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Isomorphism of Commutative Semisimple Group Algebras

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Let KG be a group algebra of the abelian group G over a field K of characteristic distinct from the orders of elements in the torsion part G_t of G (if G_t is finite, char K not divide the cardinality $|G_t|$).

In the present paper a complete system of invariants for KG is found when (*) G_t is finite; (**) G_t is an algebraic compact p-group and K is a first kind field with respect to the prime p; (***) G is a direct sum of cyclics, G_t is p-torsion and K is a first kind field with respect to p.

1. Notations and known facts

Throughout the work we assume that G is an abelian group with a subgroup of elements of infinite p-heights $G^1 = G^{p^\omega} = \bigcap_{n=1}^{\infty} G^{p^n}$ (i.e. a first p-Ulm subgroup – a first Ulm subgroup with respect to p) and K is a field whose characteristics does not divide the orders of the elements of the torsion subgroup G_t in G or the cardinality $|G_t|$, if G_t is finite. Thus by the classical results of Passman and Maschke (see [P] and [H]), KG or KG_t are semisimple group algebras, i.e. they have a trivialy Jacobson radical.

Now, we discuss the isomorphism problem for semisimple commutative group algebras over a field. The principal known facts, which may be found in [M2], [BB], [BR] or [BM2]-[NM], respectively, are these:

The K-algebra KG possesses a complete set of invariants, calculated in the terms of K and G, in the following cases:

1) K is algebraically closed (May).

2) $K = \mathbb{R}$ is a field of real numbers (Berman–Bogdan).

3) G is countable torsion with a finite number primary components (Berman–Rossa).

4) G is infinite p-torsion such that G/G^1 is a direct sum of cyclic groups and K is a field of a first kind with respect to p (Berman–Mollov–Nachev).

5) G_t is p-torsion, K is a field of a second kind with respect to p (Berman–Mollov).

In this research, necessary and sufficient conditions are given, for an isomorphism of the algebras KG and KH in the cases, when

(*) G_t is finite

(**) G_t is an infinite algebraically compact p-group, H_t is p-primary and K is a first kind field with respect to p.

(***) G is an infinite direct sum of cyclic groups, G_t and H_t are both p-primary and K is a first kind field with respect to p.

We can state the following definitions (cf. [BM2] or [P], p.684).

Definition. Suppose ε_n a primitive root of degree p^n of the unit in some extension of the field K. The field K is called a first kind field with respect to p if $K(\varepsilon_n) \neq K(\varepsilon_2)$ for some $n > 2$, where char$K \neq p$. Otherwise, K is called a second kind field with respect to p.

Definition. When char$K \neq p$, the number set $s_p(K) = \{i \in \mathbb{N} \cup \{0\} \mid K(\varepsilon_i) \neq K(\varepsilon_{i+1})\}$ is said to be a spectrum of K with respect to p. If K is algebraic closed, then $s_p(K)$ is an empty set.

Now we will prove the following

I. First main result

Theorem. (INVARIANTS) *Let G be a group for which G_t is finite. Then $KH \cong KG$ as K-algebras for any group H if and only if*

(1) H is abelian

(2) $|H_t| = |G_t|$

(3) $|(H_t)^{q^i}| = |(G_t)^{q^i}|$, for each $q \neq$ char$K$ and $i \in S_q(K)$

(4) $H/H_t \cong G/G_t$.

Proof. *Necessity*: The algebra $KG \cong KH$ is abelian, hence H is abelian. Moreover $KG \cong KH$ implies $KG_t \cong KH_t$ (see [M2,M3] or [BB,BM1]). Furthermore, using [Mol], the last isomorphism is equivalent to the equalities (2) and (3). Besides [M1], $H/H_t \cong G/G_t$.

Sufficiency: The relations (2) and (3) imply $KG_t \cong KH_t$ (see [Mol]), and that H_t is finite. So, G_t and H_t are both bounded and pure, respective in G and H, that is why $G \cong G_t \times G/G_t$ and by symmetry $H \cong H_t \times H/H_t$ (see [F], p.140, Theorem 27.5). Therefore $KG \cong KG_t \otimes_K K(G/G_t)$ and

$KH \cong KH_t \otimes_K K(H/H_t)$. By application of (4), $K(G/G_t) \cong K(H/H_t)$. Thus, $KG \cong KH$ as desired. The proof is complete. ■

II. Second main result

Recall $S(KG)$ and $U_p(KG)$ the normalized and normal unit p-components in KG. Denote by $U_p(K)$ the multiplicative p-component in K. It is well-known that $U_p(KG) = U_p(K) \times S(KG)$.

We begin with the following simple lemma.

Group Lemma. *Let A be an abelian p-group. Then A is algebraically compact if and only if A/A^1 is bounded.*

Proof. Suppose A algebraic compact. Consequently $A \cong A^1 \times A/A^1$ ([F], p.191, Exercise 7), where A^1 is a maximal divisible subgroup of A and A/A^1 is isomorphic to the its reduced part, hence A/A^1 is bounded ([F], p.199, Corollary 40.3).

Conversely, we assume that A/A^1 is bounded, i.e. $A^{p^m} \subseteq A^1$ for any natural m. Further $A^{p^m} = A^{p^{m+1}}$, i.e. $(A_r)^{p^m} = (A_r)^{p^{m+1}}$, where A_r is a reduced part of A. Thus $(A_r)^{p^m} = 1$, say A_r is bounded. Finally A is algebraic compact ([F]). The proof is verified. ■

In the articles [D1] and [D2] it is proved that when A is p-torsion and K is a first kind field with respect to p, $S(KA)$ and $U_p(KA)$ are algebraic compact if and only if A/A^1 is bounded. Hence by virtue of the Group Lemma, the following proposition holds.

Proposition. *Let A be an infinite abelian p-group and K be a field of a first kind with respect to p. The groups $S(KA)$ and $U_p(KA)$ are algebraic compact if and only if A is.*

Definition . As usually, $Z(p^\infty)$ denotes the quasicyclic group with respect to p (see [F]).

Now, we can formulate the central result in this section.

Theorem. (INVARIANTS) *Let G be a group for which G_t is an infinite algebraic compact p-primary group and K is a first kind field with respect to p. Then $KH \cong KG$ as K-algebras for any group H for which H_t is a p-group if and only if*

(1) H is abelian

(2) H_t is algebraic compact

(3) $|(H_t/(H_t)^1)^{p^i}| = |(G_t/(G_t)^1)^{p^i}|$, if $i \in s_p(K) \cup \{0\}$ and either $|(G_t/(G_t)^1)^{p^i}| \geq \aleph_0$ or $(G_t/(G_t)^1)^{p^i} = 1$

(4) $(K(\varepsilon_i) : K) = (K(\varepsilon_j) : K)$, if $i, j \in s_p(K) \cup \{0\}$ and $1 < |(G_t/(G_t)^1)^{p^i}| < \aleph_0$ and $1 < |(H_t/(H_t)^1)^{p^j}| < \aleph_0$

(5) $|(G_t)^1| = |(H_t)^1|$, *if either* $(G_t)^1$ *or* $(H_t)^1$ *is infinite or trivial.*

(6) $|G_t/(G_t)^1| < \aleph_0$ *and* $(G_t)^1 \cong Z(p^\infty)$ *if and only if* $|H_t/(H_t)^1| < \aleph_0$ *and* $(H_t)^1 \cong Z(p^\infty)$

(7) $H/H_t \cong G/G_t$.

Proof. *Necessity*: The algebra KH is commutative, therefore H is. As the above theorem, $KG \cong KH$ does imply $KG_t \cong KH_t$, hence $U_p(KG_t) \cong U_p(KH_t)$. Using the proposition, we conclude H_t is algebraic compact, i.e. $H_t/(H_t)^1$ is bounded. Further it is not difficult to see that (3), (4), (5) and (6) hold, by virtue of [NM] and [BM2]. Moreover from [M1], $G/G_t \cong H/H_t$.

Sufficiency: The groups G_t and H_t are both algebraic compact, whence it is clear that $G \cong G_t \times G/G_t$ and similarly $H \cong H_t \times H/H_t$ (cf. [F]). So, we get $KG \cong KG_t \otimes_K K(G/G_t)$ and $KH \cong KH_t \otimes_K K(H/H_t)$. The conditions (3), (4), (5) and (6) yield $KG_t \cong KH_t$ (cf. [NM] and [BM2]), and $K(G/G_t) \cong K(H/H_t)$ follows by application of (7). As a final, we establishe $KG \cong KH$, by the tensor isomorphism. The proof is over. ■

III. Third main result

Theorem. (INVARIANTS) *Let* G *be a direct sum of cyclic groups for which* G_t *is p-primary and* K *is a first kind field with respect to* p. *Then* $KH \cong KG$ *as* K*-algebras for any group* H *for which* H_t *is a* p*-group if and only if*

(1) H *is abelian*

(2) H_t *is a direct sum of cyclics*

(3) $|(H_t)^{p^i}| = |(G_t)^{p^i}|$, $i \in s_p(K) \cup \{0\}$, *if either* $|(G_t)^{p^i}| \geq \aleph_0$ *or* $(G_t)^{p^i} = 1$

(4) $(K(\varepsilon_i) : K) = (K(\varepsilon_j) : K)$, $i, j \in s_p(K) \cup \{0\}$, *if* $1 < |(G_t)^{p^i}| < \aleph_0$ *and* $1 < |(H_t)^{p^i}| < \aleph_0$

(5) $H/H_t \cong G/G_t$.

Proof. *Necessity*: Elementary, we have that $G_t \subseteq G$ is a direct sum of cyclics. As we see, $KG \cong KH$ implies $KG_t \cong KH_t$. Thus in view of [NM], the relations (3), (4) are possible and (2) is valid. Moreover, from [M1], $G/G_t \cong H/H_t$.

Sufficiency: Because G is a direct sum of cyclics, then $G/G_t \cong H/H_t$ are identity. So, G and H splits [F], i.e. $G \cong G_t \times G/G_t$ and $H \cong H_t \times H/H_t$. That is why $KG \cong KG_t \otimes_K K(G/G_t)$ and analogous $KH \cong KH_t \otimes_K K(H/H_t)$. The conditions (2), (3), (4) imply $KG_t \cong KH_t$ by [NM], and (5) implies $K(G/G_t) \cong K(H/H_t)$. Finally by the tensor isomorphism, $KG \cong KH$ as stated. The proof is over. ■

2. Concluding discussion

In this note we investigate the isomorphism problem for commutative semisimple group algebras of mixed and p-mixed (i.e. whose torsion part is p-primary) groups. Fully invariants of a such group algebra KG are was computed in the terms of G and K. An important role in the above proofs and conclussions is the fact that G is a splitting group. Thus the study of KG can be reduced to the study of KG_t.

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