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On Unified Fractional Integration Operators [1]

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We investigate two new fractional integration operators associated with a generalized $\bar{H}$-function, due to A. A. Inayat-Hussain and a general class of polynomials introduced by H. M. Srivastava. Properties of these operators associated with Mellin transforms are investigated, generalizing a number of results on various integration operators of one variable scattered in literature.

1. Introduction

In order to unify and extend the results on various fractional integration operators of one variable, the authors introduce two new fractional integration operators associated with a generalized $\bar{H}$-function due to A. A. Inayat-Hussain [7,p.4126] and a general class of polynomials due to H. M. Srivastava [25,p.1,(1)].

These operators are extensions of the operators of fractional integration defined and studied earlier by many authors notably by A. Erdélyi [3], H. Kober [15], E.R. Love [6], R.K. Saxena [19], R.K. Saxena and R.K. Kumbhat [22]-[24], S.L. Kalla [8], S.L. Kalla and R.K. Saxena [10], S.L. Kalla and V. Kiryakova [11],[12], V. Kiryakova [13],[14], M. Saigo [16], M. Saigo et al. [17],[18], and several others.

Integration operators asoociated with a multivariable H-function due to H.M. Srivastava and R Panda [26],[27,p.251,(C.1)-(C.3)] and a general class of polynomials defined by H.M. Srivastava [25,p.1,(1)] have been recently introduced and studied by R.K. Saxena, V. Kiryakova and O.P. Dave [20] and R.K. Saxena and O.P. Dave [21].

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2. The H-function and a general class of polynomials

The generalized $\bar{H}$-function, introduced by A. A. Inayat-Hussain [7] in terms of Mellin-Barnes type contour integral, is defined by

$$\bar{H}^{m,n}_{p,q}\left[z\left|\begin{matrix}(\alpha_j, A_j; a_j)_{1,n}; (\alpha_k, A_k)_{n+1,p}\\(\beta_j, B_j)_{1,m}; (\beta_k, B_k; b_k)_{m+1,q}\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\Theta(s)z^s ds, \tag{2.1}$$

where

$$\Theta(s) = \frac{\prod_{j=1}^{m}\Gamma(\beta_j - B_j s)\prod_{j=1}^{n}\Gamma(1-\alpha_j + A_j s)^{a_j}}{\prod_{k=m+1}^{q}\Gamma(1-\beta_k + B_k s)^{b_k}\prod_{k=n+1}^{p}\Gamma(\alpha_k - A_k s)} \tag{2.2}$$

which contains fractional powers of some of the Γ-functions. Here z may be real or complex but is not equal to zero and an empty product is interpretted as unity; p, q, m and n are integers such that $1 \le m \le q$, $1 \le n \le p$; $A_j (j = 1, \ldots, p)$, $B_j (j = 1, \ldots, q)$ are complex numbers. The exponents $a_j (j = 1, \ldots, n)$ and $b_k (k = 1, \ldots, q)$ can take noninteger values. When these exponents are integers, the $\bar{H}$-function reduces to the familiar H-function of Fox [4]. Analytical continuation and asymptotic expansions of the H-function have been obtained by B. L. J. Braaksma [1].

Also, due to A. A. Inayat-Hussain [7] we have

$$\bar{H}^{m,n}_{p,q}(z) = O(|z|^g) \quad \text{for small } |z|, \tag{2.3}$$

where

$$g = \min_{1\le j\le m} \Re\,(\beta_j / B_j).$$

Again, from [7],

$$\bar{H}^{m,n}_{p,q}(z) = O\left(|z|^h\right) \quad \text{for large } |z|, \tag{2.4}$$

where

$$h = \max_{1\le j\le n} \Re\,\{a_j(\alpha_j - 1/A_j)\}.$$

The following conditions [2,p.4708] are also satisfied:

$$\Omega = \sum_{j=1}^{m}|B_j| + \sum_{j=1}^{n}|a_j A_j| - \sum_{j=m+1}^{q}|b_j B_j| - \sum_{j=n+1}^{p}|A_j| > 0, \tag{2.5}$$

and

$$|\arg(z)| < \frac{1}{2}\Omega\pi. \tag{2.6}$$

The general class of polynomials, denoted by $S_n^m[x]$, due to Srivastava [25,p.1,(1)] is defined as:

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^k, \quad n = 0, 1, 2, \ldots, \tag{2.7}$$

where m is an arbitrary positive integer and the coefficients $A_{n,k}$, $n, k \geq 0$ are arbitrary real or complex constants.

3. Definitions

We define two new fractional integration operators by means of the following integral representations:

$$\begin{aligned} R[f(x)] &= R_x^{\eta\,:\,(a_j,A_j,\alpha_j)_{1,n};(a_k,A_k)_{n+1,p}\,:a}_{\,:\,(b_j,B_j)_{1,m};(b_k,B_k;\beta_k)_{m+1,q}\,:b}\,[f(x)] \\ &= x^{-\eta-1}\int_0^x t^{\eta}\,\overline{H}_{p,q}^{m,n}\left[\left(\tfrac{t}{x}\right)^{\mu}\left|\begin{matrix}(a_j,A_j;\alpha_j)_{1,n};(a_k,A_k)_{n+1,p}\\(b_j,B_j)_{1,m};(b_k,B_k;\beta_k)_{m+1,q}\end{matrix}\right.\right] \\ &\times S_b^a\left[y\left(\tfrac{t}{x}\right)^{\sigma}\right]\ f(t)\,\theta\left(\tfrac{t}{x}\right)dt, \end{aligned} \tag{3.1}$$

$$\begin{aligned} K[f(x)] &= K_x^{\delta\,:\,(c_j,C_j,\gamma_j)_{1,N};(c_k,C_k)_{N+1,P}\,:e}_{\,:\,(d_j,D_j)_{1,M};(d_k,D_k;\xi_k)_{M+1,Q}\,:f}\,[f(x)] \\ &= x^{\delta}\int_x^{\infty} t^{-\delta-1}\,\overline{H}_{P,Q}^{M,N}\left[\left(\tfrac{x}{t}\right)^{\nu}\left|\begin{matrix}(c_j,C_j;\gamma_j)_{1,N};(c_k,C_k)_{N+1,P}\\(d_j,D_j)_{1,M};(d_k,D_k;\xi_k)_{M+1,Q}\end{matrix}\right.\right] \\ &\times S_f^e\left[y'\left(\tfrac{x}{t}\right)^{\rho}\right]\ f(t)\,\theta\left(\tfrac{x}{t}\right)dt. \end{aligned} \tag{3.2}$$

The kernel $\theta(t/x)$ ocurring in (3.1) and (3.2) is supposed to be a continuous function such that the integrals make sense for wide classes of functions $f(x)$ and μ, σ, ν, ρ are positive numbers.

These operators exist under the folowing conditions:

(i) $p \geq 1, q < \infty,\ p^{-1} + q^{-1} = 1$,

(ii) $\Re\,[\eta + \mu(b_j/B_j)] > -1/q;\ j = 1, \ldots, m$,

(iii) $\Re\,[\delta + \nu(d_j/D_j)] > -1/p;\ j = 1, \ldots, M$.

(iv) $f(x) \in L_p(0, \infty)$.

The condition (iv) ensures that both operators R and K exist and also belong to $L_p(0, \infty)$.

It is interesting to observe that the operators of Kalla and Kiryakova [11],[12], Kiryakova [13],[14] can be derived as special cases of operators (3.1),(3.2).

4. The Mellin transform

The Mellin transform of $f(x)$ will be denoted by $M[f(x)]$ or by $F(s)$. We denote $s = p^{-1} + it$, where p and t are real. If $p \geq 1$, $f(x) \in L_p(0, \infty)$, then for $p = 1$,

$$M[f(x)] = F(s) = \int_0^{\infty} x^{s-1} f(x) dx \tag{4.1}$$

and

$$f(t) = \frac{1}{(2\pi w)^r} \int_{c-w\infty}^{c+w\infty} F(s) x^{-s} ds, \tag{4.2}$$

where $w = (-1)^{1/2}$ under suitable conditions on the parameters and variables.

For $p > 1$,

$$M[F(x)] = F(s) = \text{l.i.m} \int_{1/x}^{x} f(x) x^{s-1} dx, \tag{4.3}$$

where l.i.m denotes the usual limit in the mean of the L_p-spaces.

5. Theorems related to Mellin transforms

Theorem 1. *If $f(x) \in L_p(0, \infty)$, $1 \leq p \leq 2$ (or $f(x) \in M_p(0, \infty)$ with $p > 2$), $p^{-1} + q^{-1} = 1$, $\Re\, [\eta + \mu(b_j/B_j)] > -1/q; j = 1, \ldots, m$ and the integrals involved are absolutely convergent, then the following result holds:*

$$\begin{aligned} &M\left[R_x^{\eta\,:\,(a_j,A_j,\alpha_j)_{1,n};(a_k,A_k)_{n+1,p}\,:a}_{\quad :\,(b_j,B_j)_{1,m};(b_k,B_k;\beta_k)_{m+1,q}\,:b}\,[f(x)]\right] \\ &= M[f(x)]\; K_x^{-s+1\,:\,(a_j,A_j,\alpha_j)_{1,n};(a_k,A_k)_{n+1,p}\,:a}_{\quad :\,(b_j,B_j)_{1,m};(b_k,B_k;\beta_k)_{m+1,q}\,:b}[1], \end{aligned} \tag{5.1}$$

where $M_p(0, \infty)$ denotes the class of all functions $L_p(0, \infty)$ with $p > 2$ which are inverse Mellin transforms of the functions of $L_q(-\infty, \infty)$.

Proof. From (3.1) it follows that

$$\begin{aligned} M\,[R[f(x)]] &= \int_0^{\infty} x^{s-1} x^{-\eta-1} \int_0^{x} t^{\eta} f(t) \\ &\times \overline{H}_{p,q}^{m,n} [(t/x)^{\mu}]\; S_b^a\,[y(t/x)^{\sigma}]\, \theta(t/x) dt dx \end{aligned} \tag{5.2}$$

$$\begin{aligned} &= \int_0^{\infty} t^{\eta} f(t) dt \int_t^{\infty} x^{s-\eta-2} \\ &\times \overline{H}_{p,q}^{m,n} [(t/x)^{\mu}]\; S_b^a\,[y(t/x)^{\sigma}]\, \theta(t/x) dx, \end{aligned} \tag{5.3}$$

on interchanging the order of integration, which is permissible under the conditions stated in the theorem. Then the assertion follows on interpreting the above result with the help of (3.2). ■

In a similar manner, the following theorems can be established.

Theorem 2. *If* $f(x) \in L_p(0,\infty), 1 \le p \le 2$ (*or* $f(x) \in M_p(0,\infty)$ *with* $p > 2$), $p^{-1}+q^{-1}=1, \Re\,[\delta+\nu(d_j/D_j)] > -1/p; j=1,\dots,M$ *and the integrals involved are absolutely convergent, then the following result holds:*

$$M\,[K[f(x)]] = M[f(x)]\; R_{x\,:\,(d_j,D_j)_{1,M};(d_k,D_k;\xi_k)_{M+1,q}\,:f}^{\delta+s-1\,:\,(c_j,C_j,\gamma_j)_{1,N};(c_k,C_k)_{N+1,P}\,:e}[1]. \tag{5.4}$$

Theorem 3. *If* $f(x) \in L_p(0,\infty), p^{-1}+q^{-1}=1, g(x) \in L_q(0,\infty)$, $\Re\,[\eta+\mu(b_j/B_j)] > -1/q; j=1,\dots,m$ *and* $\Re\,[\delta+\nu(d_j/D_j)] > -1/p; j=1,\dots,M$, *then the following result holds:*

$$\int_0^\infty g(x)\; R\,[f(x)]\,dx = \int_0^\infty f(x)\; K\,[g(x)]\,dx. \tag{5.5}$$

6. Inversion formulae

In this section we establish two theorems for the operators defined by (3.1) and (3.2) which provide formal inversion formulas for them.

Theorem 4. *If* $f(x) \in L_p(0,\infty), 1 \le p \le 2$ (*or* $f(x) \in M_p(0,\infty)$ *with* $p > 2$), $p^{-1}+q^{-1}=1, \Re\,[\eta+\mu(b_j/B_j)] > -1/q; j=1,\dots,m$, *the integrals involved are absolutely convergent and*

$$R[f(x)] = g(x), \tag{6.1}$$

then the following result holds:

$$f(x) = \int_0^\infty t^{-1} g(t) h(x/t)\,dt, \tag{6.2}$$

where

$$h(x) = \frac{1}{2\pi w}\int_{c-w\infty}^{c+w\infty} \frac{x^{-s}}{\zeta(s)}\,ds \tag{6.3}$$

and

$$\zeta(s) = K_{x\ :\ (b_j,B_j)_{1,m};(b_k,B_k;\beta_k)_{m+1,q}\ :b}^{\eta-s+1\ :\ (a_j,A_j,\alpha_j)_{1,n};(a_k,A_k)_{n+1,p}\ :a} \quad [1]. \tag{6.4}$$

Proof. Multiply both sides of (6.1) by x^{s-1} and integrate with respect to x from 0 to ∞ and apply Theorem 1, we find that

$$M[f(x)] = \frac{Mg(x)}{\zeta(s)}. \tag{6.5}$$

Now the application of the inverse Mellin transform yields

$$\begin{aligned} f(x) &= \frac{1}{2\pi w} \int_{c-w\infty}^{c+w\infty} x^{-s} \frac{M[g(x)]}{\zeta(s)} ds \\ &= \frac{1}{2\pi w} \int_{c-w\infty}^{c+w\infty} x^{-s} \frac{1}{\zeta(s)} \int_0^\infty t^{s-1} g(t) dt\, ds \\ &= \int_0^\infty t^{-1}[g(t)][h(x/t)]dt. \end{aligned}$$

This completes the proof of Theorem 4. ■

The following theorem follows similarly by using (5.4) instead of (5.1).

Theorem 5. *If* $f(x) \in L_p(0,\infty), 1 \leq p \leq 2$ (*or* $f(x) \in M_p(0,\infty)$ *with* $p > 2$), $p^{-1} + q^{-1} = 1, \Re\,[\delta + \nu(d_j/D_j)] > -1/p; j = 1,\ldots,M$, *the integrals involved are absolutely convergent and*

$$K[f(x)] = n^*(x), \tag{6.7}$$

then the following result holds:

$$f(x) = \int_0^\infty t^{-1} n^*(t) L(x/t) dt, \tag{6.8}$$

where

$$L(x) = \frac{1}{2\pi w} \int_{c-w\infty}^{c+w\infty} \frac{x^{-s}}{U(s)} ds \tag{6.9}$$

and

$$U(s) = R_{x\ :\ (d_j,D_j)_{1,M};(d_k,D_k;\xi_k)_{M+1,Q}\ :f}^{\delta+s-1\ :\ (c_j,C_j,\gamma_j)_{1,N};(c_k,C_k)_{N+1,P}\ :e} \quad [1]. \tag{6.11}$$

7. Some general properties of the operators

Here we present some formal properties of the operators consequences of the definitions of operators (3.1) and (3.2).

(7.1)
$$x^{-1} \quad R^{\eta\ :\ (a_j,A_j,\alpha_j)_{1,n};(a_k,A_k)_{n+1,p}\ :a}_{\frac{1}{x}\ :\ (b_j,B_j)_{1,m};(b_k,B_k;\beta_k)_{m+1,q}\ :b}[f(1/x)] = K^{\eta\ :\ (a_j,A_j,\alpha_j)_{1,n};(a_k,A_k)_{n+1,p}\ :a}_{x\ :\ (b_j,B_j)_{1,m};(b_k,B_k;\beta_k)_{m+1,q}\ :b}[f(x)]\ ;$$

(7.2)
$$x^{-1} \quad K^{\delta\ :\ (c_j,C_j,\gamma_j)_{1,N};(c_k,C_k)_{N+1,P}\ :e}_{\frac{1}{x}\ :\ (d_j,D_j)_{1,M};(d_k,D_k;\xi_k)_{M+1,Q}\ :f}[f(1/x)] = R^{\delta\ :\ (c_j,C_j,\gamma_j)_{1,N};(c_k,C_k)_{N+1,P}\ :e}_{x\ :\ (d_j,D_j)_{1,M};(d_k,D_k;\xi_k)_{M+1,Q}\ :f}[f(x)]\ ;$$

(7.3)
$$x^{\lambda} \quad R^{\eta\ :\ (a_j,A_j,\alpha_j)_{1,n};(a_k,A_k)_{n+1,p}\ :a}_{x\ :\ (b_j,B_j)_{1,m};(b_k,B_k;\beta_k)_{m+1,q}\ :b}[f(x)] = K^{\eta-\lambda\ :\ (a_j,A_j,\alpha_j)_{1,n};(a_k,A_k)_{n+1,p}\ :a}_{x\ :\ (b_j,B_j)_{1,m};(b_k,B_k;\beta_k)_{m+1,q}\ :b}[x^{\lambda}f(x)]\ ;$$

(7.4)
$$x^{\lambda} \quad K^{\delta\ :\ (c_j,C_j,\gamma_j)_{1,N};(c_k,C_k)_{N+1,P}\ :e}_{x\ :\ (d_j,D_j)_{1,M};(d_k,D_k;\xi_k)_{M+1,Q}\ :f}[f(x)] = R^{\delta+\lambda\ :\ (c_j,C_j,\gamma_j)_{1,N};(c_k,C_k)_{N+1,P}\ :e}_{x\ :\ (d_j,D_j)_{1,M};(d_k,D_k;\xi_k)_{M+1,Q}\ :f}[x^{\lambda}f(x)]\ .$$

If

$$R^{\eta\ :\ (a_j,A_j,\alpha_j)_{1,n};(a_k,A_k)_{n+1,p}\ :a}_{x\ :\ (b_j,B_j)_{1,m};(b_k,B_k;\beta_k)_{m+1,q}\ :b}[f(x)] = g(x),$$

then

(7.5)
$$R^{\eta\ :\ (a_j,A_j,\alpha_j)_{1,n};(a_k,A_k)_{n+1,p}\ :a}_{x\ :\ (b_j,B_j)_{1,m};(b_k,B_k;\beta_k)_{m+1,q}\ :b}[f(cx)] = g(cx).$$

If

$$K^{\delta\ :\ (c_j,C_j,\gamma_j)_{1,N};(c_k,C_k)_{N+1,P}\ :e}_{x\ :\ (d_j,D_j)_{1,M};(d_k,D_k;\xi_k)_{M+1,Q}\ :f}[f(x)] = h(x),$$

then

(7.6)
$$K^{\delta\ :\ (c_j,C_j,\gamma_j)_{1,N};(c_k,C_k)_{N+1,P}\ :e}_{x\ :\ (d_j,D_j)_{1,M};(d_k,D_k;\xi_k)_{M+1,Q}\ :f}[f(cx)] = h(cx).$$

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