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Almost Constant Families of Functions

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If $\mathcal{F}$ is a set of functions with domain A, it is said that $\mathcal{F}$ is almost constant if there is a nonprincipal ultrafilter $\mathcal{D}$ on A such that each $f \in \mathcal{F}$ is constant on some $X \in \mathcal{D}$. We consider properties of such families and related constructions.

Let $\mathcal{D}$ be a filter over a set A and F be a set of functions from A to a set B.

Definition 1. The family $\mathcal{F}$ is constant on $\mathcal{D}$, if for every $f \in \mathcal{F}$ there is $X \in \mathcal{D}$ such that $f|X$ is a constant function.

In this article we shall consider various pairs $(\mathcal{F}, \mathcal{D})$ such that $\mathcal{D}$ is a nonprincipal ultrafilter over a set A, and $\mathcal{F}$ is a set of functions with domain A, constant on $\mathcal{D}$. If $\mathcal{F}$ is a set of functions with domain A, and there is such a pair $(\mathcal{F}, \mathcal{D})$, then $\mathcal{F}$ is called the almost constant family. As we shall see, some of such pairs appear in constructions of ultraproducts of particular types.

In the following, we shall assume notions and notations from set theory as in [3], and from model theory as in [1]. For example, every ordinal consists from all smaller ordinals. If $f: A \longrightarrow B$, and $x \in B$, then $f^{-1}[x]$ denotes the set $\{a \in A | f(a) = x\}$.

Lemma 2. *Assume k is an infinite regular cardinal number. Let $A \subseteq k$ be an unbounded subset, and $f: k \longrightarrow k$ a bounded function. Then there is unbounded $B \subseteq A$ such that $f|B$ is constant.*

Proof. By assumption there is $\lambda < k$ such that for $\alpha \in k$, $f(\alpha) \leq \lambda$. Assume that for all $x \in k$

$$A \cap f^{-1}[x] \quad \text{is bounded.}$$

Thus for every $x \leq \lambda$ there is $y_x < k$ such that

$$\text{for all } a \in A \cap f^{-1}[x], \; a < y_x.$$

By regularity of k, there is $b < k$ such that $y_x < b$ for all $x \leq \lambda$, so

$$\bigcup_{x \leq \lambda}(A \cap f^{-1}[x]) \subseteq b, \text{ i.e.}$$
$$A \cap (\bigcup_{x \leq \lambda} f^{-1}[x]) \subseteq b.$$

As $\bigcup_{x \leq \lambda} f^{-1}[x] = k$, it follows $A \subseteq b$, i.e. A is bounded in k, and this is a contradiction. Therefore, there is $x \in k$ such that $A \cap f^{-1}[x]$ is unbounded in k, so for this x, $B = A \cap f^{-1}[x]$ satisfies the lemma. ∎

As a consequence of this lemma, we have the following theorem.

Theorem 3. *Let k be an infinite regular cardinal number, and $\mathcal{F}$ be a countable set of bounded functions $f : k \longrightarrow k$. Then there is a nonprincipal ultrafilter $\mathcal{D}$ over k such that $\mathcal{F}$ is constant on $\mathcal{D}$.*

Proof. Let $\mathcal{F} = \{f_0, f_1, f_2, \ldots\}$, and C_n, $n \in \omega$, be a sequence of subsets of k such that each C_n is unbounded, and $C_0 \supseteq C_1 \supseteq C_2 \ldots$, and for each $n \in \omega$, $f_n | C_n$ is constant. Such a sequence exists by the previous lemma. Then the family $\mathcal{S} = \{C_0, C_1, C_2, \ldots\} \cup \{[a, k) | a < k\}$ has the finite intersection property, so any ultrafilter $\mathcal{D}$ containing $\mathcal{S}$ satisfies the lemma. ∎

The previous theorem is also true under some other circumstances. For example, we have the following propositions.

Proposition 4. *Let k be an infinite regular cardinal number, and $\mathcal{F}$ the set of all bounded functions $f : k \longrightarrow k$, continuous under the order topology of k. Then $\mathcal{F}$ is constant on the c.u.b. filter* $\mathrm{Cub}(k)$ *over k.*

Proof. If $f : k \longrightarrow k$ is continuous and bounded function, then for every $x \in k$, $f^{-1}[x]$ is a closed subset of k. Since f is bounded and k is regular, there is $\omega \in k$ such that $f^{-1}[\omega]$ is unbounded in k, and therefore $C = f^{-1}[\omega]$ is c.u.b. and $f | C$ is constant. ∎

It is easy to see that each continuous $f : k \longrightarrow k$ is in fact constant on a final segment of k.

Proposition 5. *Let $\mathcal{F}$ be a countable set of regressive functions $f : k \longrightarrow k$, where k is regular uncountable cardinal. Then $\mathcal{F}$ is almost constant.*

Proof. Let $\mathcal{F} = \{f_0, f_1, \ldots\}$. By use of Fodor's Lemma (Pressing-Down Lemma) there is a sequence of stationary sets: $k \supseteq C_0 \supseteq C_1 \supseteq \ldots$ such that the restriction of f_n to C_n is constant. Then $\mathrm{Cub}(k) \cup \{C_0, C_1, \ldots\}$ has the finite intersection property, so $\mathcal{F}$ is constant on any ultrafilter extending this set. ∎

In a similar way, as in the previous proposition, one can prove that the set $\mathcal{F}$ of all continuous functions $f : k \longrightarrow R$ is almost constant, where k is an uncountable cardinal such that $cf(k) > \omega$, and R is the set of reals supplied with the usual topology.

There is also a definable versions of Theorem 3. Namely, we have the following proposition.

Lemma 6. *Let $\mathcal{M} = (M, \leq, \ldots)$ be a linearly ordered model without the greatest element, $\mathcal{D}$ the Boolean algebra of all definable subsets of $\mathcal{M}$ with parameters in M, and $S = \{g_0, g_1, \ldots\}$ a countable family of bounded definable functions in $\mathcal{M}$ with parameters in M. If for all formulas of L*

$$\text{(R)} \qquad \mathcal{M} \models \forall x \leq z \quad \exists y \, \varphi \longrightarrow \exists u \quad \forall x \leq z \quad \exists y \leq u \varphi,$$

then there is nonprincipal ultrafilter $\mathcal{U}$ of $\mathcal{D}$ such that $\mathcal{S}$ is constant on $\mathcal{U}$.

First we prove that the following variant of Lemma 2.

Claim. *$X \in \mathcal{D}$ is bounded and $f : M \longrightarrow M$ is definable and bounded, then there is an unbounded $Y \subseteq X$, $Y \in \mathcal{D}$ such that $f|Y =$ const.*

Proof of Claim. There are two possibilities:

Case 1. There is $a \in M$ such that $f^{-1}[a] \cap X$ is unbounded. Then we can take $Y = f^{-1}[a] \cap X$.

Case 2. For all $a \in M$, $f^{-1}[a] \cap X$ is bounded. Thus, as X and f are definable, we can write informally

$$\mathcal{M} \models \forall x \leq m \quad \exists y \, f^{-1}[x] \cap X \subseteq \{v \in M | \; v \leq y\},$$

where $m \in M$ is a bound of f. Since $\mathcal{M}$ satisfies scheme (R), there is $u \in M$ such that

$$\mathcal{M} \models \forall x \leq m \quad \exists y \leq u \, f^{-1}[x] \cap X \subseteq \{v \in M | \; v \leq y\}.$$

Since $X = \bigcup_{x \leq m}(f^{-1}[x] \cap X) \subset \bigcup_{y \leq u}\{x \in M | x \leq y\} \subseteq \{x \in M | \; x \leq u\}$ it follows $X \subseteq \{x \in M | \; x \leq u\}$, and this contradicts to the assumption that X is unbounded. Thus, Case 2 is impossible, i.e. Claim holds.

Proof of Lemma 6. By the above claim we can construct a sequence of unbounded definable subsets of A such that

$$X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots \text{ and } g_i|X_i = \text{const}.$$

Since X_i is unbounded, $(a, \infty) \cap X_i \neq \emptyset$. Therefore, the family $\{X_i | \; i \in \omega\} \cup \{(a, \infty) | \; a \in M\}$ has the finite intersection property, hence there is an ultrafilter $\mathcal{U}$ of $\mathcal{D}$ containing all sets X_i and (a, ∞).

Let $\mathcal{M}$, $\mathcal{D}$ and $\mathcal{U}$ be as in previous lemma, and let $\mathcal{F}$ be the set of all functions definable in $\mathcal{M}$ with parameters in M. Observe that the identity function $i : M \longrightarrow M$ and $\hat{a} = \langle a|\ j \in M\rangle$ belong to $\mathcal{F}$. For $f \in \mathcal{F}$ let $f_{\mathcal{U}}$ denote the equivalence class $\{g \in \mathcal{F}|g = f \operatorname{mod}\mathcal{U}\}$. If $\mathcal{M}$ has built in Skolem functions, then the Skolem ultrapower $\mathcal{F}/\mathcal{U}$ is a model of the language L, and it has the following additional properties.

(1) By Łoš theorem for Skolem ultrapowers, the mapping $\mu : a \longrightarrow \hat{a}_{\mathcal{U}}$, $a \in M$, is an elementary embedding of $\mathcal{M}$ into $\mathcal{F}/\mathcal{U}$.

(2) For every $a \in M$, $\hat{a} \leq i \operatorname{mod}\mathcal{U}$ since $(a, \infty) \in \mathcal{U}$. Thus $i_{\mathcal{U}} \in \mathcal{F}/\mathcal{U}\backslash M$, i.e. $\mathcal{F}/\mathcal{U}$ is a proper extension of $\mathcal{M}$.

(3) If for $f \in \mathcal{F}$ there is $b \in M$ such that $f_{\mathcal{U}} \leq \hat{b}_{\mathcal{U}}$, then there is $g \in \mathcal{F}$ such that $f = g \operatorname{mod}\mathcal{U}$, and $g(i) \leq b$ for all $i \leq a$. Hence $g = \text{const} \operatorname{mod}\mathcal{U}$. Thus, there is $a \in M$ such that $f_{\mathcal{U}} = \hat{a}$. Therefore, M is an initial segment of $\mathcal{F}/\mathcal{U}$, i.e. $\mathcal{F}/\mathcal{U}$ is an end extension of $\mathcal{M}$.

The above consideration is summarized in the following

Corollary 7. *Let $\mathcal{M} = (M, \leq, \ldots)$ be a countable, linearly ordered model without the greatest element of a countable language L. If $\mathcal{M}$ has built in Skolem functions and $\mathcal{M}$ satisfies*

$$\forall x \leq z\ \exists y\, \varphi \longrightarrow \exists u\ \forall x \leq z\ \exists y \leq u \varphi,$$

where φ is a formula of L and u is a variable not occuring in φ, then $\mathcal{M}$ has a proper countable elementary end extension.

This corollary can be derived under weaker assumptions, i.e. it is not necessary to assume that the model $\mathcal{U}$ has built in Skolem functions, see [4]. However, the theorem in [4] is proved by use of a quite different technique, i.e. using the omitting types theorem. As an illustration of use of Corollary 7, we shall prove Keisler's two cardinal theorem:

Theorem 8. (J. Keisler) *Let $\mathcal{U} = (A, V, \ldots)$ be a model of a countable language L such that $\aleph_0 \leq |V| < |A|$. Then, there are models $\mathcal{B} = (B, W, \ldots)$ and $\mathcal{C} = (C, W, \ldots)$ such that $\mathcal{B} \prec \mathcal{U}$, $\mathcal{B} \prec \mathcal{C}$ and $|B| = \aleph_0, |C| = \aleph_1$.*

P r o o f. According to the downward Löwenheim-Skolem theorem we may assume that $|A| = k^+$ for some cardinal k. Let $\leq$ be a linear ordering of domain A of the order type k^+, and let $\mathcal{U}^S$ be the Skolem expansion of model $(A, \leq, V, \ldots)$, where V is an interpretation of an unary predicate symbol $P \in L$. Then $\mathcal{U}^S$ has built in Skolem functions, and since k^+ is a regular cardinal, $\mathcal{U}^S$ satisfies scheme (R). By the downward Löwenheim-Skolem theorem, there is a countable $\mathcal{B}^S \prec \mathcal{U}^S$ in which P is interpreted by a subset $W \subseteq B$. Then $\mathcal{B}^S$ has, obviously, built in Skolem functions as well, and it satisfies (R). Also, W is bouunded in $\mathcal{B}^s$ since $\mathcal{U} \models \exists x \forall y(P(x) \longrightarrow y \leq x)$. Applying Corollary 7 ω_1-times, there is an ω_1-like elementary end-extension $\mathcal{C}^S$ of $\mathcal{B}^S$. Then, $\mathcal{C} = \mathcal{C}^S|L$ is the required model.

In the literature there are known several proofs of this theorem. However, all these proofs use some form of the omitting types theorem. The proof presented in this paper does not rely on this theorem. Proofs of the similar nature can be found elsewhere. For example, in Lectures 33 and 34 in [2], where end extensions of models of set theory are discussed. Now we continue to consider some other vaariants of Lemma 2 and Theorem 3.

Lemma 9. *Let k, λ be cardinals such that k is infinite, and $\lambda > 0$, and β be an ordinal such that $|\beta|^\lambda < cf(k)$. Further, let A be a set of the cardinality k, $\mathcal{F}$ be a set of functions $f : A \longrightarrow \beta$ of the cardinality λ. Then, there is $B \subseteq A, |B| = k$, such that for all $f \in \mathcal{F}$, $f|B$ is constant.*

Proof. Let $\mathcal{F} = \{f_\alpha | \alpha < \lambda\}$, $f_\alpha : A \longrightarrow \beta$. Further, let for each $a \in A, g_a$ be defined by

$$g_a(\alpha) = f_\alpha(a), \quad \alpha < \lambda, \quad a \in A.$$

Then, $g_a : \lambda \longrightarrow \beta$, so for $\mathcal{S} = \{g_a | a \in A\}$, $|\mathcal{S}| \leq |\beta|^\lambda$. Since $|\beta|^\lambda < cf(k)$, there is g such that $|\{a \in A : g_a = g\}| = k$. If $\mathcal{B} = \{a \in A | g_a = g\}$, then for each $a \in B, f_\alpha(a) = g_a(\alpha)$, i.e. $f_\alpha|B$ is constant. ■

An immediate consequence of the above lemma is the following

Corollary 10. *Under assumptions of Lemma 4, there is a nonprincipal ultrafilter $\mathcal{D}$ over k such that $\mathcal{F}$ is constant on $\mathcal{D}$.*

Namely, $\mathcal{D}$ is any nonprincipal ultrafilter concentrated on B for A and B in the previous lemma, taking $A = k$.

We also remark the following facts concerning Lemma 9. If k is regular, then we may assume $|\beta|^\lambda < k$, instead of $|\beta|^\lambda < cf(k)$. If k is an inaccessible cardinal, we may assume β, $\lambda < k$ instead of $|\beta|^\lambda < k$. We have also the following variant of Lemma 9.

Corollary 11. *Let k be an infinite cardinal and $\lambda < cf(k)$, $\mathcal{F}$ be a family of bouunded functions $f : A \longrightarrow k$, where $|A| = k$, and $|\mathcal{F}| = \lambda$. Suppose for all cardinals $\mu < k$, $\mu^\lambda < k$, then there is $B \subseteq A$, $|B| = k$, and $\mathcal{F}$ is constant on B.*

Proof. In order to prove this corollary it suffices, according to Lemma 8, to find $\beta < k$ be such that each $f \in \mathcal{F}$ maps k into β. So let $\mathcal{F} = \{f_\alpha | \alpha < \lambda\}$, and for $\alpha < \lambda$ let $b_\alpha < k$ be such that $f_\alpha(a) \leq b_\alpha, a \in A$. Then $\sup_{\alpha<\lambda} b_\alpha < k$ as $\lambda < cf(k)$, hence we may apply Lemma 8. ■

Corollary 12. *Suppose K is an uncountable cardinal such that $cf(k) = \omega$, and let $\mathcal{F}$ be a family of functions $f : k \longrightarrow \beta$ such that $|\mathcal{F}| = k$, where β is an ordinal less than k. If for all $\lambda < k$, $|\beta|^\lambda < k$, then there is a nonprincipal ultrafilter $\mathcal{D}$ over k such that $\mathcal{F}$ is constant on $\mathcal{D}$.*

Proof. There are $\mathcal{F}_i, i \in \omega$, such that $|\mathcal{F}| = \lambda_i$, $\lambda_i < k$, and $\bigcup_{i \in \omega} \mathcal{F}_i = k$. By Lemma 8 there is a sequence of sets $k \supseteq C_0 \supseteq C_1 \supseteq \ldots$ such that $\mathcal{F}_i$ is constant on C_i. Then on any ultrafilter $\mathcal{D}$ containing $C_0, C_1, C_2, \ldots$ and $[a, k]$, $a \in k$, $\mathcal{F}$ is constant. ■

Additional hypothesis of set theory may yiield almost constant families of functions, as the following proposition shows.

Theorem 13. (under GCH) *If k is an infinite cardinal and $\mathcal{F}$ is a set of bounded functions $f : k^{++} \longrightarrow k^{++}$, $|\mathcal{F}| = k$, then there is an ultrafilter $\mathcal{D}$ over k^{++} such that $\mathcal{F}$ is constant on $\mathcal{D}$.*

Proof. Let $\mathcal{F} = \{f_\alpha | \alpha < k\}$. As $f \in \mathcal{F}$ are bounded, there is $\beta \in k^{++}$, i.e. $|\beta| \leq k^+$, such that $f_\alpha : k^{++} \longrightarrow \beta$. Define g_γ as follows:

$$g_\gamma(\alpha) = f_\alpha(\gamma), \quad \gamma < k^{++}, \quad \alpha < k.$$

Then $g_\gamma : k \longrightarrow \beta$. Therefore,

$$|\{g_\gamma : \gamma \longrightarrow k^{++}\}| \leq |\beta|^k \leq |k^+|^k = 2^k.$$

Thus, $|\{g_\gamma : \gamma < k^{++}\}| \leq 2^k = k^+$. If a map G is defined by $G(\gamma) = g_\gamma$, it follows that there is g such that $|G^{-1}[g]| = k^{++}$, i.e. $|\{\gamma < k^{++} : g_\gamma = g\}| = k^{++}$. If $A = \{\gamma < k^{++} : g_\gamma = g\}$, then for $\gamma \in A$, and arbitrary $\alpha \in k$, $f_\alpha(\gamma) = g_\gamma(\alpha) = g(\alpha)$, i.e. $f_\alpha | A$ is constant. If $\mathcal{D}$ is any nonprincipal ultrafilter concentrated on A, then $\mathcal{D}$ satisfies the conditions of the theorem. ■

The above considerations show that pairs of cardinals defined as follows may be of an interest.

Let k, λ be cardinals. The pair (λ, k) is ait acceptable, if every family $\mathcal{F}$ of bounded functions $f : k \longrightarrow k$ such that $|\mathcal{F}| \leq \lambda$ is almost constant on k. Results presented in this paper show that

1. If k is a regular cardinal, then by Theorem 3, (ω, k) is acceptable;
2. By Corollary 10, (λ, k) is acceptable, if $\lambda < cf(k)$, and for all $\mu < k, \mu^\lambda < k$;
3. Under GCH, by Theorem 13 any pair (k, k^{++}) is acceptable for infinite k.

It would be interesting to describe all acceptable pairs of cardinals.

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