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On one Absorption Law in Groupoid Theory

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It is proved: for every natural number $n \geq 5$ there is a commutative idempotent groupoid of order n which is not a quasigroup and which satisfies identity $(xy)(x(xy)) = x$.

1. It is known that any finite commutative idempotent quasigroup satisfies some "absorption law", i.e. some identity $t(x,y) = x$, where $t(x,y)$ is a term containing both variables x,y. Thus, it was natural that the following question be posed: is there any finite commutative idempotent groupoid which is not a quasigroup and which satisfies some "absorption law".

Of course, the elimination of finitary condition essentially simplifies the problem. In that case, it would be enough to choose some "suitable" law and the quotient groupoid of, let us say, two-element freely generated groupoid by the appropriate congruence relation would be a solution.

So, let us assume $G = (G, \cdot)$ is some groupoid we are looking for, thus satisfying all conditions of the problem. From now on we will call it, in general, a solution rather than repeat every now and then all conditions. The first observation is that for no two elements a,b of G it could be $a \cdot b = b$ (in opposite case $t(a,b) = b \neq a$ for any term $t(x,y)$ containing both variables). Hence the term $t(x,y)$ (from absorption law) could not be of the form $xt_1(x,y)$ unless it holds $G \models t_1(x,y) = x$. Or, to set it in other way: if $t(x,y)$ is a term (with both variables) of the least complexity such that $G \models t(x,y) = x$ then it could not be of the form $xt_1(x,y)$. Naturally, someone could notice firstly this result; reason is the following: if $t(x,y) = xt_1(x,y)$ then x must appear in $t_1(x,y)$, otherwise $G \models xy = x$ and (because of commutativity) $|G| = 1$. On the other hand, it would also hold $G \models t_1(x,y)t_1(t_1(x,y),x) = t_1(x,y)$, that is $G \models x = t_1(x,y)$. Term $t(x,y)$ could not be of the form $y \cdot t_1(x,y)$ either, for quasigroups are not taken into consideration. Hence, the simpliest terms which could be the potential candidates for the absorption law are $(xy)(x(xy)), (xy)(y(xy))$.

2. It is an easy exercise to prove that no groupoid with fewer than four elements can be a solution as well as that no four element, commutative idempotent groupoid satisfies any of identites $(xy)(x(xy)) = x$, $(xy)(y(xy)) = x$. One characterization of finite commutative idempotent groupoids which satisfy an abscription law has been given by N. Ruškuc. He has also found the following four-element solution

·	0	1	2	3
0	0	3	1	1
1	3	1	0	2
2	1	0	2	1
3	1	2	1	3

with the absorption law $(((xy)y)x((xy)y) = x$.

The next three solutions with, respectively, five, six and eight elements, all satisfying the identity $(xy)(x(xy)) = x$, will be of use later.

E x a m p l e 1. Groupoid A given by Cayley's table:

·	0	1	2	3	4
0	0	2	3	2	2
1	2	1	4	2	2
2	3	4	2	0	1
3	2	2	0	3	2
4	2	2	1	2	4

E x a m p l e 2. Groupoid B.

·	0	1	2	3	4	5
0	0	2	1	4	5	4
1	2	1	0	5	3	3
2	1	0	2	4	3	4
3	4	5	4	3	2	1
4	5	3	3	2	4	0
5	4	3	4	1	0	5

E x a m p l e 3. Groupoid C:

$\cdot$	0	1	2	3	4	5	6	7
0	0	2	1	4	3	3	7	6
1	2	1	0	5	7	3	5	4
2	1	0	2	7	7	6	5	3
3	4	5	7	3	0	1	2	2
4	3	7	7	0	4	1	0	1
5	3	3	6	1	1	5	2	2
6	7	5	5	2	0	2	6	0
7	6	4	3	2	1	2	0	7

Now, when we know that there are solutions (to the initial problem), it is natural to raise a question if there is a solution of any finite order. The following theorem offers an answer.

Theorem 2.1. *For every natural number $n \geq 5$ there is a solution of order n satisfsying ("the simpliest") absorption law $(xy)(x(xy)) = x$*

P r o o f. It is an immediate consequence of applications of next lemmas to the given examples. ∎

Lemma 2.2. *Let $G = \langle\{0, 1, \ldots, n-1\}, \cdot\rangle, n \geq 5$, be a solution satisfying the identity $(xy)(x(xy)) = x$, such that there is some i appearing in each row of its Cayley's table(in other words, it holds:$\forall j\ \exists k \quad j \cdot k = i$). Then the commutative idempotent groupoid $G_1 = \langle\{0, 1, \ldots, n+1\}, *\rangle$, where, for $k < j$:*

$$k * j = \begin{cases} k \cdot j & k, j \in n \\ i & i \neq k \in n, j = n, n+1 \\ n & k = i, j = n+1 \\ n+1 & k = i, j = n \\ i & k = n, j = n+1, \end{cases}$$

satisfies the same absorption law.

P r o o f. Let $k \in n \setminus \{i\}$. Then :

$$\begin{aligned} & (k * n) * (k * (k * n)) = i * (k * i) = i \cdot (k \cdot i) = \\ = \quad & (k \cdot j)(k \cdot (k \cdot j)) = k, \text{ where } j \in n, \text{ and } k \cdot j = i, \end{aligned}$$

$$(k * n) * (n * (k * n)) = i * (n * i) = i * (n+1) = n.$$

Analogously: $(k * (n+1)) * (k * (k * (n+1))) = k$

$$(k * (n+1)) * ((n+1) * (k * (n+1))) = n+1.$$

$$(i * n) * (i * (i * n)) = (n+1) * (i * (n+1)) = (n+1) * n = i,$$

$$(i * n) * (n * (i * n)) = (n+1) * (n * (n+1)) = (n+1) * i = n.$$

Similarly:

$$(i * (n+1)) * (i * (i * (n+1))) = i,$$

$$(i * (n+1)) * ((n+1) * (i * (n+1))) = n+1.$$

Finally:

$$(n * (n+1)) \star (n * (n * (n+1))) = i * (n * i) = i * (n * i) = i * (n+1) = n,$$

$$(n * (n+1)) * ((n+1) * (n * (n+1))) = i * ((n+1) * i) = i * n = n+1.$$

Let us note that element i also appears in each row of Cayley's table of (new groupoid) G_1. ■

Lemma 2.3. *(lemma of duplication). Let $G = \langle\{0,1,\ldots,n-1\},\cdot\rangle, n \geq 5$, be a solution satisfying the identity $(xy)(x(xy)) = x$. Then the commutative idempotent groupoid $\langle G_1 = \{0,1,\ldots,2n-1\}, *\rangle$ where, for $i < j$:*

$$i * j = \begin{cases} i \cdot j & i,j \in n \\ i \cdot (j-n) + n & i \in n, j \geq n \wedge j \neq i+n \\ i \cdot (j+1-n) + n & i \in n-1, j = i+n \\ i \cdot (j-1-n) + n & i = n-1, j = 2n-1 (= i+n) \\ (i-n) \cdot (j-n) & i,j \geq n, \end{cases}$$

satisfies the same absorption law.

Proof. Let $i,j \in n, i \neq j$. Since $i \cdot j \neq i,j$, we have:

$$\begin{aligned} & (i * (j+n)) * (i * (i * (j+n))) = (i \cdot j + n) * (i * (i \cdot j + n)) = \\ = \; & (i \cdot j + n) * (i \cdot (i \cdot j) + n) = (i \cdot j) \cdot (i \cdot (i \cdot j)) = i; \end{aligned}$$

$$\begin{aligned} & (i * (j+n)) * ((j+n) * (i * (j+n))) = \\ = \; & (i \cdot j + n) * ((j+n) * (i \cdot j + n)) = \\ = \; & (i \cdot j + n) * (j \cdot (i \cdot j)) = (i \cdot j) \cdot (j \cdot (i \cdot j)) + n = j + n. \end{aligned}$$

Now let $i \in n-1$.

$$(i*(i+n))*(i*(i*(i+n))) =$$

$$= \quad (i\cdot(i+1)+n)*(i*(i\cdot(i+1)+n)) =$$

$$= \quad (i\cdot(i+1)+n)*(i\cdot(i\cdot(i+1))+n) =$$

$$= \quad (i\cdot(i+1))\cdot(i\cdot(i\cdot(i+1))) = i,$$

$$(i*(i+n))*((i+n)*(i*(i+n))) =$$

$$= \quad (i\cdot(i+1)+n)*((i+n)*(i\cdot(i+1)+n)) =$$

$$= \quad (i\cdot(i+1)+n)*(i\cdot(i\cdot(i+1))) =$$

$$= \quad (i\cdot(i+1))\cdot(i\cdot(i\cdot(i+1)))+n = i+n.$$

$$\begin{aligned} & ((n-1)*(2n-1))*((n-1)*((n-1)*(2n-1))) = \\ = \quad & ((n-1)\cdot(n-2)+n)*((n-1)*((n-1)\cdot(n-2)+n)) = \\ = \quad & ((n-1)\cdot(n-2)+n)*((n-1)\cdot((n-1)\cdot(n-2))+n) = \\ = \quad & ((n-1)\cdot(n-2))\cdot((n-1)\cdot((n-1)\cdot(n-2))) = n-1, \end{aligned}$$

$$\begin{aligned} & ((n-1)*(2n-1))*((2n-1)*((n-1)*(2n-1))) = \\ = \quad & ((n-1)\cdot(n-2)+n)*(((n-1)+n)*((n-1)\cdot(n-2)+n)) = \\ = \quad & ((n-1)\cdot(n-2)+n)*((n-1)\cdot((n-1)\cdot(n-2))) = \\ = \quad & ((n-1)\cdot(n-2))\cdot((n-1)\cdot((n-1)\cdot(n-2)))+n = \\ = \quad & (n-1)+n = 2n-1. \end{aligned}$$

Finally, for $i, j \in n, i \neq j$:

$$\begin{aligned} & ((i+n)*(j+n))*((i+n)*((i+n)*(j+n))) = \\ = \quad & (i\cdot j)*((i+n)*(i\cdot j)) = \\ = \quad & (i\cdot j)*(i\cdot(i\cdot j)+n) = (i\cdot j)\cdot(i\cdot(i\cdot j))+n = i+n. \end{aligned}$$

Analogously:

$$((i+n)*(j+n))*((j+n)*((i+n)*(j+n))) = j+n.$$

■

There is nothing left for us to do but to summarize the facts. By solution we will mean a solution with absorption law $(xy)(x(xy)) = x$.

Groupoid A "generates" solutions of odd orders ≥ 5, by applications of lemma 2.2. sufficiently many times (note that element 2 appears in each row and see the remark in the end of the proof of lemma 2.2) as well as solution of orders $2^k r, k \geq 1, r$ -odd number ≥ 5(this time we need both lemmas).

Groupoid B "generates" solutions of orders $3.2^k, k \geq 1$, and C solutions of orders $2^k, k \geq 3$(in both cases lemma 2.3 does the whole work.

The next lemma explains why we need (besides Lemma 2.2.) lemma of duplication.

Lemma 2.4. *There is no finite commutative idempotent groupoid of even order satisfying absorption law $(xy)(x(xy)) = x$ such that some element appears in each row of its Cayley's table'*

Proof. Suppose that G is a finite commutative idempotent groupoid satisfying the law $(xy)(x(xy)) = x$ and that (some) element i appears in each raw. Then, for every element j there exist k such that $j \cdot k = i$, whence $i \cdot (j \cdot i) = (j \cdot k) \cdot (j \cdot (j \cdot k)) = j$. Thus any element appears in i-th row exactly once. Let us note further that, for $p \neq i, i \cdot p = q$, it has to be $i \cdot q = p$. For, in opposite case, let us say $i \cdot q = r (\neq p)$, it would follow that q appears twice in i-th row (because of $q \cdot r = (i \cdot p)(i \cdot (i \cdot p)) = i$ and $i \cdot r = (q \cdot r) \cdot ((q \cdot r) \cdot q) = q)$. But the last remark just means that there are even number of elements different from i, i.e. that $|G|$ is an odd number. ■

Lemma 2.5. *There is no finite commutative idempotent grupoid which is not a quasigroup and which satisfies the absorption law $(xy)(y(xy)) = x$.*

Proof. Suppose G is a finite commutative idempotent grupoid and $G \models (xy)(y(xy)) = x$. But then also $G \models y(x(xy)) = x$ and, because of $|G| < \infty, G$ would be a quasigroup. ■

3. In the end of this text we just want to note that the result mentioned in the begining for finite comutative idempotent quasigroups does not hold in general (i.e. for all commutative idempotent quasigroups). For instance, in $\langle Ra, \circ \rangle$, where Ra is the set of rational numbers and $\circ$ is defined by $a \circ b = \frac{a+b}{2}$, for any term $t(x, y)$ containing both variables x, y and any $a, b(\in Ra), a < b$, it holds: $a < t(a, b) < b$.

One general method for constructing counterexamples is based on the fact that for any term $t(x, y)$ with both variables x, y, there is a commutative idempotent quasigroup $K_{t(x,y)}$ which does not satisfy identity: $t(x, y) = x$. Let us take for domain the set of natural numbers. Since we want to have $K_{t(xy)} \models \neg(t(x, y) = x)$, we will define operation so that $K_{t(x,y)} \models t(0, 1) \neq 0$. We start determining "multiplication" by $0 \cdot 1 = 2$ and following "the structure" of the term $T(x, y)$ we continue so that whenever we have different a, b, "product"(if not already defined) is some c greater than a and b. Once we provide $t(0, 1) \neq 0$, we fulfill Cayley's table so that idempotency, commutativity and property "being quasigroup" are preserved. This can be done (in many ways) since at each step there are only finitely many restrictive conditions while we have at disposal infinitely many elements. Finally, let $T_{(x,y)}$ be the set of all terms containing both variables. Quasigroup $K = \Pi_{t \in T_{x,y}} K_{t(x,y)}$ certainly does not satisfy any absorption law for it contains isomorphic copy of each of its factors.

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