Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

New Series Vol.11, 1997, Fasc. 1-2

Stationary-Periodic Waves in Discrete Media with Cubic Nonlinearity

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Presented by P. Kenderov

In the present paper periodic and solitary solutions of a generalized and modified version of the classic KDV equation have been found analytically for medium with negative and mixed type dispersion. One-soliton solution and its long-wave approximation: a rational solution, have been also found depending on the wave length and medium's dispersive characteristics.

1. Introduction

The nonlinear evolution partial differential equation

$$u_t + 6u^2 u_x + \alpha u_{xxx} - \beta u_{xxxxx} = 0 \tag{1}$$

describes the evolution of nonlinear waves in medium having negative type dispersion ($\alpha > 0, \beta > 0$) or mixed one ($\alpha < 0, \beta > 0$). We call (1) generalized modified version of the KDV equation because for $\alpha = 0$ and $\beta \neq 0$ it belongs to the generalized equations of KDV and when $\alpha \neq 0$ and $\beta = 0$ it is of the class of the modified versions of the classic Korteveg-de-Vries equations. A number o authors have investigated (numerically or analytically) separately the modified equation or th egeneralized one. Kawahara [1] analyzed numerically a weakly nonlinear analogue of (1) and found th eexistence of two types of oscillatory monotonous-solitary waves, namely compressive and rare fictive ones, corresponding to medium with negative dispersion and such one with positive dispersion.

In the present analysis of the periodic wave packs the Toda's idea [2] has been used, summarized further by Parker [3] for presenting solitar waves as a boundary case for the periodic ones and known as the nonlinear principle of superposition. As it was shown by P a r k e r, there is no superposition for the evolution equation ILW in the common sense of the linear theory, but superposition of their forms. This is due to the different (in the general case) phase velocities of the periodic and solitary waves.

2. Periodic waves

We search for a particular solution of (1) in the form of a "running" wave

$$u(x,t) = f(z), \quad z = k(x - \omega t) + C_0, \tag{2}$$

where in the general case $k \in \mathbb{C}$ is the wave number, ω is the phase velocity and the integration constant C_0 (that could be complex in the general case) characterizes the phase shift. After substituting $u(x,t)$ from (2) into (1) and integrating once, we obtain the nonlinear equation for the uknown function $f(z)$:

$$3f^3(z) - \omega f(z) + C = \beta k^4 f^{iv}(z) - \alpha k^2 f''(z). \tag{3}$$

Despite the lack of dynamic sense for the constant C, below we show that for periodic solutions (generating solitary waves under certain conditions) this constant is different from zero. We look for a periodic solution of (3) in the form

$$f(z) = A\, cn^2(z,m) + B, \tag{4}$$

where A and B are constants for which it is supposed that $A \neq 0$ and $cn(z,m)$ is the second elliptic Jacobi's fuinction with modulus m, for which $m^2 \leq 1$. Instead of $cn(z,m)$, we could use each of the other two elliptic functions $sn(z,m)$, having in mind the following relations

$$sn^2(z,m) + cn^2(z,m) = m^2 sn^2(z,m) + dn^2(z,m) = 1.$$

In the general case A and B are complex, as well as the phase variable z. If we substitute $f(z)$ from (4) into (3) and make use of the basic differential equation for cn:

$$\left[cn'(z,m)\right]^2 = \left[1 - cn^2(z,m)\right]\left[(1-m^2) + m^2 cn^2(z,m)\right], \tag{5}$$

the left and the right-hand side of (3) transform themselves identically into two polynomials of sixth degree with regard to $cn(z,m)$. Actually, the condition for identity of these polynomials is simultaneously a condition for the existence of a nontrivial biperiodic solution of type (4). The coefficients in front of the corresponding degrees form the following algebraic system for the unknown constants and parameters

$$(6)\quad \begin{aligned} &A(A^2 - 40\beta k^4 m^4) = 0,\\ &A[40\beta k^4 m^2(2m^2 - 1) - 2\alpha k^2 m^2 + 3AB] = 0,\\ &A[8\beta k^4(17m^4 - 17m^2 + 2) - 4\alpha k^2(2m^2 - 1) - 9B^2 + \omega] = 0,\\ &C = 2Ak^2(1 - m^2)[4\beta k^2(2m^2 - 1) - \alpha] + \omega B - 3B^3. \end{aligned}$$

Real biperiodic solutions could be realized not only when A, B and k are real under the condition $m, \omega \in \mathbb{R}$. In the general case, i.e. for complex values of z and $m^2 \le 1$,

$$(7)\qquad cn^2(z,m) = \frac{\mathcal{P}\left(z(e_1 - e_3)^{-1/2}, G_2, G_3\right) - e_1}{\mathcal{P}\left(z(e_1 - e_3)^{-1/2}, G_2, G_3\right) - e_3},$$

$cn^2(z,m)$ takes complex values. Therefore, if the constants A and B are complex, then evidently $z = k(x - \omega t) + C_0$ must take only such complex values for which $Acn^2(z,m) = \bar{B}$, that could be a restrictive solution without any physical adequacy. Let us suppose that $\beta < 0$ and $k = ik_0, k_0 \in \mathbb{R}$, hence $\beta k^4 > 0$ and then A will be purely imaginary number and also B would be found to be the same, according to the second equation of (6). According to (4) this means that the imaginary constants are binded with the phase variable z because it is necessary $A_0[\Re\,(cn^2) - \Im\,(cn^2)] + B_0 = 0$, where $A = iA_0$, $B = iB_0$; $A_0, B_0 \in \mathbb{R}$. We could come to analogical conclusion under the assumption that $\beta < 0$ but $k \in \mathbb{R}$, i.e. it is impossible $f(z)$ to be a real function. Within the context of the preceding considerations, we will analyze the solutions of the system obtained in the case that $\beta > 0$. When $k \in \mathbb{R}$ the solutions of (6) for A, B, ω, C determine the following real biperiodic solutions of (1):

(8)
$$u(x,t) = \pm 2k^2m^2(10\beta)^{1/2}cn^2[k(x - \omega t) + C_0] \pm 3(10\beta)^{-1/2}[\alpha - 20k^2\beta(2m^2 - 1)],$$

when the following dispersion relations are fulfilled

$$(9)\ \omega = (10\beta)^{-1}[\alpha - 20\beta k^2(2m^2 - 1)]^2 + 4\alpha k^2(2m^2 - 1) + 8\beta k^4(17m^4 - 17m^2 + 2);$$

$$(10)\quad \begin{aligned} C\ &= \pm 4(10\beta)^{1/2}k^4m^2(1 - m^2)[4\beta k^2(2m^2 - 1) - \alpha]\\ &+ B[6B^2 + 4\alpha k^2(2m^2 - 1) - 8\beta k^4(17m^4 - 17m^2 + 2)], \end{aligned}$$

where

$$(11)\qquad B = \pm(90\beta)^{-1/2}[\alpha - 20\beta k^2(2m^2 - 1)].$$

As the integrating constant $C(k,m)$ plays a key role for determining the character of the dispersive waves, so $B(k,m)$ is of decisive importance for determining the class of biperiodic waves that can generate solitary waves. In fact the biperiodic with respect to x waves, obtained in (8) are two families of two-parameter wave packs. It is not difficult to find for them that in the case of medium having negative dispersion ($\alpha > 0, \beta > 0$) for every $m \in [0.93, 1]$ the phase velocity $\omega(k,m) > 0$, i.e. the waves are propagating in the positive direction on the $0x$-axis, whille in the case of medium with mixed type dispersion ($\alpha < 0, \beta > 0$) for every $m \in [0.37, 1]$ and $\beta > |\alpha\ (2m^2 - 1)/[4k^2(17m^4 - 17m^2 + 2)]$ the phase velocity $\omega(k,m) < 0$ and the periodic wave is propagating in negative direction of $0x$.

The biperiodic solution (8) is not valid for every permissible value of the phase variable z, i.e. it is necessary to exclude the two-fold poles of $cn^2(z,m)$ out of its permissible arguments. Taking into account (7), we have the relation

$$k(x - \omega t) + C_0 \neq 2nK(m) + i(2s+1)K(\sqrt{1-m^2}). \tag{12}$$

where n and s are arbitrary integer numbers. We assume the phase shift C_0 real in the case under consideration, so that for $x \in \mathbb{R}$ and $t \geq 0$ the left-hand side of (12) is real, while the right-hand side is complex in the general case, because $K(m)$ represents the normal elliptic Jacobi's integral of first kind which is real:

$$K(m) = \int_0^{\pi/2} (1 - m^2 \sin^2 \varphi)^{-1/2} d\varphi.$$

Apparently for $\beta > 0, k \in \mathbb{R}$ and real phase shift C_0, the relation (12) is satisfied for every integer n and s, while for $C_0 \in \mathbb{C}$, (12) gives the restrictions for the phase variable.

So (8), together with the dispersion relations (9) and (10), determines two wave packs, symmetrically situated towards the abscissa. The wave lengths are $\sigma = 2\pi/k$, the primitive periods are two: $T_1 = 4K(m), T_2 = 2K(m) + 2iK(\sqrt{1-m^2})$, and their crests are in the points with abscissae $4nK(m)$, where $n = 0, \pm 1, \pm 2, \ldots$ This solution is bounded because for real z and $m^2 \in [0,1]$ the relation $0 \leq cn(z,m) \leq 1$ is valid.

3. Real solitary waves

The degeneration of the Jacobi's elliptic functions realizes when one or both primitive periods tend to ∞. In the first case, equivalent to the boundary values $m = 0$ and $m = 1$, we get a solitary wave solution of (1), i.e. for which

$|u(x,t)| < \infty$ when $|z| \to \infty$. For $m = 0$ solution (8) degenerates into a constant and this case is not interesting. Let $m = 1$ and take into account the equality

$$cn(z,1) = \mathrm{sech}(z),$$

reminding again that the phase variable $z = k(x - \omega t) + C_0$ takes real values for every $x \in \mathbb{R}$ and every $t \geq 0$. The biperiodic solution from (8) degenerates into a solitary solution of the form:

$$(13) \quad u(x,t) = \pm 2k^2(10\beta)^{1/2}\mathrm{sech}^2[k(x - \omega_1 t) + C_0] + (90\beta)^{1/2}(\alpha - 20\beta k^2),$$

if the following dispersion relations are fulfilled

$$(14) \quad \omega_1 = \omega(k,1) = (10\beta)^{-1}(\alpha - 20\beta k^2)^2 + 4k^2(\alpha + 4\beta k^2),$$

$$(15) \quad C_1 = C(k,1) = B(k,1)[6B^2(k,1) + 4k^2(\alpha - 4k^2\beta)],$$

and $B(k,m)$ is given in (11).

Let us consider again the periodic solution from (8) taken with a sign "+". Comparing the amplitudes of the periodic and the solitary wave , we will see that the second one has a greater amplitude than the periodic wave for every wave length σ, i.e. for every positive value of the wave number. When $m^2 \to 1$, the crests of the periodic and the solitary wave coincide as the solitary-wave forms are well separated for greater wave lenghts. This is the regime of the "great amplitude", for which the nonlinear effects are the most considerable. And vice versa, for smaller modulus $m^2 — 0$ and smaller wave lenghts the exponential solitary forms coincide more and more but their amplitudes increase. The propagation velocities of the periodic and the solitary wave are different in the general case. The phase velocity ω of the periodic wave from (8) depends on the wave number k, the modulus m and also, on the medium's dispersion characterstics. For example, if $\alpha > 0$ and $\beta > 0$ (i.e. medium with negative dispersion and $m \in [(1/2 + \alpha/(40\beta k^2))^{1/2}, 1]$), then for every $k > (\alpha/20\beta)^{1/2}\omega(m)$ from (8) is an increasing function of m and therefore,

$$(16) \quad \omega(k,m) < \omega(k,1),$$

i.e. the solitary wave has a greater phase velocity than the periodic one. For medium having mixed type dispersion ($\alpha < 0, \beta > 0$) and $0 < k < (|\alpha|/20\beta)^{1/2}$ for every m for which $0 < m < [1/2+\alpha/(40\beta k^2)^{1/2}]\omega(m)$ is a decreasing function of m and hence,

$$(17) \quad \omega(k,m) > \omega(k,1).$$

A considerable part of the nonlinear evolution equations possess this remarkable property: the periodic nonlinear wave can be presented as s superposition of solitary-wave forms, having in general different phase velocities. This principle was called by Toda [2] a nonlinear principle of superposition. For the long-wave evolution equation ILW it was shown by Boyd [4] and Parker [3] that even the solitary wave evolves like a sort of a "model" function, which repeating itself at equal intervals, generates a periodic wave.

4. One soliton wave in medium with negative dispersion

In the previos section it was found that in media with negative ($\alpha > 0, \beta > 0$) and mixed type ($\alpha < 0, \beta > 0$) dispersion, we have periodic and solitary wave packs generated, while in media with positive dispersion ($\alpha < 0, \beta < 0$) they can not realize themselves. In the case of a medium with negative dispersion we will show that for a given wave number k there exists a solitary solution of (1). Actually, if

$$k = \sqrt{\alpha/(20\beta)}, \tag{18}$$

supposing $\alpha > 0$ and $\beta > 0$, the solution of (13) takes the form:

$$u(x,t) = \alpha(10\beta)^{-1/2}\mathrm{sech}^2[\alpha(20\beta)^{-1/2}(x - 6\alpha^2(25\beta)^{-1}t) + C_0], \tag{19}$$

where $C_0 = \mathrm{const}$ and $C_0 \in \mathbb{R}$. This solution is one-soliton. A common and very typical pecularity of the nonlinear waves reveals for them: a dependence between the amplitude and the phase velocity and also that the impulses with greater amplitudes propagate considerably faster but are narrower. Moreover, they have one-way propagation in the positive direction of the real abscissa because $\omega = 6\alpha^2/25\beta > 0$.

From physical point of view, possible soliton type solutions of (1) in medium with mixed dispersion ($\alpha < 0, \beta > 0$) would be also of interest. In this case the nullification of the integration constant C_1 takes place for purely imaginary values of the wave number

$$k = i\sqrt{(-\alpha)/(20\beta)} \tag{20}$$

for which the phase variable $z + C_0$ is complex or purely imaginary depending on the choice of the free constant C_0. This follows from the fact that the phase velocity ω_1 from (14).is real: $\omega_1 = -\alpha^2/5$. Let C_0 take purely imaginary values, i.e. $C_0 = iC_2$ and $C_2 \in \mathbb{R}$, then the analytical solitary solution (9) has a degeneration due to the obvious relations

$$\mathrm{sech}^2(iy) = \mathrm{cosec}^2 y.$$

In this case we obtain a real singular solution of (1) in the form:

$$(21)\quad u(x,t) = \pm 2k^2(10\beta)^{1/2}\mathrm{cosec}^2[k(x-\omega_1 t)+C_0] + 3(10\beta)^{1/2}(\alpha - 20\beta k^2),$$

if the dispersion relations (14) and (15) are fulfilled.

The presence of singular (or nonsingular, for $C_0 \in \mathbb{R}$) rational solutions of the nonlinear evolution equations is an unusual their property. It consists in the following - in all the cases when such solutions exist, a transformation is defined between the evolution with t, described by the partial differential equation, and the evolution of a dynamic system with finite number degrees of freedom. Such rational solutions have been analyzed for the classical equation of KDV by Airault, McKean and Moser [5] as well as by Ablowitz and Satsuma [6.]

There it is shown that the rational solution for the equation KDV can be obtained as a long-wave transition in the one-soliton solution. We have a similar result in (15). A purely imaginary value of the wave number k from (14) is equivalent to a long-wave boundary transition, because $\sigma = 2\pi/\Re\,(k) \to \infty$ when $\Re\,(k) \to 0$.

To avoid twofold poles of the rational solution from (15), it is necessary to restrict the phase variable $z = k(x - \alpha^2 t/5) + C_2$, namely:

$$(22)\quad C_3 + 2n)\sqrt{5\beta/(-\alpha)} < x - \alpha^2 t/5 < [C_3 + 2(n+1)]\sqrt{5\beta/(-\alpha)}$$

for $n = 0, \pm 1, \pm 2, \dots$ and $C_3 = 2C_2 - \pi$.

5. Concluding remarks and summaries

In the present paper we have found analytically stationary-periodic solutions and their restrictions for a generalized and modified equation of KDV. An "elliptic" procedure initiated by Kano and Nakayama [7] has been used for analyzing equations of the form

$$(23)\quad u_t + P(u, n+1)_x = u_{(2s+1)x},$$

where $P(u, n+1)_x$ is a polynomial of u of $(n+1)$-st order. Two classes of biperiodic solutions (8) have been obtained. Apparently they can be applied for periodic and solitary wave solutions in the boundary case $\alpha = 0$, i.e. for the equation

$$(24)\quad u_t + 6u^2 u_x = \beta u_{xxxxx},$$

but in this case soliton and rational solutions can not be generated according to (19) and (21). The case $\beta = 0$ and $\alpha \neq 0$ is the well-known modified equation of KDV

$$u_t + 6u^2 u_x + \alpha u_{xxx} = 0. \tag{25}$$

This equation occurs with a sign "-" in front of the nonlinear term and the existence of a soliton solution has been proved first by Wadati [8]. Later on, Ablowitz, Kruskal and Segur [9] have shown its integrability by the help of Miura's transformation [10], giving a dependence between the modified and the classical equation of KDV.

The analytical solutions of (1) found, can be modified easily for the equation (1) with a sign "-" in front of the nonlinear term, i.e. for the nonlinear equation

$$u_t - 6u^2 u_x + \alpha u_{xxx} = \beta u_{xxxxx}, \tag{26}$$

because after the following transformation

$$t \mapsto -t, x \mapsto -x, u \mapsto u, \alpha \mapsto -\alpha, \beta \mapsto -\beta,$$

equation (1) coincides with (26).

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Received 08.05.1995