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Continious Dependence of Solutions of Quasidifferential Equations with Impulses

V. A. Plotnikov, P. M. Kitanov***

Presented by Bl. Sendov

In this article on quasidifferential impulse equations we consider the continuous dependence of the solutions on the initial conditions as well as the mappings defined by these equations.

1. Introduction

Work [1] defines the concept of quasidifferential equation, which is a generalization of the concept of R - solution [2,3] of differential inclusion. Works [1,4] define and prove theorems for exsistence and uniqueness of the solution for quasidifferential equations and show that these equations define an irreversible dynamical system in a metric space.

A lot of research has been carried out recently in differential equations with impulses. Some basic results and reference points can be found in [5,7]. This article on quasidifferential impulse equations is going to consider the continious dependence of the solutions on the initial conditions and the mappings defined by these equations.

2. Statement of the problem

Let X be complete metric space with distance function $\delta(.,.)$. Then

$$\varphi : [0, \tau) \times [t_0, t_0 + T) \times X \mapsto X$$

defines a local quasimovement, i.e. the following conditions are satisfied.

Condition D:

1) an axiom of initial conditions: $\varphi(0, t_0, x_0) = x_0$;
2) an axiom of equasifitting:
$$\delta(\varphi(\tau_1+\tau_2, t, x), \varphi(\tau_2, t+\tau_1, \varphi(\tau_1, t, x))) = o(\tau_1+\tau_2);$$
3) an axiom of continuity, i.e. for every $\varepsilon > 0$ there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $\delta(\varphi(\tau_1, t, x_1), \varphi(\tau_2, t, x_2)) < \epsilon$ when
$$\delta(x_1, x_2) < \delta_1, \ |\tau_1 - \tau_2| < \delta_2.$$

Definition 1. A *quasidifferential equation* in the metric space is called the equation

$$\delta(x(t+\Delta), \varphi(\Delta, t, x, (t))) = o(\Delta), \ x(t_0) = x_0. \tag{1}$$

A *solution* of equation (1) is called a continuity map

$$x : [t_0, T] \mapsto X,$$

which satisfies (1) for $t \in [t_0, T]$.

Let us consider in the domain

$$Q = \{\Delta \in [0, \tau), \ t \in [t_0, t_0 + T]; \ D \subset X\}$$

a quasidifferential equation with impulses

$$\delta(x(t+\Delta), \varphi(\Delta, t, x(t))) = 0(\Delta), \quad t \neq \tau_i, \quad x(t_0) = x_0 \tag{2}$$
$$x(\tau_i + 0) = \psi_i(x(\tau_i)),$$

where

$$\psi_i : X \mapsto X, \quad x(\tau_i) = x(\tau_i - 0).$$

3. Main results

Theorem 1. *Let the following conditions be fulfilled in the domain Q:*

1) the map $\varphi(\Delta, t, x))$ satisfies condition D, Lipshitcz's condition w.r to Δ with constant λ, and in x the condition

$$|\delta(x, y) - \delta(\varphi(\Delta, t, x), \varphi(\Delta, t, y))| \leq \Delta \ \gamma \ \delta(x, y); \tag{3}$$

2) the map $\psi_i(x)$ satisfies Lipshitsz's condition with constant λ. Then for the solutions $x(t)$ and $y(t)$ of quasidifferential equation with impulses as defined in (2), the following estimation is correct:

$$\delta(x(t), y(t)) \leq \lambda^{i(t_0,t)} e^{\gamma(t-t_0)} \delta_0, \tag{4}$$

where $i(t_0, t)$ is the number of impulses on the interval $[t_0, t]$, $i(t_0, T) < \infty$,

$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad \delta_0 = \delta(x_0, y_0).$$

P r o o f. Let us denote:

$$\delta_i^- = \delta(x(\tau_i), y(\tau_i)), \quad \delta_i^+ = \delta(x(\tau_i + 0), \; y(\tau_i + 0)).$$

Let us divide the interval $[t_0, \tau_i]$ into m parts.

In the moment $t \in [t_k, t_{k+1}] \subset [t_0, \tau_i]$, the error estimate due to initial conditions is:

$$\begin{aligned} &\delta(x(t), y(t)) \\ &\le \delta(\varphi(t - t_k), t_k, x(t_k), \varphi(t - t_k, t_k, y(t_k))) + 0(\Delta) \le (1 + \Delta\gamma)\delta(x(t_k), y(t_k)) + 0(\Delta) \\ &= (1 + \Delta\gamma)\delta(\varphi(\Delta, t_{k-1}, x(t_{k-1})), \varphi(\Delta, t_{k-1}, y(t_{k-1}))) + (1 + \Delta\gamma)o(\Delta) + o(\Delta) \\ &\le (1 + \Delta\gamma)^2\delta(x(t_{k-1}, y(t_{k-1})) + o(\Delta)(1 + \Delta\gamma) + o(\Delta)). \end{aligned}$$

It is easy to check that:

$$\begin{aligned} \delta(x(t), y(t)) &\le (1 + \Delta\gamma)^{k+1}\delta_0 + [(1 + \Delta\gamma)^k + \ldots + (1 + \Delta\gamma) + 1] \, o(\Delta) \\ &\le \left(1 + \tfrac{(\tau_1 - t_0)}{m}\gamma\right)^m \delta_0 + \tfrac{(1+\Delta\gamma)^{k+1} - 1}{\Delta\gamma} \, o(\Delta) \\ &\le e^{\gamma(\tau_1 - t_0)}\delta_0 + \tfrac{o(\Delta)}{\Delta} \, \tfrac{e^{\gamma(t - t_0)}}{\gamma}. \end{aligned}$$

Therefore,

$$\delta(x(t), y(t)) \le e^{\gamma(t - t_0)}\delta_0, \; t \in [t_0, \tau_1]. \tag{5}$$

From (5) and from the fact that $\psi_1(x)$ satisfies Lipshitsz's condition w.r to x after the first impulse $t = \tau_1$, we obtain

$$\begin{aligned} \delta_1^+ &= \delta(x(\tau_1 + 0), y(\tau_1 + 0)) = \delta(\psi_1(x(\tau_1)), \psi_1(y(\tau_1))) \\ &\le \lambda\delta(x(\tau_1), y(\tau_1)) \le \lambda e^{\gamma(\tau_1 - t_0)}. \end{aligned} \tag{6}$$

Similarly, within the interval $[\tau_1, \tau_2]$ and from (5), (6) we have

$$\delta_2^- \le \lambda \; e^{\gamma(\tau_2 - \tau_1)} \; e^{\gamma(\tau_1 - t_0)} \; \delta_0 = \lambda \; e^{\gamma(\tau_2 - t_0)} \; \delta_0$$
$$\delta_2^+ \le \lambda^2 \; e^{\gamma(\tau_2 - t_0)} \; \delta_0.$$

If we continue further, we will obtain (4) which is a statement of the error due to the initial conditions. ■

Theorem 2. *Let in domain Q be given quasidifferential equations*

$$\delta(x(t+\Delta),\varphi_1(\Delta,t,x(t))) = o(\Delta), \quad x(t_0) = x_0 \tag{7}$$

$$\delta(y(t+\Delta),\varphi_2(\Delta,t,y(t))) = o(\Delta), \quad y(t_0) = y_0. \tag{8}$$

Let us suppose then that the maps $\varphi_1(\Delta,t,x)$ and $\varphi_2(\Delta,t,x)$ satisfy condition 1) by Theorem 1 and in addition satisfy

$$\delta(\varphi_1(\Delta,t,x),\varphi_2(\Delta,t,x)) \le \eta\,\Delta, \;\; \delta(x_0,y_0) = \delta_0. \tag{9}$$

Then in interval $[t_0,T]$ the following error estimate is correct

$$\delta(x(t),y(t)) \le e^{\gamma(t-t_0)}\delta_0 + \frac{e^{\gamma(t-t_0)}-1}{\gamma}\,\eta. \tag{10}$$

Proof. Let us divide the interval $[t_0,T]$ into m parts, then for $t\in[t_k,t_{k+1}]$ we have

$$\begin{aligned}
\delta(x(t),y(t)) &= \delta(\varphi_1(t-t_k),t_k,x(t_k)),\varphi_2(t-t_k,t_k,y(t_k))+o(\Delta)\\
&\le \delta(\varphi_1(t-t_k,t_k,x(t_k)),\varphi_1(t-t_k,t_k,y(t_k)))\\
&\quad+\delta(\varphi_1(t-t_k,t_k,y(t_k)),\varphi_2(t-t_k,t_k,y(t_k)))+o(\Delta)\\
&\le (1+\Delta\gamma)\delta(x(t_k),y(t_k))+\Delta\eta+o(\Delta)\\
&= (1+\Delta\gamma)\delta(\varphi_1(\Delta,t_{k-1},x(t_{k-1})),\varphi_2(\Delta,t_{k-1},y(t_{k-1})))+o(\Delta)(1+\Delta\gamma)+\Delta\eta+o(\Delta)\\
&\le (1+\Delta\gamma)[\delta(\varphi_1(\Delta,t_{k-1},x(t_{k-1})),\varphi_1(\Delta,t_{k-1},y(t_{k-1})))\\
&\quad+\delta(\varphi_1(\Delta,t_{k-1},y(t_{k-1})),\varphi_2(\Delta,t_{k-1},y(t_{k-1})))]+\Delta\eta+(1+\Delta\gamma)o(\Delta)+o(\Delta)\\
&\le (1+\Delta\gamma)^2\delta(x(t_{k-1}),y(t_{k-1}))+(1+\Delta\gamma)\Delta\eta+\Delta\eta+(1+\Delta\gamma)o(\Delta)+o(\Delta).
\end{aligned}$$

It is easy to check that:

$$\begin{aligned}
\delta(x(t),y(t)) &\le (1+\Delta\gamma)^{k+1}\delta(x_0,y_0)+[(1+\Delta\gamma)^k+\ldots+(1+\Delta\gamma)+1]\,\Delta\eta\\
&\quad+[(1+\Delta\gamma)^k+\ldots+(1+\Delta\gamma)+1]\,o(\Delta)\\
&\le e^{\gamma(t-t_0)}\,\delta_0+\tfrac{(1+\Delta\gamma)^{k+1}-1}{\Delta\gamma}\,\Delta\eta+\tfrac{[(1+\Delta\gamma)^{k+1}-1]}{\Delta\gamma}\,o(\Delta).
\end{aligned}$$

However,

$$\frac{[(1+\Delta\gamma)^{k+1}-1]}{\Delta\gamma}\,o(\Delta)\;\mapsto\;0.$$

Thus we obtain (10). ■

Now let us consider the equations with impulses

$$\text{(11)} \qquad \delta(x(t+\Delta), \varphi_1(\Delta, t, x(t))) = o(\Delta), \quad t \neq \tau_i, \quad x(t_0) = x_0$$

$$x(\tau_i + 0) = \psi_{1i}(x(\tau_i)),$$

$$\text{(12)} \qquad \delta(y(t+\Delta), \varphi_2(\Delta, t, y(t))) = o(\Delta), \quad t \neq \tau_i, \quad y(t_0) = y_0$$

$$y(\tau_i + 0) = \psi_{2i}(y(\tau_i)).$$

Theorem 3. *Let the conditions set in Theorem 1 and Theorem 2 be fulfilled in the domain Q, and in addition*

$$\text{(13)} \qquad \delta(\psi_{1i}(x), \psi_{2i}(x)) \le \eta.$$

Then for the solutions $x(t)$ and $y(t)$ of equations (11) and (12) the following estimate is correct:

$$\text{(14)} \qquad \delta(x(t), y(t)) \le \lambda^{i(t_0,t)} e^{\gamma(t-t_0)}(\delta_0 + \frac{\eta}{\gamma}) + C\eta,$$

where C is a constant.

Proof. According to Theorem 2 for the error estimate before the first impulse we have

$$\delta_1^- \le e^{\gamma(\tau_1 - t_0)}\delta_0 + \frac{e^{\gamma(\tau_1-t_0)} - 1}{\gamma}\eta$$

and after the impulse we have

$$\delta_1^+ = \delta(\psi_{11}(x(\tau_1)), \psi_{21}(y(\tau_1))) \le \delta(\psi_{11}(x(\tau_1)), \psi_{11}(y(\tau_1)))$$
$$+\delta(\psi_{11}(y(\tau_1)), \psi_{21}(y(\tau_1))) \le \lambda\delta_1^- + \eta.$$

Analogously,

$$\delta_2^- \le e^{\gamma(\tau_2-\tau_1)}\delta_1^+ + \frac{e^{\gamma(\tau_2-\tau_1)}-1}{\gamma}\eta \le e^{\gamma(\tau_2-\tau_1)}(\lambda\delta_1^- + \eta) + \frac{e^{\gamma(\tau_2-\tau_1)}-1}{\gamma}\eta$$
$$= \lambda e^{\gamma(\tau_2-t_0)}(\delta_0 + \tfrac{\eta}{\gamma}) - \lambda\eta e^{\frac{\gamma(\tau_2-\tau_1)}{\gamma}} + \eta\frac{e^{\gamma(\tau_2-\tau_1)}}{\gamma} - \frac{\eta}{\gamma}$$
$$\le \lambda e^{\gamma(\tau_2-\tau_0)}\,(\delta_0 + \tfrac{\eta}{\gamma}) + C_1\eta,$$

$$\delta_2^+ = \delta(x(\tau_2+0), y(\tau_2+0)) = \delta(\psi_{12}(x(\tau_2)), \psi_{22}(y(\tau_2))) \le \lambda\delta_2^- + \eta$$
$$\le \lambda^2 e^{\gamma(\tau_2-t_0)}(\delta_0 + \tfrac{\eta}{\gamma}) - \lambda^2\eta\frac{e^{\gamma(\tau_2-\tau_1)}}{\gamma} + \lambda\eta\frac{e^{\gamma(\tau_2-\tau_1)}}{\gamma} - \frac{\lambda\eta}{\gamma} + \eta,$$
$$\delta_2^+ \le \lambda^2 e^{\gamma(\tau_2-t_0)}(\delta_0 + \tfrac{\eta}{\gamma}) + C_2\eta.$$

Continuing this process, we obtain for $t \in (t_i, t_{i+1})$

$$\delta(x(t), y(t)) \leq \lambda^{i(t_0,t)} e^{\gamma(t-t_0)}(\delta_0 + \frac{\eta}{\gamma}) + C\eta,$$

where C and C_i are constants independent from δ_0 and η. ∎

4. Examples

Example 1. Let $X = R^n$ and

$$\phi_1(\Delta, t, x) = x + \Delta f(t, x),$$
$$\phi_2(\Delta, t, x) = x + \Delta(f(t, x) + R(t, x))$$
$$\phi_{1i}(x) = x + I_i(x), \quad \phi_{2i}(x) = x + I_i(x) + R_i(x),$$

then from Theorem 3 we can obtain the corresponding theorem for differential equation with impulses

(15) $$\dot{x} = f(t, x),\ t \neq \tau_i, \quad \Delta x|_{t=\tau_i} = I_i(x), \quad x(t_0) = x_0,$$

(16) $$\dot{y} = f(t, x) + R(t, x),\ t \neq \tau_i, \quad \Delta y|_{t=\tau_i} = I_i(x) + R_i(x), \quad y(t_0) = y_0.$$

It is obviously, that a similar result would hold for every complete linear metric space X, i.e. that the equations (15) and (16) can easily be, for example, equations in a Banach space.

Example 2. Let $X = \mathrm{comp}(R^n)$, where this space is the space of all nonempty compacts sets within R^n.

Let $\mathrm{conv}(R^n)$ be the space of nonempty convex and compact sets within R^n.

$$\phi_1(\Delta, t, X) = \bigcup_{z\in X} (z + \textstyle\int_t^{t+\Delta} F(t, z)dt),$$
$$\phi_2(\Delta, t, X) = \bigcup_{z\in X} \left(z + \textstyle\int_t^{t+\Delta}(F(t, z) + R(t, z))dt\right),$$
$$\psi_{2i}(X) = \bigcup_{z\in X} (z + I_i(z)),$$
$$\psi_{2i}(X) = \bigcup_{z\in X} (z + I_i(z) + R_i(z)),$$

where

$$F : R^1 \times R^n \mapsto \mathrm{conv}(R^n)$$
$$R : R^1 \times R^n \mapsto \mathrm{conv}(R^n)$$
$$I_i : R^n \mapsto \mathrm{conv}(R^n)$$
$$R_i : R^n \mapsto \mathrm{conv}(R^n)$$

From Theorem 3 we arrive to the following corresponding theorem for differential inclusions with impulses

(17) $$\dot{x} \in F(t,x),\ t \neq \tau_i, \quad \Delta x|_{t=\tau_i} \in I_i(x), \quad x(t_0) \in X_0$$

(18) $$\dot{y} \in F(t,x) + R(t,x),\ t \neq \tau_i, \quad \Delta x|_{t=\tau_i} \in I_i(x) + R(x), \quad y(t_0) \in Y_0.$$

The solutions of differential inclusions (17) and (18) should be understood to represent R-solution [2,3].

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* *Dept. Optimal Control*
Odessa State University
2, Petra Velikova Street,
270 000 Odessa
UKRAINE

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** *Dept. Mathematics,*
"Neofit Rilski" Southwestern University
2700 Blagoevgrad
BULGARIA