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An Orthogonal Tangential Group of Transformations of a 4-Dimensional Riemannian Manifold [1]

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In this paper we find the orthogonal group of transformations in the tangent space of a 4-dimensional Riemannian manifold (M, g) which keep its system of curvature conditions (Γ) invariant.

1. Introduction

Let (M, g) be a 4-dimensional Riemannian manifold with curvature tensor R and E^2 be a 2-dimensional subspace of the tangent space M_p at a point $p \in M$. We call the orthonormal base e_1, e_2, e_3, e_4 of M_p a *special base*, if e_1 is an arbitrary vector and e_1, e_2, e_3, e_4 are eigen vectors of the Jacobi operator R_{e_1}.

In our papers [1] and [2] we have investigated some problems about the following skew-symmetric curvature operator

$$\kappa_{E^2}(u) = R(X, Y)u, \ u \in M_p,$$

where X, Y is an orthonormal base of E^2. The characteristical equation of κ_{E^2} is

$$\lambda^4 + J_2\lambda^2 + J_4 = 0,$$

where $J_2(p; E^2)$ and $J_4(p; E^2)$ are the nonzero characteristical coefficients. We have proved that the non-flat Einstein non-real space forms with the property

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$J_4(p; E^2) = 0$, for any E^2 at a point $p \in M$, can be characterized by the following curvature conditions:

$$(1) \qquad (\Gamma) \quad \begin{cases} R_{1213} = R_{1214} = R_{1314} = 0 \\ R_{1223} = R_{1224} = 0 \\ R_{1234} = R_{1423} = R_{1324} = 0 \\ K_{13} + K_{23} = 0, \ K_{13} \neq 0 \\ K_{12}^2 = K_{13}^2 + R_{1323}^2, \ R_{1323} \neq 0 \\ K_{12} \text{ is a global constant.} \end{cases}$$

Our purpose is to find out all of the orthogonal transformations which keep the conditions (Γ) invariant. Moreover, we shall investigate the changing of the functions $K_{13}(p; E^2)$ and $R_{1323}(p; E^2)$ under the acting of these transformations from a special base e_1, e_2, e_3, e_4 to another special base $\overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{e_4}$.

2. Preliminary considerations

It can be proved immediately that the following transformations (we shall call them $\varphi-$ and $\psi-$ transformations, respectively):

$$(2) \qquad (\varphi) \quad \left| \begin{array}{l} \overline{e_1} = \cos\varphi \, e_1 - \sin\varphi \, e_2 \\ \overline{e_2} = \sin\varphi \, e_1 + \cos\varphi \, e_2 \\ \overline{e_3} = e_3 \\ \overline{e_4} = e_4 \qquad \varphi \in [0, 2\pi] \end{array} \right.$$

$$(3) \qquad (\psi) \quad \left| \begin{array}{l} \overline{e_1} = \cos\psi \, e_1 - \sin\psi \, e_2 \\ \overline{e_2} = \sin\psi e_1 + \cos\psi \, e_2 \\ \overline{e_3} = e_4 \\ \overline{e_4} = e_3 \qquad \varphi \in [0, 2\pi] \end{array} \right.$$

keep the conditions (Γ) invariant. Besides, the transformations (φ) and (ψ) change the functions $K_{13}(p; E^2)$ and $R_{1323}(p; E^2)$ in the following way:

a) by φ-transformations:

$$(4) \qquad \left| \begin{array}{l} \overline{K_{13}} = \cos 2\varphi \, K_{13} - \sin 2\varphi \, R_{1323} \\ \overline{R_{1323}} = \sin 2\varphi \, K_{13} + \cos 2\varphi \, R_{1323}, \end{array} \right.$$

b) by ψ-transformations:

$$(5) \qquad \left| \begin{array}{l} \overline{K_{13}} = -\cos 2\psi \, K_{13} + \sin 2\psi \, R_{1323} \\ \overline{R_{1323}} = -\sin 2\psi \, K_{13} - \cos 2\psi \, R_{1323}. \end{array} \right.$$

We shall use the following lemmas:

Lemma 1. *The functions $K_{13}(p; E^2)$ and $R_{1323}(p; E^2)$ are linearly independent.*

Proof. Let us presume there exists a function $f(p)$ of the point $p \in M$ so that

$$K_{13} = f(p)\, R_{1323} \tag{6}$$

is true. If we change the sign of (2) we get

$$K_{13} = -f(p)\, R_{1323}.$$

Then it follows $K_{13} = 0$ which is impossible. ■

Lemma 2. *The functions $K_{12}(p; E^2)$, $K_{13}(p; E^2)$ and $R_{1323}(p; E^2)$ are linearly independent.*

Proof. Let us presume there exist the functions $\lambda_1(p)$, $\lambda_2(p)$ and $\lambda_3(p)$ so that

$$\lambda_1 K_{12} + \lambda_2 K_{13} + \lambda_3 R_{1323} = 0.$$

Then we have consequently

$$\begin{aligned} &\lambda_1 K_{12} = -(\lambda_2 K_{13} + \lambda_3 R_{1323}); \\ &\lambda_1^2(K_{13}^2 + R_{1323}^2) = \lambda_2^2 K_{13}^2 + \lambda_3^2 R_{1323}^2 + 2\lambda_2\lambda_3 K_{13} R_{1323}; \\ &(\lambda_2^2 - \lambda_1^2)K_{13} + (\lambda_3^2 - \lambda_1^2)R_{1323}^2 + 2\lambda_2\lambda_3 K_{13} R_{1323} = 0. \end{aligned} \tag{7}$$

From (7) we obtain $\lambda_2^2 - \lambda_1^2 = 0$; $\lambda_3^2 - \lambda_1^2 = 0$; $\lambda_2\lambda_3 = 0$ as the other case leads to a contradiction with Lemma 1.

Hence we get $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and Lemma 2 is proved. ■

Let us consider an arbitrary orthogonal transformation of M_p:

$$\left|\begin{aligned} \overline{e_1} &= \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 \\ \overline{e_2} &= \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3 + \beta_4 e_4 \\ \overline{e_3} &= \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3 + \gamma_4 e_4 \\ \overline{e_4} &= \delta_1 e_1 + \delta_2 e_2 + \delta_3 e_3 + \delta_4 e_4 \end{aligned}\right. \tag{8}$$

with det $= \varepsilon$, $\varepsilon = \pm 1$.

Remark 1. The character of our further geometrical investigations allows us to confine the consideration to the case $\varepsilon = 1$ only.

The matrix

$$(9) \qquad \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{pmatrix}$$

of the transofrmation (8) is an orthogonal matrix and then the following equations hold:

$$(10) \qquad \sum_{i=1}^{4} \alpha_i^2 = \sum_{i=1}^{4} \beta_i^2 = \sum_{i=1}^{4} \gamma_i^2 = \sum_{i=1}^{4} \delta_i^2 = 1.$$

$$(11) \qquad \sum_{i=1}^{4} \alpha_i\beta_i = \sum_{i=1}^{4} \alpha_i\gamma_i = \sum_{i=1}^{4} \alpha_i\delta_i = \sum_{i=1}^{4} \beta_i\gamma_i = \sum_{i=1}^{4} \beta_i\delta_i = \sum_{i=1}^{4} \gamma_i\delta_i = 0.$$

Moreover, all of the symbols $(\alpha\beta)_{ij}$, $(\alpha\gamma)_{ij}$, $(\alpha\delta)_{ij}$, $(\beta\gamma)_{ij}$, $(\beta\delta)_{ij}$, $(\gamma\delta)_{ij}$, $i < j$, satisfy the well known Plücker's equation, which for $(\alpha\beta)_{ij}$ has the form

$$(12) \qquad (\alpha\beta)_{12}(\alpha\beta)_{34} + (\alpha\beta)_{23}(\alpha\beta)_{14} + (\alpha\beta)_{31}(\alpha\beta)_{24} = 0,$$

where for convenience we have used the notations

$$\begin{array}{ll} (\alpha\beta)_{ij} = \alpha_i\beta_j - \alpha_j\beta_i & (\beta\gamma)_{ij} = \beta_i\gamma_j - \beta_j\gamma_i \\ (\alpha\gamma)_{ij} = \alpha_i\gamma_j - \alpha_j\gamma_i & (\beta\delta)_{ij} = \beta_i\delta_j - \beta_j\delta_i \\ (\alpha\delta)_{ij} = \alpha_i\delta_j - \alpha_j\delta_i & (\gamma\delta)_{ij} = \gamma_i\delta_j - \gamma_j\delta_i. \end{array}$$

Using (10) and (11) it can be proved directly

$$(13) \qquad \sum_{1\le i<j\le 4} (\alpha\beta)_{ij}^2 = 1.$$

Analogously,

$$(14) \qquad \begin{aligned} \sum_{1\le i<j\le 4} (\alpha\gamma)_{ij}^2 &= \sum_{1\le i<j\le 4} (\alpha\delta)_{ij}^2 = \sum_{1\le i<j\le 4} (\beta\gamma)_{ij}^2 \\ &= \sum_{1\le i<j\le 4} (\beta\delta)_{ij}^2 = \sum_{1\le i<j\le 4} (\gamma\delta)_{ij}^2 = 1. \end{aligned}$$

3. Orthogonal transformations keeping (Γ) invariant

Let e_1, e_2, e_3, e_4 and $\overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{e_4}$ be both special bases in M_p. Let the arbitrary orthogonal transformation (8) leave the conditions (Γ) invariant, i.e. the relations

$$(15)\quad \left\{ \begin{array}{l} \overline{R_{1213}} = \overline{R_{1214}} = \overline{R_{1314}} = 0 \\ \overline{R_{1223}} = \overline{R_{1224}} = 0 \\ \overline{R_{1234}} = \overline{R_{1423}} = \overline{R_{1324}} = 0 \\ \overline{K_{13}} + \overline{K_{23}} = 0 \\ \overline{K_{12}}^2 = \overline{K_{13}}^2 + \overline{R_{1323}}^2 \\ \overline{K_{12}} = K_{12} \end{array} \right.$$

be satisfied. Having in mind the well-known properties of the curvature tensor R and the systems (8) and (15), we obtain:

$$\begin{aligned} O &= \overline{R_{1213}} = R(\overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{e_4}) = R\Big(\sum_{i=1}^{4} \alpha_i e_i, \sum_{j=1}^{4} \beta_j e_j, \sum_{k=1}^{4} \gamma_k e_k, \sum_{l=1}^{4} \delta_l e_l\Big) \\ &= \lambda_1 K_{12} + \lambda_2 K_{13} + \lambda_3 R_{1323}, \end{aligned}$$

where

$$\left| \begin{array}{l} \lambda_1 = (\alpha\beta)_{12}(\alpha\gamma)_{12} + (\alpha\beta)_{34}(\alpha\gamma)_{34} \\ \lambda_2 = (\alpha\beta)_{13}(\alpha\gamma)_{13} + (\alpha\beta)_{24}(\alpha\gamma)_{24} - (\alpha\beta)_{23}(\alpha\gamma)_{23} - (\alpha\beta)_{14}(\alpha\gamma)_{14} \\ \lambda_3 = (\alpha\beta)_{13}(\alpha\gamma)_{23} + (\alpha\beta)_{23}(\alpha\gamma)_{13} - (\alpha\beta)_{14}(\alpha\gamma)_{24} - (\alpha\beta)_{24}(\alpha\gamma)_{14}. \end{array} \right.$$

According to Lemma 2, $\lambda_1 = \lambda_2 = \lambda_3 = 0$, i.e. the system

$$(16)\quad \left| \begin{array}{l} (\alpha\beta)_{12}(\alpha\gamma)_{12} + (\alpha\beta)_{34}(\alpha\gamma)_{34} = 0 \\ (\alpha\beta)_{13}(\alpha\gamma)_{13} + (\alpha\beta)_{24}(\alpha\gamma)_{24} - (\alpha\beta)_{23}(\alpha\gamma)_{23} - (\alpha\beta)_{14}(\alpha\gamma)_{14} = 0 \\ (\alpha\beta)_{13}(\alpha\gamma)_{23} + (\alpha\beta)_{23}(\alpha\gamma)_{13} - (\alpha\beta)_{14}(\alpha\gamma)_{24} - (\alpha\beta)_{24}(\alpha\gamma)_{14} = 0 \end{array} \right.$$

holds. Analogously, for any $\overline{R_{ijkl}} = 0$ in (15), according to Lemma 2, we obtain the systems:

$$(17)\quad \left| \begin{array}{l} (\alpha\beta)_{12}(\alpha\delta)_{12} + (\alpha\beta)_{34}(\alpha\delta)_{34} = 0 \\ (\alpha\beta)_{13}(\alpha\delta)_{13} + (\alpha\beta)_{24}(\alpha\delta)_{24} - (\alpha\beta)_{23}(\alpha\delta)_{23} - (\alpha\beta)_{14}(\alpha\delta)_{14} = 0 \\ (\alpha\beta)_{13}(\alpha\delta)_{23} + (\alpha\beta)_{23}(\alpha\delta)_{13} - (\alpha\beta)_{14}(\alpha\delta)_{24} - (\alpha\beta)_{24}(\alpha\delta)_{14} = 0, \end{array} \right.$$

$$(18)\quad \left| \begin{array}{l} (\alpha\gamma)_{12}(\alpha\delta)_{12} + (\alpha\gamma)_{34}(\alpha\delta)_{34} = 0 \\ (\alpha\gamma)_{13}(\alpha\delta)_{13} + (\alpha\gamma)_{24}(\alpha\delta)_{24} - (\alpha\gamma)_{23}(\alpha\delta)_{23} - (\alpha\gamma)_{14}(\alpha\delta)_{14} = 0 \\ (\alpha\gamma)_{13}(\alpha\delta)_{23} + (\alpha\gamma)_{23}(\alpha\delta)_{13} - (\alpha\gamma)_{14}(\alpha\delta)_{24} - (\alpha\gamma)_{24}(\alpha\delta)_{14} = 0, \end{array} \right.$$

$$(19)\quad \begin{vmatrix} (\alpha\beta)_{12}(\beta\gamma)_{12} + (\alpha\beta)_{34}(\beta\gamma)_{34} = 0 \\ (\alpha\beta)_{13}(\beta\gamma)_{13} + (\alpha\beta)_{24}(\beta\gamma)_{24} - (\alpha\beta)_{23}(\beta\gamma)_{23} - (\alpha\beta)_{14}(\beta\gamma)_{14} = 0 \\ (\alpha\beta)_{13}(\beta\gamma)_{23} + (\alpha\beta)_{23}(\beta\gamma)_{13} - (\alpha\beta)_{14}(\beta\gamma)_{24} - (\alpha\beta)_{24}(\beta\gamma)_{14} = 0, \end{vmatrix}$$

$$(20)\quad \begin{vmatrix} (\alpha\beta)_{12}(\beta\delta)_{12} + (\alpha\beta)_{34}(\beta\delta)_{34} = 0 \\ (\alpha\beta)_{13}(\beta\delta)_{13} + (\alpha\beta)_{24}(\beta\delta)_{24} - (\alpha\beta)_{23}(\beta\delta)_{23} - (\alpha\beta)_{14}(\beta\delta)_{14} = 0 \\ (\alpha\beta)_{13}(\beta\delta)_{23} + (\alpha\beta)_{23}(\beta\delta)_{13} - (\alpha\beta)_{14}(\beta\delta)_{24} - (\alpha\beta)_{24}(\beta\delta)_{14} = 0, \end{vmatrix}$$

$$(21)\quad \begin{vmatrix} (\alpha\beta)_{12}(\gamma\delta)_{12} + (\alpha\beta)_{34}(\gamma\delta)_{34} = 0 \\ (\alpha\beta)_{13}(\gamma\delta)_{13} + (\alpha\beta)_{24}(\gamma\delta)_{24} - (\alpha\beta)_{23}(\gamma\delta)_{23} - (\alpha\beta)_{14}(\gamma\delta)_{14} = 0 \\ (\alpha\beta)_{13}(\gamma\delta)_{23} + (\alpha\beta)_{23}(\delta\gamma)_{13} - (\alpha\beta)_{14}(\gamma\delta)_{24} - (\alpha\beta)_{24}(\gamma\delta)_{14} = 0, \end{vmatrix}$$

$$(22)\quad \begin{vmatrix} (\alpha\delta)_{12}(\beta\gamma)_{12} + (\alpha\delta)_{34}(\beta\gamma)_{34} = 0 \\ (\alpha\delta)_{13}(\beta\gamma)_{13} + (\alpha\delta)_{24}(\beta\gamma)_{24} - (\alpha\delta)_{23}(\beta\gamma)_{23} - (\alpha\delta)_{14}(\beta\gamma)_{14} = 0 \\ (\alpha\delta)_{13}(\beta\gamma)_{23} + (\alpha\delta)_{23}(\beta\gamma)_{13} - (\alpha\delta)_{14}(\beta\gamma)_{24} - (\alpha\delta)_{24}(\beta\gamma)_{14} = 0. \end{vmatrix}$$

From the last equation in (15) we get

$$\begin{aligned} K_{12} = \overline{K_{12}} &= R(\overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{e_4}) = R(\sum_{i=1}^{4} \alpha_i e_i; \sum_{j=1}^{4} \beta_j e_j, \sum_{k=1}^{4} \gamma_k e_k, \sum_{l=1}^{4} \delta_l e_l) \\ &= \mu_1 K_{12} + \mu_2 K_{13} + \mu_3 R_{1323}, \end{aligned}$$

where

$$\begin{vmatrix} \mu_1 = (\alpha\beta)_{12}^2 + (\alpha\beta)_{34}^2 \\ \mu_2 = (\alpha\beta)_{13}^2 + (\alpha\beta)_{24}^2 - (\alpha\beta)_{23}^2 - (\alpha\beta)_{14}^2 \\ \mu_3 = 2[(\alpha\beta)_{13}(\alpha\beta)_{23} - (\alpha\beta)_{14}(\alpha\beta)_{24}]. \end{vmatrix}$$

Thus we have $\mu_1 = 1$, $\mu_2 = 0$, $\mu_3 = 0$ which are equivalent to the system

$$(23)\quad \begin{vmatrix} (\alpha\beta)_{12}^2 + (\alpha\beta)_{34}^2 = 1 \\ (\alpha\beta)_{13}^2 + (\alpha\beta)_{24}^2 - (\alpha\beta)_{23}^2 - (\alpha\beta)_{14}^2 = 0 \\ (\alpha\beta)_{13}(\alpha\beta)_{23} - (\alpha\beta)_{14}(\alpha\beta)_{24} = 0. \end{vmatrix}$$

Using (23) and the first equation in (16), we get

$$(\alpha\beta)_{13}^2 + (\alpha\beta)_{14}^2 + (\alpha\beta)_{23}^2 + (\alpha\beta)_{24}^2 = 0$$

and

$$(24)\quad (\alpha\beta)_{13} = (\alpha\beta)_{14} = (\alpha\beta)_{23} = (\alpha\beta)_{24} = 0.$$

These equations are equivalent to the system:

$$(25)\qquad \left|\begin{array}{l}\alpha_1\beta_3=\alpha_3\beta_1\\ \alpha_1\beta_4=\alpha_4\beta_1\end{array}\right. \qquad \left|\begin{array}{l}\alpha_2\beta_3=\alpha_3\beta_2\\ \alpha_2\beta_4=\alpha_4\beta_2.\end{array}\right.$$

From the last two equations in (25), it follows

$$\begin{array}{ll}\alpha_3=\rho\,\alpha_2 & \beta_3=\rho\,\beta_2\\ \beta_4=\rho'\,\alpha_2 & \beta_4=\rho'\,\beta_2.\end{array}$$

Then we obtain

$$(26)\qquad (\alpha\beta)_{34}=\alpha_3\beta_4-\alpha_4\beta_3=\rho\rho'(\alpha_2\beta_2-\alpha_2\beta_2)=0$$

and using the first equation in (23) we get $(\alpha\beta)^2_{12}=1$, i.e. $(\alpha\beta)_{12}=\varepsilon$, $\varepsilon=\pm1$.

According to Remark 1 we consider only the case $\varepsilon=1$, i.e.

$$(27)\qquad (\alpha\beta)_{12}=1.$$

That means at least one of α_1 and α_2 is not zero.

Let $\alpha_1\neq 0$ (the case $\alpha_2\neq 0$ is analogous). From the first and the third equations in (25) we get

$$(28)\qquad \beta_3=\frac{\alpha_3\beta_1}{\alpha_1},$$

$$(29)\qquad \alpha_3(\alpha_2\beta_1-\alpha_1\beta_2)=0.$$

Now from (27), (28), (29) it follows

$$(30)\qquad \alpha_3=\beta_3=0.$$

Analogously from the second and fourth equations in (25) we obtain

$$(31)\qquad \alpha_4=\beta_4=0.$$

Putting (26) and (27) in the first equations of the systems (16), (17), (19), (20) and (21) we get the system

$$(32)\qquad \left|\begin{array}{l}\alpha_1\gamma_2=\alpha_2\gamma_1\\ \alpha_1\delta_2=\alpha_2\delta_1\\ \beta_1\gamma_2=\beta_2\gamma_1\\ \beta_1\delta_2=\beta_2\delta_1\\ \gamma_1\delta_2=\gamma_2\delta_1.\end{array}\right.$$

Again from the first and third equations in (32) we can express

$$\gamma_2 = \frac{\alpha_2\gamma_1}{\alpha_1} \tag{33}$$

$$\gamma_1(\alpha_1\beta_2 - \alpha_2\beta_1) = 0. \tag{34}$$

Then (27), (33) and (34) give us

$$\gamma_1 = \gamma_2 = 0. \tag{35}$$

Analogously from the second and the fourth equations in (32) we get

$$\delta_1 = \delta_2 = 0. \tag{36}$$

According to (30), (31), (35) and (36) the matrix of the transformation (8) is

$$\begin{pmatrix} \alpha_1 & \alpha_2 & 0 & 0 \\ \beta_1 & \beta_2 & 0 & 0 \\ 0 & 0 & \gamma_3 & \gamma_4 \\ 0 & 0 & \delta_3 & \delta_4 \end{pmatrix}. \tag{37}$$

Thus we have the relations:

$$\begin{array}{ll} (\alpha\gamma)_{13} = \alpha_1\gamma_3 & (\alpha\delta)_{13} = \alpha_1\delta_3 \\ (\alpha\gamma)_{14} = \alpha_1\gamma_4 & (\alpha\delta)_{14} = \alpha_1\delta_4 \\ (\alpha\gamma)_{23} = \alpha_2\gamma_3 & (\alpha\delta)_{23} = \alpha_2\delta_3 \\ (\alpha\gamma)_{24} = \alpha_2\gamma_4 & (\alpha\delta)_{24} = \alpha_2\delta_4. \end{array} \tag{38}$$

Putting (38) in the second and third equations of (18) we get respectively:

$$(\alpha_1^2 - \alpha_2^2)(\gamma_3\delta_3 - \gamma_4\delta_4) = 0 \tag{39}$$

$$2\alpha_1\alpha_2(\gamma_3\delta_3 - \gamma_4\delta_4) = 0. \tag{40}$$

Let us presume

$$\gamma_3\delta_3 - \gamma_4\delta_4 \neq 0. \tag{41}$$

Then from (39) and (40) it follows

$$\alpha_1^2 - \alpha_2^2 = 0, \quad \alpha_1\alpha_2 = 0$$

and hence we obtain $\alpha_1 = \alpha_2 = 0$, which because of (2) is not possible. It means that (41) is not true, i.e.

(42) $$\gamma_3\delta_3 - \gamma_4\delta_4 = 0.$$

The matrix (37) is orthogonal one and from the last equation of (11) it follows

(43) $$\gamma_3\delta_3 + \gamma_4\delta_4 = 0.$$

Then (42) and (43) give us

(44) $$\gamma_3\delta_3 = 0$$

(45) $$\gamma_4\delta_4 = 0.$$

From (10), (35) and (36) we get

(46) $$\gamma_3^2 + \gamma_4^2 = 1$$

(47) $$\delta_3^2 + \delta_4^2 = 1.$$

For γ_3 and δ_3, from (44) the three possibilities follow:

a) $\gamma_3 = 0,\ \delta_3 \neq 0$;
b) $\gamma_3 \neq 0,\ \delta_3 = 0$;
c) $\gamma_3 = \delta_3 = 0$,

and for γ_4 and δ_4, from (45) it follows:

d) $\gamma_4 = 0,\ \delta_4 \neq 0$
e) $\gamma_4 \neq 0,\ \delta_4 = 0$
f) $\gamma_4 = \delta_4 = 0$.

Having in mind (46) and (47), from all of the cases ad), ae), af), bd), be), bf), cd), ce), cf), the only possible are:

1) the case ae): $\gamma_3 = 0,\ \gamma_4 \neq 0,\ \delta_3 \neq,\ \delta_4 = 0$.

From (46) and (47) it follows

$$\gamma_4^2 = 1, \quad \delta_3^2 = 1$$

and the matrix of the transformation has the form:

(48) $$\begin{pmatrix} \alpha_1 & \alpha_2 & 0 & 0 \\ \beta_1 & \beta_2 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon' \\ 0 & 0 & \varepsilon'' & 0 \end{pmatrix},$$

$$\alpha_1\beta_2 - \alpha_2\beta_1 = \varepsilon,$$
$$\varepsilon' = \pm 1, \ \varepsilon'' = \pm 1.$$

2) the case bd): $\gamma_3 \neq 0, \gamma_4 = 0, \delta_3 = 0, \delta_4 \neq 0$.
Again from (46) and (47) it follows

$$\gamma_3^2 = 1, \quad \delta_3^2 = 1$$

and the matrix of the transformation has the form

$$\begin{pmatrix} \alpha_1 & \alpha_2 & 0 & 0 \\ \beta_1 & \beta_2 & 0 & 0 \\ 0 & 0 & \varepsilon' & 0 \\ 0 & 0 & 0 & \varepsilon'' \end{pmatrix}, \tag{49}$$

$$\alpha_1\beta_2 - \alpha_2\beta_1 = \varepsilon,$$
$$\varepsilon' = \pm 1, \ \varepsilon'' = \pm 1.$$

Remark 2. For our geometrical investigations it is enough only to consider the case $\varepsilon = \varepsilon' = \varepsilon'' = 1$.

For α_1, α_2, β_1, β_2 using (10), (11), (30), (31) we obtain the system

$$\left| \begin{array}{l} \alpha_1^2 + \alpha_2^2 = 1 \\ \beta_1^2 + \beta_2^2 = 1 \\ \alpha_1\beta_1 + \alpha_2\beta_2 = 0. \end{array} \right. \tag{50}$$

Then from (27) and (50) we get

$$\begin{array}{lll} \alpha_1 = \rho\cos\varphi & \alpha_2 - \rho\sin\varphi & \\ \beta_1 = \rho\varepsilon\sin\varphi & \beta_2 = \rho\varepsilon\cos\varphi, & \rho = \pm 1, \ \varepsilon = \pm 1. \end{array}$$

Remark 3. Because of the character of our geometrical investigations, we are not interested in the case $\rho = \varepsilon = -1$.

Finally from (48), (49) and (51), we can conclude that the matrix of the orthogonal transformation (8) has either the form

$$\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and hence (8) coincides with the φ-transformations (2), or the form

$$\begin{pmatrix} \cos\psi & -\sin\psi & 0 & 0 \\ \sin\psi & \cos\psi & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and transformations (8) coincides with the ψ-transformations (3).

In such a way we have proved the following

Theorem. *The transformations* (φ) *and* (ψ) *are the only orthogonal transformations which keep the curvature system* (Γ) *invariant and the functions* $K_{13}(p; E^2)$ *and* $R_{1323}(p; E^2)$ *are changed in the following way:*

i) by φ-transformations

$$\left| \begin{array}{l} \overline{K_{13}} = \cos 2\varphi\ K_{13} - \sin 2\varphi\ R_{1323} \\ \overline{R_{1323}} = \sin 2\varphi\ K_{13} + \cos 2\varphi\ R_{1323}, \quad \varphi \in [0, 2\pi]; \end{array} \right.$$

ii) by ψ-transformations

$$\left| \begin{array}{l} \overline{K_{13}} = -\cos 2\psi\ K_{13} + \sin 2\psi\ R_{1323} \\ \overline{R_{1323}} = -\sin 2\psi\ K_{13} - \cos 2\psi\ R_{1323}, \quad \psi \in [0, 2\pi]. \end{array} \right.$$

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