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Continuous Selections for Mappings with Generalized Ordered Domain [1]

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To the memory of Prof. D. Doitchinov

Every l.s.c. closed-and-convex valued mapping $\Phi : X \to 2^Y$, where X is a generalized ordered space and Y is a reflexive Banach space has a single-valued continuous selection.

1. Introduction

The main purpose of the present paper is to establish the following:

Theorem 1.1. *Every l.s.c. closed-and-convex valued mapping $\Phi : X \to 2^Y$, where X is a generalized ordered space (GO-space) and Y is a reflexive Banach space, has s single valued continuous selection.*

Our proof of Teorem 1.1 proceeds as follows. We construct a generalized-ordered paracompactification πX of X in Section 5. This construction is based on results of L. Gillman and Henriksen [3] and R. Engelking and D. Lutzer [2].

Next we extend the mapping Φ to an l.s.c. closed-and-convex valued mapping $\widehat{\Phi} : \pi X \to 2^Y$ and then the well-known E. Michael selection theorem completes the proof of Theorem 1.1. To show that the required extension $\widehat{\Phi}$ exists, we apply the method from [5] where a particular case of Theotem1.1 is established, namely the case when X is the space of all countable ordinals in the order topology.

It is easy to see that the convex-valuedness of Φ is essential for the conclusion of Theorem 1.1. In view of the well-known E. Michael [4] characterization

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of paracompactness by means of selections, the requrement on Y to be reflexive is essential as well (there are non-paracompact GO-spaces). In fact our proof of Theorem 1.1 imposes the following two requirements on the Banach space Y:

(i) Y has (an equivalent) locally uniformly rotund norm $\|.\|$ (for a reflexive Y the existence of such a norm is guaranteed by the well-known Troyanski renorming theorem [6]).

(ii) Every bounded closed convex subset of Y is weakly compact (a well-known property of reflexive Ys).

Yet Theorem 1.1 is closely related to the following:

Conjecture 1.2.. (M. Choban, V. Gutev, St. Nedev) *Every l.s.c., closed-and-convex valued mapping $\Phi : X \to 2^Y$, where X is ($\tau-$) collectionwise normal and countably paracompact space and Y is a Hilbert (or reflexive Banach) space (of weight $\leq \tau$), then Φ has a single-valued continuous selection.*

2. Notation and terminology

All the spaces considered in this paper are at least Hausdorff and our topological terminology corresponds to that in [1,2]. If A is a set, $|A|$ denotes its cardinality, 2^A denotes the set of all non empty subsets of A. A set-valued mapping $\Phi : X \to 2^Y$ between the topological spaces X and Y is said to be lower semi-continuous (l.s.c. for short) at the point $x_0 \in X$ if for every open subset U of Y that intersects $\Phi(x_0)$ (i.e. $U \cap \Phi(x_0) \neq \emptyset$), the set $\Phi^{-1}(U) = \{x \in X : \Phi(x) \cap U \neq \emptyset\}$ is a neighbourhood of x_0, i.e. x_0 is an interior point of $\Phi^{-1}(U)$. Φ is said to be l.s.c.provided Φ is l.s.c. at every point of X.

Let $(X, <)$ be a linearly ordered set, $x, y \in X$ and $x < y$. Under a (semi-open) interval $(x, y]$ we understand $(x, y] = \{z \in X : x < z \leq y\}$. The intervals $(x, y), [x, y), [x, y]$ are defined in a similar way. The set $\{z \in X : z < x\}$ we denote by $X_{<x}$. The symbols $X_{\leq x}, X_{>x}, X_{\geq x}$ have similar meanings. If $L \subset A \subset X$ then the set L is said upper (respectively lower) cofinal in A if $A \subset \cup\{X_{\leq x} : x \in L\}$ ($A \subset \cup\{X_{\geq x} : x \in L\}$, respectively). Let A be a non-empty subset of X. Denote u-cf$(A) = \min\{|L| : L \subset A, L$ is upper-cofinal in$A\}$ and l-cf$(A) = \min\{|L| : L \subset A, L$ is lower-cofinal in $A\}$. A subset B of X is called order-convex if $[x, y] \subset B$ for every $x, y \in B$.

A generalized ordered space (GO-space) is a triple $(X, T, <)$, where $<$ is a linear ordering of the set X and T is a topology on X with a base consisting of order-convex sets. A space is GO-space iff it is a subspace of a linearly ordered space (see, e.g.[2]).

A pair $c = (A, B)$ of subsets A, B of a GO-space X is called:

- a left gap if A and B are closed subsets of X, $A \cup B = X$, $A \cap B = \emptyset$, $x < y$ whenever $x \in A$ and $y \in B$ and A has no largest element.

- a left D-gap if c is a left gap and either $A = \emptyset$ or there is a discrete in X subset $L \subset A$ which is upper cofinal in A.

The right gap and right D-gap are defined analogously.

3. Elementary properties of gaps

Lemma 3.1. *Let (A, B) be a left D-gap in a GO-space X and $A \neq \emptyset$. Then there exists a closed and discrete in X upper cofinal in A subset $L = \{x_\alpha : \alpha \in Ord, |\alpha| < u - cf(A)\}$ with the following properties:*

1. if $\alpha < \beta$, then $x_\alpha < x_\beta$;

2. if $H \subseteq L$ and $|H| = |L|$, then H is upper cofinal in A.

Proof. There exists a discrete and closed in X subset P in A of cardinality $u - cf(A)$. We consider that $P = \{y_\alpha : \alpha < u - cf(A)\}$. We put $x_0 = y_0$. If $\alpha \geq 1$ and the points $\{x_\beta : \beta < \alpha\}$ are constructed then there exists the point $x_\alpha \in P$ such that $x_\alpha > y_\alpha$ and $x_\alpha > x_\beta$ for each $\beta < \alpha$. ■

Lemma 3.2. *Let (A, B) be a left gap in a GO-space X and $A \neq \emptyset$. The following statements are equivalent:*

1. (A, B) is not a left D-gap.

2. If L is a closed upper cofinal subset in A, U is an open subset in X and $L \subseteq U$ then $A \cap X_{\geq x} \subset U$ for some $x \in A$.

3. Every continuous mapping $f : A \to Y$ in a metrizable space Y is eventually constant, i.e. there exists a point $b \in A$ such that $f(x) = f(b)$ for each $x \in A \cap X_{>b}$.

4. If L and M are closed upper cofinal subsets in A, then $L \cap M \neq \emptyset$.

5. If $F_1, F_2, ...$ are closed upper cofinal sets in A, then $F = \cap\{F_n : n \in N = \{0, 1, 2, ...\}\}$ is upper cofinal in A.

Proof. Suppose that (A, B) is a left D-gap in X. Fix a discrete upper cofinal set L in A of cardinality $u - cf(A)$ such that if $H \subseteq L$ and $|H| = |L|$ then H is upper cofinal in A. It is obvious that $|L| \geq \aleph_0$. There are subsets $L_1, L_2, ...$of L such that $L_n \cap L_m = \emptyset$ if $n \neq m$ and $|L_n| = |L|$ for each $n \in N$. Then $L_1, L_2, ...$are closed upper cofinal subsets in A. This proves the implications 4→1,3→1 and 2→1. The imlications 5→4→2→4 and 3→4 are obvious.

Now we prove the implication 1→3. It is obvious that $u - cf(A) \geq \aleph_1$. Fix a continuous mapping $f : A \to Y$ in a metric space Y with a metric d. In A fix an upper cofinal set H of cordinality $u - cf(A)$. Suppose that for every $y \in H$ there are $c(y), b(y) \in A \cap X_{>y}$ such that $c(y) < b(y)$ and $f(c(y)) \neq f(b(y))$. The sets $\{c(y) : y \in H\}$ and $\{b(y) : y \in H\}$ are upper cofinal in A. Let $H_n = \{y \in H : d(f(c(y)), f(b(y))) \geq 2^{-n}\}$. Then $H_n \subseteq H_{n+1}$ and $H = \cup\{H_n : n \in N\}$. If every H_n is not upper cofinal in A then there exists the set $\{c_n \in A : n \in N\}$

such that $x \leq c_m$ for every $x \in H_m$ and $m \in N$. Then $\{c_n : n \in N\}$ is upper cofinal in A and $u - cf(A) = \aleph_0$. Therefore H_m is upper cofinal in A for some $m \in N$. We construct, by induction, the set $L = \{l_\alpha \in A : \alpha < u - cf(A)\}$ such that:

1. If α is an even ordinal number then there exists $y_\alpha \in H_m$ such taht $l_\alpha = c(y_\alpha)$ and $l_{\alpha+1} = b(y_\alpha)$.

2. If $\alpha < \beta < u - cf(A)$ then $l_\alpha < l_\beta$.

3. For every $\alpha < u - cf(A)$ the set $\{l_\beta : \beta < \alpha\}$ is discrete in A.

Fix $y_0 \in H_m$ and put $l_0 = c(y_0)$ and $l_1 = b(y_0)$. Suppose that $\alpha < u - cf(A)$ and the set $L_\alpha = \{l_\beta : \beta < \alpha\}$ has already been constructed. Let's show that L_α is discrete in A. Fix $x \in A$ and $0 < \varepsilon < 2^{-m-2}$. We put $U_1 = \{y \in A : d(f(x), f(y)) < \varepsilon\}$. There exists an order-convex open subset U of x such that $x \in U \subset U_1$. Then $U \cap L_\alpha$ is finite. Hence L_α is a discrete subset of A. Now we construct points l_α and $l_{\alpha+1}$. We consider two possible cases.

Case 1. $\alpha = \beta + 1$.

Tere exists an element $y_\alpha \in H_m$ such that $y_\alpha > l_\beta$. We put $l_\alpha = c(y_\alpha)$ and $l_{\alpha+1} = b(y_\alpha)$.

Case 2. α is a limit number.

There exists a piont $y_\alpha \in H_m$ such that $l_\beta < y_\alpha$ for every $\beta < \alpha$. In this case we put $l_\alpha = c(y_\alpha)$ and $l_{\alpha+1} = b(y_\alpha)$.

The set $L = \{l_\alpha : \alpha < u - cf(A)\}$ is thus constructed. By construction L is a discrete upper cofinal subset in A.Therefore (A, B) is a left D-gap. Now we turn to the implication 1$\rightarrow$5. Suppose $F_0, F_1, F_2, \ldots$ are closed upper cofinal subset in A, $F_0 \supset F_1 \supset F_2 \ldots$ (by virtue of 4)) and $F = \cap\{F_n : n \in N\} = \emptyset$. We construct the upper cofinal set $H = \{x_\alpha : \alpha < \alpha_0 = u - cf(A)\}$ with properties:

1. $x_0 \in F_0$ and $x_n \in F_n$ for every $n \in N$.

2. If α is a limit ordinal number, then $x_\alpha \in F_0$ and $x_{\alpha+n} \in F_n$ for each $n \in N$.

3. If $\alpha < \beta < \alpha_0$, then $x_\alpha < x_\beta$.

The set H is closed and discrete in A. If $\alpha < \alpha_0$ and $x \in cl(\{x_\beta : \beta < \alpha\}) \backslash \{x_\beta : \beta < \alpha\}$ then $x \in F$. Hence (A, B) is a left D-gap. The proof is complete. ■

The implication 1$\rightarrow$3 of Lemma 3.2 is a generalization of [3; Theorem 1.4].

Remark. Analogous results hold true for the right gaps as well.

Lemma 3.3. *Let X be a GO-space that has no largest element and $(X, \emptyset)$ be not a left D-gap. Then for every l.s.c. closed-valued mapping $\Phi : X \rightarrow 2^Y$ to a matrizable space Y the following are equivalent:*

1. For some $x \in X$ the mapping $\Phi \mid X_{\geq x} : X_{\geq x} \to 2^Y$ has a single-valued continuous selection.

2. For some closed upper cofinal subspace H the mapping $\Phi \mid H : H \to 2^Y$ has a single-valued continuous selection..

3. There exists $x \in X$ such that $\cap\{\Phi(y) : y \geq x\} \neq \emptyset$.

Proof. The implications 3→1→2 are obvious.

Let H be a closed upper cofinal set in X and $f : H \to Y$ .be a continuous single-valued selection of the mapping $\Phi \mid H$. It is clear that $(H, \emptyset)$ is not a D-gap in H, because any discrete upper cofinal subset in H is discrete and upper cofinal in X .as well. By Lemma 3.2 there exists a point $x_0 \in H$ such that $f(x) = f(x_0)$ for every $x \geq x_0$. Fix an open base $\{U_n : n \in N\}$ at $f(x_0)$, with $U_{n+1} \subseteq U_n$ for every $n \in N$. The set $\Phi^{-1}(U_n)$ is open in X and $H \subseteq \Phi^{-1}(U_n)$. In vertue of Lemma 3.2 there exists a point x_n such that $X_{\geq x_n} \subseteq \Phi^{-1}(U_n)$. Because $u - cf(H) = u - cf(X) \geq \aleph_1$ there exists a point $x \in H$ such that $x > x_n$ for every $n \in N$. Therefore $X_{\geq x} \subseteq \cap\{\Phi^{-1}(U_n) : n \in N\}$. By construction $f(x_0) = f(x) \in cl\Phi(y)$ for each $y \geq x$. Hence $f(x_0) \in \cap\{\Phi(y) : y \geq x\}$. The proof is complete. ■

Lemma 3.4. *Let X be a GO-space with no largest element and $(X, \emptyset)$ be not a left D-gap. Then for every open cover $\{U_n : n \in N\}$ of X there exist $x \in X$ and $m \in N$ such that $X_{\geq x} \subseteq \cup\{U_i : i \leq m\}$.*

Proof. We assume that $U_n \subseteq U_{n+1}$ for every $n \in N$. Suppose $X_{\geq x} \setminus U_n \neq \emptyset$ for each $x \in X$ and each $n \in N$. Then the sets $F_n = X \setminus U_n$ are closed and upper cofinal in X. By construction $\cap\{F_n : n \in N\} = \emptyset$. Lemma 3.2 completes the proof. ■

Lemma 3.5. *Let X be a GO space with no largest element and with $u - cf(X) \geq \aleph_1$. Then every decreasing (increasing) real-valued function $f : X \to R$. is eventually constant.*

Proof. Let $f(x) \leq f(y)$ if $x \geq y$. Put $b = \inf\{f(x) : x \in X\}$. Fix a sequence$\{b_n \in R : n \in N\}$ such that $b < b_{n+1} < b_n$ for each $n \in N$ and $b = \lim_{n\to\infty} b_n$. For every $n \in N$ there exists $x_n \in X$ such that $b \leq f(x_n) < b_n$. By the assumption, $u - cf(X) \geq N_1$. there exists a point $x \in X$ such that $x \geq x_n$ for each $n \in N$. Then $f(y) = f(x) = b$ for every $y \geq x$ and $+\infty > b > -\infty$. The proof is complete. ■

4. Banach spaces with locally uniformly rotund norms

A norm $\|.\|$ in a Banach space Y is locally uniformly rotund if ($\|y_n\| = \|y\|$ for each $n \in N$ and $\lim_{n\to\infty} \|y_n + y\| = 2\|y\|$) implies ($\lim_{n\to\infty} \|y_n - y\| = 0$).

By vertue of the well-known theorem of S. Troyanski [6] every reflexive Banach space has an equivalent locally uniformly rotund norm.

Let Y be a Banach space with a locally uniformly rotund norm $\|.\|$. For all $r \geq 0$ we put $B(r) = \{y \in Y : \|y\| < r\}$ and $S(r) = \{y \in Y : \|y\| = r\}$. If $r > 0$ then for every point $y \in S(r)$, at the point y there exists a unique closed tangent hyperplane $\pi(y, r)$ of $S(r)$.

Lemma 4.1. *Let Y be a reflexive Banach space with a locally uniformly rotund norm $\|.\|$. Then for every non-empty closed convex subset L of Y there exists a unique point $m(L) \in L$ such that $\|m(L)\| = \inf\{\|y\| : y \in L\}$.*

Proof. In a reflexive Banach space every bounded closed convex subset is weakly compact. Hence the point $m(L)$ exists in L. It is clear that the point $m(L)$ is unique. The proof is complete. ■

Lemma 4.2. *Let Y be a Banach space with a locally uniformly rotund norm $\|.\|$. Then for every $r > 0, y \in S(r)$ and $\varepsilon > 0$ there exists $\delta = \delta(r, y, \varepsilon) > 0$ such that $x \in S(r)$ and $\|x - y\| \geq \varepsilon$ implies $d(y, \pi(x, r)) = \min\{\|y - z\| : z \in \pi(x, r)\} \geq \delta$.*

Proof. Suppose that there exists a sequence $\{x_n \in S(r) : n \in N\}$ for which $\|y - x_n\| \geq \varepsilon$ and $d(y, \pi(x_n, r)) < 2^{-n}$ for each $n \in N$. Choose $z_n \in \pi(x_n, r)$ such that $\|y - z\| < 2^{-n}$. Then $\lim_{n\to\infty} \|y - z_n\| = 0$. On the other hand, $\frac{1}{2}(z_n + x_n) \in \pi(x_n, r), \left\|\frac{1}{2}(z_{n+}x_n)\right\| > r$ and $2r \geq \|x_n + y\| = \|x_n + y_n - z_n + z_n\| \geq \|x_n + z_n\| - \|z_n - y\| \geq 2r - \|z_n - y\|$. Hence $\lim_{n\infty} \|y + x_n\|$ $2r$ and $\|y\| = \|x_n\| = r$ for each $n \in N$. Thus, by the local uniform rotundity of the norm, $y = \lim_{n\to\infty} x_n$ and the proof is over. ■

Lemma 4.3. *Let (A, B) be a left gap in a generalized ordered space X, Y be a Banach space with a locally uniformly rotund norm $\|.\|$, $r > 0, \Phi : A \to 2^Y$ be a l.s.c. closed- and convex-valued mapping and $\|m(\Phi(x))\| = r$ for every $x \in A$. If (A,B) is not a left D-gap then there exists $x_0 \in A$ such that $m(\Phi(x)) = m(\Phi(x_0))$ for each $x \in \{y \in A : y \geq x_0\}$.*

Proof. Suppose that for every $x \in A$ there exists $c(x), b(x) \in A$ such that $x \leq c(x) < b(x)$ and $\|m(\Phi(c(x))) - m(\Phi(b(x)))\| = r(x) > 0$. For some $m \in N$ the set $H_m = \{x \in A : r(x) \geq 2^{-m}\}$ is upper cofinal in A. We construct the upper cofinal set $F = \{l_\alpha : \alpha < \alpha_0 = u - cf(A)\}$ with properties:

1. If $\alpha < \beta < \alpha_0$ then $l_\alpha < l_\beta$.
2. For every even ordinal number α there exists $x_\alpha \in H_m$ such that $l_\alpha = c(x_\alpha)$ and $l_{\alpha+1} = b(x_{\alpha+1})$.

Fix $\alpha < \alpha_0$. We prove that $F_\alpha = \{l_\beta : \beta < \alpha\}$ is closed in A. Let $x_0 \in A$ and $2\delta = \delta(r, m(\Phi(x_0)), 2^{-m})$ (see Lemma 4.2.). Put $U = \{y \in Y : \|y - m(\Phi(x_0))\| < \delta\}$ and $V = \Phi^{-1}(U)$. Then V is open in X and $x_0 \in V$. In virtue of Lemma 4.2 $V \cap \{c(x) : x \in H_m\} = \oslash$ or $V \cap \{b(x) : x \in H_m\} = \oslash$. Hence if $x_0 \in W \subseteq V$ and W is an order-convex open subset of X then $|W \cap F_\alpha| \leq 1$. Therefore F is a discrete closed subset of X and (A,B) is a left D-gap. The proof is complete. ∎

Lemma 4.4. *Let (A,B) be a left gap that is not a left D-gap in a GO-space X, let Y be a Banach space with a locally uniformly rotund norm $\|.\|$ and let $\Phi : X \to 2^Y$ be a l.s.c closed-and-convex valued mapping. Then there exists $x_0 \in A$ so that $\cap\{\Phi(x) : x \in A, x \geq x_0\} \neq \oslash$.*

Proof. Let for every $x \in A$, $\xi(x) = \inf\{\|y\| : y \in \Phi(x)\}$. There is a (unique) point $y(x) \in \Phi(x)$ with $\|y(x)\| = \xi(x)$ (by Lemma 4.1). For every $r > 0$ the set $\{x \in A : \xi(x) < r\} = \Phi^{-1}(B(r)) \cap A$ is open in A. By Lemma 3.5 there are $x_1 \in A$ and $m \in N$ such that $H = A_{\geq x_1} \subset \Phi^{-1}(B(m))$. For every $x \in H$ let $\eta(x) = \sup\{\xi(y) : y \in H, y \geq x\}$ (note, $\xi(x) < m$ for every $x \in H$). Obviously the function η is decreasing on H. By Lemma 3.4 there are $r_0 \in R$ and $x_2 \in H$ such that $\eta(x) = \eta(x_2) = r_0$ for every $x \geq x_2$. By construction, the set $F_n = \{x \in A : x \geq x_2, \xi(x) \geq r_0 - 2^{-n}\}$ is closed and upper-cofinal in A. So, (Lemma 3.2), the set $F = \cap\{F_n : n \in N\}$ is closed and upper cofinal in A. By virtue of Lemma 4.3 there is $x_3 \in F$ so that $y(x) = y(x_3)$ for every $x \in L = F_{\geq x_3}$. Hence, $y : L \to Y$ is a single-valued continuous selection of $\Phi \mid L$ and L is closed and upper cofinal in A. Thus, by Lemma 3.3, for same $x_4 \in A$, the mapping $\Phi \mid A_{\geq x_4}$ has a single-valued continuous selection $f : A_{\geq x_4} \to Y$. By Lemma 3.2 there is $x_0 \in A$, $x_0 \geq x_4$ such that $f(x) = f(x_0)$ for every $x \geq x_0$. Therefore $f(x_0) \in \cap\{\Phi(x) : x \in A, x \geq x_0\}$, and the proof is complete. ∎

GO-paracompactification πX of the GO-space X

From the results in [2] and Lemma 3.2 one easily derives the following theorem:

Theorem 5.1. *A generalized ordered space X is paracompact if and only if every left gap in X is a left D-gap and every right gap in X is a right D-gap.* theorem

Let now X be a GO-space with respect to the linear ordering " $<$" on X. To every left gap $c = (A_c, B_c)$, that is not a left D-gap, we associate an "ideal" (i.e. non belonging to X) point ξ_c and to every right gap $c = (A_c, B_c)$ that is not a right D-gap we associate an "ideal" point η_c. By πX we denote the set that is made up by all the points in X, all the points ξ_c and all the points η_c. The linear ordering $<$ extends on πX in an obvious manner. We preserve the symbol $<$ to denote this extended ordering as well.

We define a GO topology on πX by assigning a local base to every point in πX. So, let $\xi \in \pi X$ if $\xi \in X$, then the set $U \subset \pi X$ is in the local base of ξ

under discussion if both U is order-convex in πX and $U \cap X$ is an open order convex neighborhood of ξ in X. If $\xi = \xi_c$, then the local base of ξ consists of all the semi-open intervals $(\eta, \xi]$ in πX, where $\eta \in A_c$. Finally, if $\xi = \eta_c$ then the local base under question consists of all the semi-open intervals $[\xi, \eta)$ in πX, where $\eta \in B_c$.

It is obvious that after this assignment of local bases, πX becomes a GO-space, containing X as a dense subspace. Moreover, the next proposition follows easily by Theorem 2.1.

Proposition 5.2. *πX, endowed with the above described topology, is a paracompact GO-space.*

6. Proof of Theorem 1.1

As it was announced in Introduction, we first define an l.s.c. closed-and-convex valued extension $\widehat{\Phi} : \pi X \to 2^Y$ of Φ. The mapping Φ is defined as follows: For $x \in Y$ we let $\widehat{\Phi}(x) = \Phi(x)$. For $\zeta \in \pi X \setminus X$ we consider the possibilities (i) and (ii) that can occur: (i)-$\zeta = \xi_c$ for a left gap $c = (A, B)$ of X that is not a left D-gap of X; then we let $\widehat{\Phi}(\zeta) = cl(\cup\{(\cap\{\Phi(y) : y \in A_{\geq x}\}) : x \in A\})$ and (ii)-$\zeta = \eta_c$ for a right gap $c = (A, B)$ of X that is not a right D-gap; then we let $\widehat{\Phi}(\zeta) = Cl(\cup\{(\cap\{\Phi(y) : y \in B_{\leq x}\}) : x \in B\})$.

Now a straight forward verification (based mostly on Lemma 4.4 and the properties of the convex subsets of Y) shows that $\widehat{\Phi}$ satisfies all our requirements. Let us, for instance, show that $\widehat{\Phi}$ is l.s.c. at at point $x_0 \in X$. Assume $V \subset Y$ is open and $V \cap \widehat{\Phi}(x_0) = V \cap \Phi(x_0) \neq \emptyset$. Pick $y_0 \in V \cap \Phi(x_0)$ and let $\varepsilon > 0$ be such that $B_\varepsilon(y_0) \subset Y$. Yet, let U be an order convex neighbourhood of x_0 in X with $\Phi(x) \cap B_{\varepsilon/2}(y_0) \neq \emptyset$ for every $x \in U$. Define the mapping $\varphi : U \to 2^Y$ by the formula $\varphi(x) = cl(\Phi(x) \cap B_{\varepsilon/2}(y_0))$. Obviously, φ is l.s.c. If now $\zeta \in \pi X \setminus X$ lies in an open interval (in πX) with endpoints in U and if, say, $\zeta = \xi_c$, where $c = (A, B)$ is a left-gap in X that is not a left D-gap, then there is $x_1 \in A \cap U$ with $\cap\{\varphi(x) : x \in A_{\geq x_1}\} \neq \emptyset$ (in vertue of Lemma 4.4 applied to the mapping φ). Since, obviously, $\widehat{\Phi}(\zeta) \supset \cap\{\varphi(x) : x \in A_{\geq x_1}\}$, it follows that $\Phi(\zeta) \cap V \neq \emptyset$.

If now, $\widehat{U} = U \cup \{\zeta \in \pi X \setminus X : \zeta$ is in an open interval with endpoints in U$\}$, then $\widehat{U}$ is a neighbourhood of x_0 in πX with $\widehat{\Phi}(x) \cap V \neq \emptyset$ for every $x \in \widehat{U}$. The cases $x_0 \in \pi X \setminus X$ and $x_0 = \xi_c$ or $x_0 = \eta_c$ can be treated in a similar way.

Finally, as it was already mensioned, Michael's selection theorem [4] completes the proof of Theorem 1.1.

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