37) that $r_{i_1}(\hat{x}) = r_j(\hat{x})$, $j \neq i_q$ on $V(\hat{x}_j)$ in contradiction to [Coe]. Consequently $\partial\xi_1/\partial x_j = 0, \quad j \neq i$ on θ, i.e. ξ_1 is independent of $x_j, \quad j \neq i_q$ and

$$(0.39)\qquad \xi_1\left(x_{i_1}, x_{i_2}, \ldots, x_{i_\sigma}\right) = 0$$

is the equation of θ.

The class of Nelson–type examples discussed here has the merit to be of interest in quantum theory. In this case, the notion of simultaneous measurability (compatibility) is basic for a number of developments. Recall that an observable according to von Neumann [4] is considered to be a self–adjoint operator or equivalently its canonically associated PV measure. More generally, e.g. [5], to an observable one associates a positive operator valued (POV) measure defined on the measure space of the possible observed values. Denoting by $F(\Delta)$ (correspondingly by $G(\Delta')$) the POV measures of the first (corresp. second) observable and taking the whole measure space to be R^1, one has the following Definition [5]. Two observables are called compatible if there exists a POV measure $M(\Delta \times \Delta'), \Delta \times \Delta' \in R^1 \times R^1$ such that $M(\Delta \times R^1) = F(\Delta), \quad M(R^1 \times \Delta') = G(\Delta')$.

In the case of PV measures it is known that the above definition is equivalent [5] to strong commutativity. (For an extension of the notation of strong commutativity to maximal symmetric operators – via commutativity of the generated semigroups – which is equivalent to compatibility, see [6], [7]).

An application of the Theorem from Section 3 to the special case of second order differential operators with constant coefficients which represent quantum observables is given in [7]. The case of higher order derivatives in the Theorem corresponds to observables representing powers of some other observables according to the standard quantization scheme. The application of the Theorem then shows that the simultaneous measurability of quantum observables depends on the geometric properties of the boundary of the Euclidean domain where the motion of the particle is confined.

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