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On the Spectrum of the Couple of Symmetric Homogeneous Operators

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There are problems in modern science and engineering in which for the exact description of studied phenomena the linearization of nonlinear mathematical models is not sufficient. In such cases the application of homogeneous and polynomial operators instead of linear operators can be convenient. For these operators many notations and properties are similar to the notations and properties which are known for linear operators. For example, the notations of the spectrum and the numerical range can be transferred from the linear analysis to homogeneous operators as it was done in [9] and [11].

In the present article we extend to homogeneous operators further notations as the symmetry, the selfadjointness and the normality. The main result is the Theorem 3.7 in which it is proven that the spectrum of a couple of symmetric homogeneous operators contains nonzero points.

1. Notations and definitions

Throughout this paper let X, Y denote Banach spacees and X^* the dual space of X. By the symbol $\langle \cdot, \cdot \rangle$ we mean the pairing between X and X^*. In case of a Hilbert space we use the same symbol for the inner product. For the norm or weak convergence of the sequence $\{x_n\} \subset X$ to a point $x_0 \in X$ we use the symbols $x_n \to x_0$ or $x_n \overset{w}{\to} x_0$ respectively.

Let $\mathbf{R}$ and $\mathbf{C}$ be the spaces of real and complex numbers respectively. Further we denote $S_1(0) = \{x \in X : ||x|| = 1\}$ the unit sphere in X.

Definition 1.1. We shall say that an operator $F : X \to Y$ is

(a) a *positive homogeneous operator of degree* k if there is a number $k \in \mathbf{R}$ such that $F(tx) = t^k.F(x)$ for any $x \in X$ and any $t \in \mathbf{R}$, $t > 0$;

(b) a *homogeneous operator of degree k* on a real (resp. complex) space X if k is an integer and the equality $F(tx) = t^k.F(x)$ holds for any $t \in \mathbf{R}$ (resp. $t \in \mathbf{C}$), $t \neq 0$ and any $x \in X$;

(c) for any continuous positive homogeneous operator $F : X \to Y$ of the degree $k > 0$ we define the norm by $||F|| = \sup_{x \in S_1(0)} ||F(x)||$.

R e m a r k 1.2. It is easy to show that for the operator from the Definition 1.1(c) for any $x \in X$ the following estimation holds: $||F(x)|| \leq ||F||.||x||^k$.

Definition 1.3 We shall say that a homogeneous operator P from X into a Banach space Y of the degree $k \geq 1$ is a *homogeneous polynomial operator* if there is a k-linear symmetric operator $\mathcal{P} : X \times X \times \ldots \times X \to X$, (i.e., $\mathcal{P}(x_1, x_2, \ldots, x_k)$ is linear in any variable x_j, $j = 1, 2, \ldots, k$ and it does not change its values under arbitrary permutation of all variables) such that $\mathcal{P}(x, x, \ldots, x) = P(x)$ for any $x \in X$.

Operator $\mathcal{P}$ is called the *polar operator* to the operator P.

For any continuous k-linear operator $\mathcal{P}$ we define the norm by

$$||\mathcal{P}|| = \sup_{x_i \in S_1(0)} ||\mathcal{P}(x_1, x_2, \ldots, x_k)||, \qquad i = 1, 2, \ldots, k.$$

R e m a r k 1.4. Polynomial operators have some properties which are similar to properties of linear operators. For example, it is easy to prove that for any polynomial operator $P : X \to Y$ the following conditions are equivalent:

(a) P is continuous at the point zero;

(b) P is continuous at any point;

(c) P is bounded (i.e. the norm $||P||$ is finite);

(d) P is uniformly continuous on every bounded set;

(e) P is Fréchet differentiable at any point;

Definition 1.5. We shall say that a homogeneous operator $F : X \to X^*$ is:

a) *positive* if for any $x \in X$, $x \neq 0$ it holds $\langle Fx, x \rangle > 0$;

b) *positively defined* if there is a number $c \in \mathbf{R}$, $c > 0$ such that

$$\inf_{x \in S_1(0)} Re\langle Fx, x \rangle = c > 0.$$

Definition 1.6. An operator $F : X \to Y$ is called *hemi-continuous at* $x_0 \in X$, if for any sequence $\{t_n\} \subset \mathbf{R}$, $t_n \to 0$, and for any $h \in X$ it holds $F(x_0 + t_n h) \xrightarrow{w} F(x_0)$ in the weak topology of the space Y.

We shall say that Gateaux derivative F' of a differentiable operator $F : X \to X^*$ is hemi–continuous at a point $x \in X$, if for any sequence $\{t_n\} \subset [0, +\infty)$ such that $t_n \to 0$ and for arbitrary points $h \in X$, $y \in X$ the sequence $\{F'(x + t_n h)y\} \subset X^*$ converges in the weak*–topology of the space X^* to the point $F'(x)y \in X^*$.

In the paper [3] the notation of the adjoint operator to a nonlinear operator was introduced and in [4] some of its properties were studied. We recall this notation for the convenience of the reader in the following definition.

Definition 1.7. ([3], Definition 3.) Let $D \subset X$ be an open set which is starshaped with respect to the origin (i.e., for any $x \in D$ and all $t \in [0, 1]$ it holds $tx \in D$).

Let the operator $F : D \subset X \to X^*$ have Gateaux derivative $F'(x)$ at any point $x \in D$ and let F satisfy the following conditions:

(1) $F(0) = 0$;

(2) The function $\langle F'(tx)h,\, x\rangle$ of the variable $t \in [0, 1]$ is integrable for arbitrary (but fixed) points $x \in D$, $\quad h \in X$.

Let us suppose, further, that for any $x \in D$ there exists unique point $x^*(x) \in X^*$ such that for all $h \in X$ it holds

$$\langle x^*(x),\, h\rangle = \int_0^1 \langle F'(tx)h,\, x\rangle\, dt.$$

Then the operator $F^* : D \subset X \to X^*$ defined for $x \in D$ by $F^*(x) = x^*(x)$ we call the *adjoint operator* to the operator F.

R e m a r k 1.8. In case of a real Banach space X the adjoint operator F^* from Definition 1.7 can be written in the form

$$F^*(x) = \int_0^1 [F'(tx)]^*\,(x)\, dt,$$

where $[F'(tx)]^*$ denotes the adjoint operator to the continuous linear operator $F'(tx)$.

R e m a r k 1.9. According to [4] (Theorem 2.6) there exists the adjoint operator F^* to a nonlinear operator F if F satisfies the conditions (1) and (2) from the Definition 1.7 and moreover, F has a hemi–continuous Gateaux derivative. Then F and F^* are both hemi–continuous and the following estimation holds

$$||F^*(x)|| \leq ||x|| \int_0^1 ||F'(tx)||\, dt, \quad x \in X.$$

The following proposition is a direct consequence of the Definition 1.7 and results from [3] and [4] applied to the class of homogeneous operators.

Proposition 1.10. *Let $F : X \to X^*$ be a homogeneous operator of the degree $k \geq 1$ having hemi–continuous Gateaux derivative F'. Then for any $x \in X$ the following assertions hold:*

(1)

$$F^*(x) = \frac{1}{k}\,[F'(x)]^*\, x,$$

where F^* is the adjoint operator to the operator F and $[F'(x)]^*$ is the adjoint operator to the continuous linear operator $F'(x)$;

(2)

$$F(x) = H(x) + R(x),$$

where $H, R : X \to X^*$ are hemi–continuous operators which can be written as

$$H(x) = \frac{1}{k+1}\,[F(x) + kF^*(x)],$$

$$R(x) = \frac{k}{k+1}\,[F(x) - F^*(x)];$$

(3) H is a potential operator, $H = grad\,\varphi$, where

$$\varphi(x) = \frac{1}{k+1}\langle F(x), x\rangle$$

and the operator R fulfills the equality

$$\langle R(x), x\rangle = \frac{2ki}{k+1}\, Im\,\{\langle F(x), x\rangle\}:$$

(4)

$$||F^*(x)|| \leq \frac{1}{k}||F'(x)||.||x||;$$

(5) If $S, T : X \to X^*$ are homogeneous operators with their adjoints operators S^*, T^* then for any $\lambda \in \mathbf{C}$ it holds $(S - \lambda T)^* = S^* - \overline{\lambda} T^*$.

2. Symmetric and selfadjoint operators

Definition 2.1. Let $M \subset X$ be an open set which is starshaped with respect to the origin. We shall say that an operator $F : M \to X^*$ satisfying $F(0) = 0$ is *symmetric* on M if there exists the adjoint operator F^* to F and for any $x \in M$ it holds $F(x) = F^*(x)$.

Let X be a Hilbert space. We shall say that $F : X \to X$ is *selfadjoint* if F is symmetric on the whole space X.

Definition 2.2. Let an operator $F : X \to X^*$ have Gateaux derivative $F'(x_0)$ at a given point $x_0 \in X$. We shall say that the operator F is *normal at the point* x_0 if for any $h \in X$ holds the equality

$$||F'(x_0)h|| = ||\left[F'(x_0)\right]^* h||.$$

(Here $[F'(x_0)]^*$ is the adjoint to the linear continuous operator $F'(x_0)$.)

We shall say that F is a *normal operator on an open set* $D \subset X$ if it is normal at any point $x \in D$.

R e m a r k 2.3. If X is a real Banach space and $F : X \to X^*$ has a hemi-continuous Gateaux derivative on M then F is symmetric on M if and only if F is potential on M.

P r o o f. It follows from [4] (Theorem 2.18) and [13] (Theorem 6.3), p. 76). ■

An important example of a selfadjoint homogeneous operator is a continuous symmetric homogeneous polynomial operator.

E x a m p l e 2.4. Let k be a natural number and $I \subset \mathbf{R}$ be a compact interval. Let $K(t, t_1, t_2, \ldots, t_k)$ be a real quadratically integrable function on the Cartesian product I^{k+1} such that it does not change its values under arbitrary permutation of variables $t, t_1, t_2, \ldots, t_k$. Then the operator $S : L^2(I) \to L^2(I)$ defined on the real Hilbert space $L^2(I)$ by

$$(Sx)(t) = \int_{I^k} K(t, t_1, t_2, \ldots, t_k).x(t_1).x(t_2). \ldots .x(t_k)\, dt_1 dt_2 \ldots dt_k$$

is selfadjoint.

R e m a r k 2.5. Unlike of the linear case, for a nonlinear operator F on a Hilbert space with its adjoint F^*, the equality $F = F^{**}$ does not hold generally even if the operator F is normal as the following example shows.

E x a m p l e 2.6. Let E_2 be two–dimensional Euclidean space and let $F : E_2 \to E_2$ be a homogeneous operator of the degree $k = 2$ defined by

$$F(x) = (x_1^2 - x_2^2\,,\, 2x_1x_2) \qquad \text{for any} \qquad x = (x_1, x_2) \in E_2.$$

Then

$$F'(x) = \begin{pmatrix} 2x_1 & -2x_2 \\ 2x_2 & 2x_1 \end{pmatrix} \qquad \text{and} \qquad [F'(x)]^* = \begin{pmatrix} 2x_1 & 2x_2 \\ -2x_2 & 2x_1 \end{pmatrix}.$$

Using proposition 1.10 we obtain the adjoint operator

$$F^*(x) = \frac{1}{2}\,[F'(x)]\,x^* = \frac{1}{2}\begin{pmatrix} 2x_1 & 2x_2 \\ -2x_2 & 2x_1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^2 + x_2^2 \\ 0 \end{pmatrix}.$$

Further it holds

$$[F^*(x)]' = \begin{pmatrix} 2x_1 & 2x_2 \\ 0 & 0 \end{pmatrix}, \qquad \left\{[F^*(x)]'\right\}^* = \begin{pmatrix} 2x_1 & 0 \\ 2x_2 & 0 \end{pmatrix}.$$

Hence, using Proposition 1.10 again, we have

$$F^{**}(x) = \frac{1}{2}\begin{pmatrix} 2x_1 & 0 \\ 2x_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^2 \\ x_2x_1 \end{pmatrix},$$

so that $F(x) \neq F^{**}(x)$.

At the same time F is normal on E_2 (see Definition 2.2) because $||F'(x)h|| = 2||x||.||h|| = ||\,[F'(x)]^*\,||$ for any $x, h \in E_2$.

Proposition 2.7. *Let X be a Hilbert space and let $F : X \to X^*$ be a homogeneous operator of the degree $k \geq 1$ with its adjoint F^* having hemi-continuous Gateaux derivative. Then the equality $F = F^{**}$ holds if and only if for any $x \in X$ the following condition (*) is satisfied*

$$[F'(x)]^* = [F^*(x)]' \qquad (*)$$

P r o o f. Properties of homogeneous operator (see Proposition 1.10) imply the following equalities

$$\langle Fx, h\rangle = \langle \frac{1}{k}F'(x)x, h\rangle = \frac{1}{k}\langle x, [F'(x)]^*\,h\rangle,$$

and

$$\langle F^{**}x, h\rangle = \langle \frac{1}{k}\left\{[F^*(x)]'\right\}^*\,x, h\rangle = \frac{1}{k}\langle x, [F^*(x)]'\,h\rangle,$$

which are valid for any $x, h \in X$.

The assertion follows now immediately. ∎

3. Properties of the approximative spectrum of a selfadjoint homogeneous operator

Definition 3.1. ([9], Definition 3.1) Let $S, T : X \to X^*$ be homogeneous operators of the degree k. We call the *numerical range of the couple* (S,T) the set $W(S,T)$ of complex numbers defined by

$$W(S,T) = \left\{ \frac{\langle Sx, x\rangle}{\langle Tx, x\rangle} : x \in S_1(0),\ \langle Tx, x\rangle \neq 0 \right\}.$$

It is evident that if X is a Hilbert space, S is a linear continuous and T is the identity operator on X then we obtain the well known Hausdorff's definition of the numerical range.

Proposition 3.2. *([11], Proposition 3.14) Let $S, T : X \to X^*$ be positive homogeneous operators. Let S be continuous and T be positively defined.*

Then the following assertions hold:

(1) W(S,T) is a bounded set and for any $\lambda \in W(S,T)$ it holds

$$|\lambda| \leq \frac{||S||}{c}, \qquad \textit{where} \qquad c = \inf_{x \in S_1(0)} Re\, \{\langle Tx, x\rangle\},$$

(2) If, in addition, S and T are polynomial operators then $W(S,T)$ is a convex set. (A generalization of Hausdorff's and Toeplitz's theorem on the convexity of the numerical range).

Definition 3.3. ([11], Definition 3.1) Let $S, T : X \to Y$ be positive homogeneous operators. By the *approximative spectrum* (briefly spectrum) of the couple (S,T) we understand the set $\sigma(S,T)$ of complex numbers defined as follows

$$\sigma(S,T) = \left\{ \lambda \in \mathbf{C} : \quad \inf_{x \in S_1(0)} ||Sx - \lambda Tx|| = 0 \right\}.$$

Definition 3.4. ([11], Definition 3.8) We shall say that $\lambda_0 \in \mathbf{C}$ is the *eigenvalue of a couple* (S,T) of positive homogeneous operators $S, T : X \to Y$, if there is a point $x_0 \in S_1(0) \subset X$ such that $Sx_0 - \lambda_0 Tx_0 = 0$. The point x_0 is called the *eigenvector* of the couple (S,T) related to λ_0.

The set of all eigenvalues of the couple (S,T) we denote $\Lambda(S,T)$.

Proposition 3.5. *Let $S,T : X \to X$ be continuous homogeneous operators on a Hilbert space X. Suppose S is symmetric on $S_1(0) \subset X$ and T is positive. Then the following assertions hold:*

(1) $W(S,T) \quad \subset \quad [m,M]$;

(2) $\sigma(S,T) \quad \subset \quad \overline{W(S,T)}$, *where*

$$m = \inf_{x\in S_1(0)} \frac{\langle Sx,x\rangle}{\langle Tx,x\rangle}, \qquad M = \sup_{x\in S_1(0)} \frac{\langle Sx,x\rangle}{\langle Tx,x\rangle}.$$

Proof. According to 3.2 $W(S,T)$ is a connected set. To prove (1) it is sufficient to show that $\langle Sx,x\rangle$ possesses real values only because $\langle Tx,x\rangle$ is positive due to the assumption. Indeed, using definition 2.2 we have $Sx = S^*x$ for any $x \in S_1(0)$ and, according to Proposition 1.10, it holds

$$\begin{aligned}\langle S^*x,x\rangle &= \langle \frac{1}{k}[S'(x)]^*x,x\rangle = \frac{1}{k}\langle x,S'(x)x\rangle = \\ &= \overline{\langle \frac{1}{k}S'(x)x,x\rangle} = \overline{\langle S(x),x\rangle} = \langle S(x),x\rangle.\end{aligned}$$

It shows that the expression $\dfrac{\langle S(x),x\rangle}{\langle T(x),x\rangle}$ is real and thus $W(S,T)$ is a real interval in $[m,M]$.

The assertion (2) is a direct consequence of Theorem 3.13 in [11]. ∎

Definition 3.6. Let $S,T : X \to X^*$ be homogeneous operators. Supposing that sets $\sigma(S,T)$ and $W(S,T)$ are nonempty and bounded we define the number

$$r_{\sigma(S,T)} = \sup_{\lambda\in\sigma(S,T)} |\lambda|$$

as the *spectral radius* of the couple (S,T) and the number

$$r_{W(S,T)} = \sup_{\lambda\in W(S,T)} |\lambda|$$

as the *numerical radius* of the couple (S,T).

Theorem 3.7. *Let X be a Hilbert space and let $S,T : X \to X^*$ be homogeneous operators of the degree $k \geq 1$ which are both symmetric on*

$S_1(0) \subset X$. Suppose, further, that S and T have hemi-continuous Gateaux derivatives on $S_1(0)$ and let T be positive.

Then the following assertions hold:

(1) *The couple (S,T) has only a real approximative spectrum $\sigma(S,T)$ which lies in the interval $[m, M]$, where*

$$m = \inf_{x\in S_1(0)} \frac{\langle S(x), x\rangle}{\langle T(x), x\rangle}, \qquad M = \sup_{x\in S_1(0)} \frac{\langle S(x), x\rangle}{\langle T(x), x\rangle},$$

and both boundary points m and M belong to the spectrum $\sigma(S,T)$.

(2) *If there is a point $y \in S_1(0)$ such that the number $\lambda = \dfrac{\langle S(y), y\rangle}{\langle T(y), y\rangle}$ is equal to m or M then λ is an eigenvalue of the couple (S,T) with its eigenvector y.*

(3) $r_{\sigma(S,T)} = r_{W(S,T)}$.

P r o o f. Applying Proposition 3.5 (2) we obtain

$$\sigma(S,T) \quad \subset \quad \overline{W(S,T)} \quad \subset \quad [m, M].$$

Suppose $\lambda = m$. Being symmetric on $S_1(0)$, S and T satisfy the inequality

$$\langle S'(x)z \; - \; \lambda T'(x)z, \; z\rangle \geq 0 \qquad \text{for any} \quad x \in S_1(0), \quad z \in X.$$

This enables to derive by rutine argument Schwarz's inequality

$$|\langle [S'(x) \; - \; \lambda T'(x)]z, \; y\rangle|^2 \; \leq$$
$$\langle [S'(x) \; - \; \lambda T'(x)]z, \; z\rangle . \langle [S'(x) \; - \; \lambda T'(x)]y, \; y\rangle,$$

which holds for any $y, z \in X$, $\quad x \in S_1(0)$.

Putting $z = x$ and $y = \dfrac{1}{k} \, [S'(x) \; - \; \lambda T'(x)x]$, we obtain

$$||S(x) \; - \; \lambda T(x)||^4 \; \leq$$
$$\leq k.\langle S(x) \; - \; \lambda T(x), \; x\rangle .||S'(x) \; - \; \lambda T'(x)||^3.k^{-3}.||x||^2. \qquad (**)$$

The assumption $\lambda = m \; = \; \inf_{x\in S_1(0)} \dfrac{\langle Sx, x\rangle}{\langle Tx, x\rangle}$ implies that

$$\inf_{x\in S_1(0)} \langle Sx \; - \; \lambda Tx, \; x\rangle \; = \; 0.$$

The expression $||S'x - \lambda T'x||$ is bounded on $S_1(0)$ because both derivatives are homogeneous and hemi–continuous. Taking infimum on $S_1(0)$ in the inequality (**) we obtain $\inf_{x\in S_1(0)} ||Sx - \lambda Tx|| = 0$, which means that $\lambda = m \in \sigma(S,T)$.

Similarly it is possible to show that $\lambda = M \in \sigma(S,T)$ and (1) is proven.

To prove (2) we assume that $\lambda = \dfrac{\langle Sy,y\rangle}{\langle Ty,y\rangle} = m$ and define an operator $P : X \to X$ by

$$Px = Sx - \lambda Tx.$$

Then P is symmetric on $S_1(0)$ and $\langle Py, y\rangle = 0$.

At the same time it holds $\inf_{x\in S_1(0)} \dfrac{\langle Px,x\rangle}{\langle Tx,x\rangle} \geq 0$, which implies that $\langle Px, x\rangle \geq 0$ for any $x \in X$.

Let $t \in \mathbf{R}$, $x \in X$ be arbitrary points. Let us put $\alpha = t\langle Py, x\rangle$.

Because P is homogeneous and Gateaux–differentiable on $S_1(0)$ then, according to Taylor's Theorem (see [13], Theorem 4.8), it holds the following

$$\begin{aligned} 0 \ &\leq\ \langle P(y+\alpha x),\, y+\alpha x\rangle = \\ &=\ \langle P(y) + \alpha P'(y)x + \circ(\alpha), y+\alpha x\rangle = \\ &=\ \langle Py,y\rangle + \overline{\alpha}\langle Py,x\rangle + \alpha\langle P'(y)x,y\rangle + \alpha\overline{\alpha}\langle P'(y)x,x\rangle + \\ &\quad + \langle \circ(\alpha), y+\alpha x\rangle = \\ &=\ t\overline{\langle Py,x\rangle}\langle Py,x\rangle + t\langle Py,x\rangle\langle P'(y)x,y\rangle + \\ &\quad + t^2\langle Py,x\rangle\overline{\langle Py,x\rangle}\langle P'(y)x,x\rangle + \langle \circ(t\langle Py,x\rangle), y + t\langle Py,x\rangle x\rangle. \end{aligned}$$

Further, because P is symmetric on $S_1(0)$, due to Proposition 1.10 (1) it holds

$$\langle P'(y)x,y\rangle = \langle x, [P'(y)]^* y\rangle = \langle x, kP^*(y)\rangle = k\langle x, P(y)\rangle = k.\overline{\langle P(y),x\rangle}.$$

Using this and the previous inequality we obtain

$$0 \leq t|\langle Py,x\rangle|^2 \left[1 + k + t\langle P'(y)x,x\rangle + \langle \circ(1), y + t\langle Py,x\rangle x\rangle\right].$$

Further, it is obvious that the sign of the right hand side depends on t because, for sufficiently small $t \in \mathbf{R}$, the expression in square brackets is a positive real number. It implies that $\langle Py, x\rangle = 0$ for any $x \in X$, so that $Py = 0$.

Hence $\lambda = m$ must be an eigenvalue of the couple (S, T).
The proof that $\lambda = M$ is also an eigenvalue is analogous.
The assertion (3) follows immediately from (1) and (2). ∎

Corollary 3.8. *Let X be a Hilbert space and let $S, T : X \to X$ be continuous homogeneous symmetric polynomial operators. Suppose T is positive. Then it holds:*

(1) The approximative spectrum $\sigma(S,T)$ of the couple (S,T) consists of real numbers only which lie in the interval $[m, M]$, where

$$m = \inf_{x \in S_1(0)} \frac{\langle Sx, x\rangle}{\langle Tx, x\rangle}, \qquad M = \sup_{x \in S_1(0)} \frac{\langle Sx, x\rangle}{\langle Tx, x\rangle}.$$

Moreover, both m and M belong to $\sigma(S,T)$;

(2) $\overline{W(S,T)} = [m, M]$;

(3) $r_{\sigma(S,T)} = r_{W(S,T)}$.

P r o o f. The assertion (1) follows from Theorem 3.7. To prove (2) we mention that, according to Proposition 3.2, the set $W(S,T)$ is convex. Due to Theorem 3.7 it holds $m, M \in \sigma(S,T)$. Because Theorem 3.13 in [11] implies that $\sigma(S,T) \subset \overline{W(S,T)}$ we obtain $\overline{W(S,T)} = \langle m, M\rangle$. The assertion (3) is the same as in Theorem 3.7. The proof is finished. ∎

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