Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol.11, 1997, Fasc. 3-4

On the Integrability Conditions and Structure Properties for Almost Contact Metric Manifolds

M. Hristov

Presented by P. Kenderov

The relations between the integrability conditions for almost complex and for almost contact structures on differentiable manifold are given. The locally structure of the basic classes of almost contact metric manifold with respect to product of an almost Hermitian manifold and the real line is established.

1. Introduction

Let M be a 2n-dimensional differentiable manifold with *almost Hermitian structure* (J, g), where J is an almost complex structure tensor field of type (1.1), g is a definit metric tensor field, such that

$$J^2 = -id, \ g(x, y) = g(Jx, Jy), \ x, y \in T_pM.$$

By using the algebraic decomposition of the set of 3-covariant tensors on an almost Hermitian vector space, having the symmeries of the covariant derivative of the Kähler 2-form, into multually orthogonal and $U(n)$-invariant irreducible factors, Gray and Hervella have classified the set

$$W = \{M^{2n}(J, g), \ n \geq 3\}$$

of almost Hermitian manifolds into sexteen classes [1]. In Table 1 the four basic classes of such a manifolds are characterized by the equations for the essential (which may be zero) components of the well known fundamental tensors, arrizing on an almost Hermitian manifold, with respect to the complex bases of the complexification T_p^cM of the tangential spaces

$$T_pM = \text{ span } \{e_\alpha, Je_\alpha, \alpha \in I\} - \text{ orthonormal}$$

$$T_p^c M = T_p^{10} \oplus T_p^{01}$$

$= \text{ span } \{Z_\alpha = e_\alpha - iJe_\alpha, \alpha \in I\} \oplus \text{ span } \{Z_{\overline{\alpha}} = e_\alpha + iJe_\alpha, \overline{\alpha} \in \overline{I} = \text{ span } \{Z_\alpha, \alpha \in I \cup \overline{I}$

The index sets I and $\overline{I}$ are $\{1, 2, \dots, n\}$ and $\{\overline{1}, \overline{2}, \dots \overline{n}\}$ respectively.

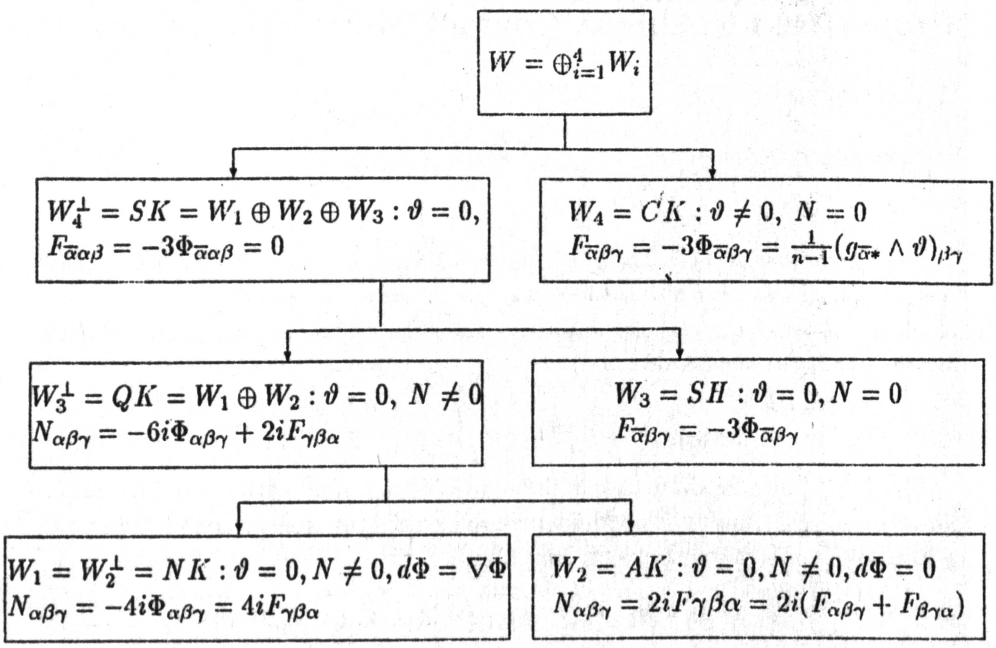

Table 1.

Here $\Phi\{\Phi_{\alpha\overline{\beta}}\}$: $\Phi(x, y) = g(x, Jy)$ is the fundamental (Kähler) 2-form, $F = -\nabla\Phi\{F_{\alpha\beta\gamma}\}$, ($\nabla$-Levi-Chivita connection), $d\Phi\{\Phi_{abc}\}$ the exterior differential of Φ, $N = [J, J]$ $\{N_{\alpha\beta}^{\overline{\sigma}}, N_{\alpha,\beta,\gamma} = N_{\alpha\beta}^{\overline{\sigma}} g_{\gamma\overline{\sigma}}\}$ the Nijenhuis tensor field and $\vartheta\{\vartheta_\beta, \vartheta_{\overline{\beta}} = \overline{\vartheta_\beta}\}$ is the Lee form:

$$\vartheta = \frac{-1}{2(n-1)} \delta\Phi \circ J, \text{ where } \delta\Phi(Z) = -\sum_{\alpha=1}^n [(\nabla_{e_\alpha}\Phi)(e_\alpha, Z) + (\nabla_{Je_\alpha}\Phi)(Je_\alpha, Z)],$$

$$\vartheta_\beta = \frac{i}{4(n-1)} \sum_{\alpha=1}^n F_{\overline{\alpha}\alpha\beta} = \frac{-3i}{4(n-1)} \sum_{\alpha=1}^n \Phi_{\overline{\alpha}\alpha\beta}.$$

2. Integrability conditions for almost complex structures

The four basic classes of Hermitian manifolds: $N = [J, J] = 0$ are the classes K of Kähler manifolds, W_3 of special Hermitian manifolds, W_4 of locally conformal Kähler manifolds and $W_3 \oplus W_4$. The well known Newlender-Nirenberg's theorem implies that the vanishing of the Nijenhuis tensor field N is necessary and sufficient condition an almost complex structure J (resp. manifold $M^{2n}(J)$) to be a complex structure (resp. manifold), i.e. to carry differentiable structure , compatible with the pseudogroup of holomorphic transformations in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$

$$H_{2n} = \{f : \mathbb{R}^{2n}(J_\circ) \to \mathbb{R}^{2n}(J_\circ) : f_* \circ J_\circ = J_\circ f_*\},$$

where $J_\circ$ is the standard almost complex structure on

$$\mathbb{R}^{2n} = \{(x^\alpha, y^\alpha = x^{n+\alpha}), \alpha \in I\} : J_\circ \begin{pmatrix} \partial/\partial x^\alpha \\ \partial/\partial y^\alpha \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial/\partial x^\alpha \\ \partial/\partial y^\alpha \end{pmatrix}, \alpha \in$$

If we put $[Z_a, Z_b] = C^c_{ab} Z_c$, then we have

$$(2.1) \qquad \begin{pmatrix} [Z_\alpha & Z_\beta] \\ [Z_\alpha & Z_{\overline{\beta}}] \end{pmatrix} = \begin{pmatrix} C^\sigma_{\alpha\beta} & -\frac{1}{4} N^{\overline{\sigma}}_{\alpha\beta} \\ C^\sigma_{\alpha\overline{\beta}} & C^{\overline{\sigma}}_{\alpha\overline{\beta}} \end{pmatrix} \begin{pmatrix} Z_\sigma \\ Z_{\overline{\sigma}} \end{pmatrix}.$$

Frölicher in [2] has proved that the vanishing of N on a real-analytical manifold with real-analytical almost complex structure J is necessary and sufficient condition the Pfaffian system

$$(2.2) \qquad (J^j_k + i\delta^j_k) dx^k = 0, \; j, k \in \{1, 2, \ldots, 2n\},$$

to be integrable, i.e. the null space $\mathcal{NJ}$ of the set $\mathcal{J}$ of 1-forms $\omega^j = (J^j_k + i\delta^j_k) dx^k \in \chi^* U$ to be an involutive differentiable distribution, or equivalently the set $\mathcal{J} = \{\omega^1, \ldots, \omega^{2n}\}$ to be a differential ideal. Here δ^j_k are the Kroneker's symbols and J^j_k :

$$J(\partial/\partial x^k) = J^j_k \partial/\partial x^j, \; J^S_k J^j_S = -\delta^j_k$$

are the components of J with respect to local real coordinate system $(U, \{x^\alpha, y^\alpha = x^{n+\alpha}\}), \alpha \in I$.

Indeed, the vector fields $\xi^k = (J^k_s - i\delta^k_s)\partial/\partial x^s \in \chi U$ belongs to $\mathcal{NJ}$, the integral curves $(x^1(t), x^2(t), \ldots x^{2n}(t)$ of ξ_k are $\mathbb{C}$- valued, i.e. J^j_k are uniquely extendible on complex arguments and rank $(J^j_k - i\delta^j_k)$ =rank $(J^j_k - i\delta^j_k) = n$, i.e. $\mathcal{NJ}$ is spanned on ξ^k. The Frobenius conditions for the totally integrability of (2.2), written in the sence $\mathcal{J}$ to be differential ideal, are $d\omega^j = \omega^j_i \wedge \omega^i$ for some

$\omega_i^j \in \chi^* U$, i.e. $d\omega^j(\xi^k.\xi^l) = 0$ for $\xi^k, \xi^l \in \mathcal{N}\mathcal{J}$. These conditions are expressible as

$$(\partial J_k^j/\partial x^l - \partial J_l^j/\partial x^k)(J_p^k - i\delta_p^k)(J_q^l - i\delta_q^l) = (J_p^k - i\delta_p^k)N_{qk}^j = 0$$

and are equivalent to the vanishing of N [2]. Further, for arbitrary local coordinate system $(U, \{x^\alpha, y^\alpha = x^{n+\alpha}\})$, $\alpha \in I$, there exists open subset V and $\mathbb{R}$-analytical transformation

$$\psi : U \to V$$

$$(x^\alpha, x^{n+\alpha} \to u^\beta(x^\alpha, x^{n+\alpha}), v^\beta(x^\alpha, x^{n+\alpha})),\ \alpha, \beta \in I$$

with almost pluryharmonical functions u^β, v^β, such that for the corresponding almost holomorphical functions $z^\beta = u^\beta + iv^\beta$ the system

$$dz^\beta = 0,\ \beta \in I \tag{2.3}$$

is equivalent to (2.2). Realy, the null spaces of (2.2) and (2.3) coincide iff $dz^\beta(\xi^k) = 0$ and $\omega^k(\partial/\partial \overline{z^\beta}) = 0$ or equivalently the Jacobian of ψ to be nonvanishing and u^β, v^β to satisfy

$$dv^\beta = -J_k^j(\partial u^\beta/\partial x^j)dx^k \iff dv^\beta = -Jdu^\beta.$$

Locally, on open subset U at every point, the functions u^β, v^β are uniquelly determined - u^β is a solution of the elliptic differential equation $dJdu^\beta = 0$, [2], [5], which is necessary and sufficient condition to be the integral, defining the conjgated function v^β, not depending on the integral way. $(V = \psi(U), \{u^\beta, v^\beta\}), \beta \in I$ is local coordinate system, and so $(V, \{z^\beta = u^\beta + iv^\beta\})$, $\beta \in I$ is local complex coordinate system iff $N \equiv 0$ [2].

From the above explanations and (2.1), we get the following

Corollary 2.1. *Let $M^{2n}(J)$ be a real analytical almost complex structure manifold with real analytical almost complex structure J. The following implication is valid $N \equiv 0 \Rightarrow C_{\alpha\beta}^\sigma = C_{\alpha\overline{\beta}}^{\overline{\sigma}} = -\overline{C_{\beta\overline{\alpha}}^{\sigma}} \equiv 0$, i.e. locally there exists $\mathbb{R}$-analytical transformation ψ on M*

$$(U, \{x^\alpha, y^\alpha = x^{n+\alpha}\}) \xrightarrow[dv^\beta - Jdu^\beta]{\psi} (V, \{u^\beta(x^\alpha, x^{n+\alpha}), v^\beta(x^\alpha, x^{n+\alpha})\}),$$

$$1-1 \updownarrow \quad dJdu^\beta = i\partial\overline{\partial}u^\beta = 0 \quad \updownarrow 1-1$$

$$(U, \{\xi^\alpha = x^\alpha + iy^\alpha\}) \xrightarrow[\substack{almost\\ holomorphic}]{\psi^c} (V, \{z^\alpha = u^\alpha + iv^\alpha\}),\ \alpha, \beta \in I,$$

so that

$$T^c_p M = T^{10}_p \oplus T^{01}_p = \mathrm{span}\{Z_\alpha = \partial/\partial z^\alpha, \alpha \in I\} \oplus \mathrm{span}\{Z_{\overline{\alpha}} = \partial/\partial z^{\overline{\alpha}}, \overline{\alpha} \in \overline{I}\}$$

$$= \mathrm{span}\{Z_\alpha = \partial/\partial z,\ \alpha \in I \cup I\},$$

or simply $N \equiv 0 \Longrightarrow$ *(2.1) has vanishing matrix.*

The Newlender-Nirenberg's theorem we formulate in the form

Theorem 2.1. *Let* $M^{2n}(J)$ *be almost complex manifold. The following conditions are equivalent:*

i) $N^{\overline{\sigma}}_{\alpha\beta} = 0$;

ii) $M^{2n}(J)$ *is complex manifold;*

iii) J *is Lie-parallel on* $T^{10}_p M$: $(\mathcal{L}_U J)V = 0,\ U, V \in T^{10}_p M$ *(*$\mathcal{L}$ *is the Lie-differentiation);*

3. Almost contact metric manifolds

In this paragraph we shall give analogous description of the basic classes of almost contact metric manifolds, introduced by Ganchev, Alexiev [3].

Let M be a $2n+1$-dimensional differentiable manifold with *almost contact metric structure* (φ, ξ, η, g), where φ is a tensor field of type (1,1), ξ is a vector field, η is a 1-form, g is a definit metric tensor field, such that

$$\varphi^2 = -id + \eta \otimes \xi,\ \varphi(\xi) = 0,\ \eta(\xi) = 1,\ g = g \circ \varphi + \eta \otimes \eta.$$

With $M^{2n+1}(\varphi, \xi, \eta, g)$ are associated the following fundamental objects (∇ is the Levi-Civita connection of the metric g):

i) operators: $h = -\varphi^2,\ v = \eta \otimes \xi,\ A = \nabla\xi$;

ii) distributions: $D = \{h(T_pM) = \mathrm{Ker}\ v_p, p \in M\}$, $\{\xi\} = \{v(T_pM) = Im v_p, p \in M\}$ contact and vertical distributions respectively and $T_pM = h(T_pM) \oplus v(T_pM)$ - orthogonal and $U(n) \times 1$-invariant.

iii) tensors: $\Phi : \Phi(x, y) = g(x, \varphi y)$ (fundamental 2-form); $\nabla\varphi; \nabla\eta; \nabla\Phi = -F; d\eta; d\Phi; \mathcal{L}_\xi g; \mathcal{L}\varphi; \mathcal{L}\eta; N = [\varphi, \varphi] + 2d\eta \otimes \xi([\varphi, \varphi]$ is the Nijenhuis tensor field, formed with φ :

$$[\varphi, \varphi](x, y) = [\varphi x, \varphi y] - \varphi[x, \varphi y] - \varphi[\varphi x, y] + \varphi^2[x, y], x, y \in T_pM.$$

iv) traces [3]: With $F = -\nabla\Phi$ are associated the following forms:

$$f(z) = g^{\alpha\beta} F(e_\alpha, e_\beta, z), f^*(z) = g^{\alpha\beta} F(e_\alpha, \varphi e_\beta, z), \omega(z) = F(\xi, \xi, z),$$

where $e_\alpha, e_{n+\alpha} = \varphi e_\alpha, e_{2n+1} = \xi,\ \alpha \in I$ is an orthonormal basis of T_pM.

$$\vartheta = -\frac{\delta\eta}{2n}\eta + \frac{1}{2(n-1)}\delta\Phi \circ \varphi(\text{ Lee-form }),$$

$$\delta\eta = -f^*(\xi) = -\sum_{\alpha=1}^{n}\left[(\nabla_{e_\alpha}\eta)e_\alpha + (\nabla_{\varphi e_\alpha})\varphi e_\alpha\right],$$

$$\delta\Phi(z) = f(z) + \omega(z) = -\sum_{\alpha=1}^{n}\left[(\nabla_{e_\alpha}\Phi)(e_\alpha, z) + (\nabla_{\varphi e_\alpha}\Phi)(\varphi e_\alpha, z)\right].$$

The complexification T_p^cM of the tangential space T_pM is decomposable as follows [4]:

$$T_p^cM = D_p^{10} \oplus D_p^{01} \oplus Im\ v_p,$$

where $D_p^{10}(D_p^{01})$ is $+i(-i)$-eigenspace of the operator φ_p and $D_p^c = D_p^{10} \oplus D_p^{01}$ is the compolexification of the contact distribution $D_p =$ Ker $v_p, p \in M$. For any orhtonormal basis $\{e_\alpha, \varphi e_\alpha, \xi\}_{\alpha\in I}, I = \{1, 2, \ldots, n\}$ of T_pM, the vectors $Z_\alpha = e_\alpha - i\varphi e_\alpha$ (resp. $Z_{\overline{\alpha}} = \overline{Z_\alpha}), \alpha \in I$ from a basis for $D_p^{10}(D_p^{01})$, Im η_p is spanned by $Z_\circ = \xi$ and so T_p^cM (resp. D_p^c) is spanned by the complex frame fields $Z_A, A \in I \cup \overline{I} \cup I_\circ$ (resp.$Z_a,\ a \in I \cup \overline{I}$), where $\overline{I} = \{\overline{1}, \overline{2}, \ldots, \overline{n}\}$ and $I_\circ = \{\circ = \overline{\circ}\}$. Unless otherwise started, Greek small letters will be run through the index-set I, Latin small-through $I \cup \overline{I}$ and Latin capital-through $I \cup \overline{I} \cup I_\circ$.

The relations between the essential complex components (i.e. which may not be zero) of the tensors above, with respect to the complex basis $\{Z_A\}_{A\in I\cup\overline{I}\cup I_\circ} : g\{g_{\alpha\overline{\beta}}, g_{\circ\circ} = 1\}, \Phi\{\Phi_{\alpha\overline{\beta}}\}, d\eta\{\eta_{AB}\}, d\Phi\{\Phi_{ABC}\}, \mathcal{L}_\xi g\{(\mathcal{L}_\xi g)_{AB}\}$, $N\{N_{\alpha\beta}^{\overline{\sigma}}, N_{\alpha\beta}^{\circ}, N_{\overline{\alpha}\circ}^{\overline{\sigma}}, N_{\overline{\alpha}\circ}^{\circ}\}$ are easily expressible, by using their properties [4], [9].

Theorem 3.1. *The essential complex characteristic equations of each of the basic classes W_i of almost contact metric manifolds, so that*

$$W = \{M^{2n+1}(\varphi, \xi, \eta, g), n \geq 3\} \oplus_{i=1}^{12} W_i$$

-orthogonal, $U(n) \times 1$ - invariant and irreducible, are the described in Table 2.

class	*characteristic* $\mathbb{C}$ – *equations*	$\delta\eta =$	$\delta\Phi \circ \varphi =$	$\delta\Phi(\xi) =$
W_1	$F_{\circ\circ\alpha} = \omega_\alpha = -2i\eta_{\alpha\circ} = -iN_{\alpha\ circo} = i(\mathcal{L}_\xi g)_{\alpha\circ}$	0	$\omega \circ \varphi$	0
W_2	$F_{\overline{\alpha}\beta\circ} = \frac{f(\xi)}{2n} g_{\overline{\alpha}\beta} = -i\eta_{\overline{\alpha}\beta} = F_{\beta\overline{\alpha}\circ}$	0	0	$f(\xi)$
W_3	$F_{\overline{\alpha}\beta\circ} = -i\frac{f^*(\xi)}{2n} g_{\overline{\alpha}\beta} = -\frac{i}{2}(\mathcal{L}_\xi g)_{\overline{\alpha}\beta} = F_{\beta\overline{\alpha}\circ}$	$-f^*(\xi)$	0	0
W_4	$F_{\overline{\alpha}\beta\circ} = -i\eta_{\overline{\alpha}\beta} = F_{\beta\overline{\alpha}\circ}$	0	0	0
W_5	$F_{\overline{\alpha}\beta\circ} = -\frac{3}{2}i\Phi_{\overline{\alpha}\beta\circ} = -\frac{i}{2}(\mathcal{L}_\xi g) - \overline{\alpha}\beta = -F_{\beta\overline{\alpha}\circ}$	0	0	0
W_6	$F_{\alpha\beta\circ} = -\frac{i}{2}(\mathcal{L}_\xi g)_{\alpha\beta} = -\frac{i}{2}N_{\circ\alpha\beta} = F_{\beta\alpha\circ}$	0	0	0
W_7	$F_{\alpha\beta\circ} = -i\eta_{\alpha\beta} = -\frac{i}{4}N_{\alpha\beta\circ} = -F_{\beta\alpha\circ}$	0	0	0
W_8	$F_{\circ\alpha\beta} = -3\Phi_{\circ\alpha\beta} = N_{\circ\alpha\beta}$	0	0	0
W_9	$F_{\overline{\alpha}\beta\gamma} = -3\Phi_{\overline{\alpha}\beta\gamma} = 2(\Phi_{\overline{\alpha}^*} \wedge \vartheta)_{\beta\gamma}$	0	$f^\circ\varphi$	0
W_{10}	$F_{\overline{\alpha}\beta\gamma} = -3\Phi_{\overline{\alpha}\beta\gamma}$	0	0	0
W_{11}	$4iF_{\alpha\beta\gamma} = -4i\Phi_{\alpha\beta\gamma} = N_{\alpha\beta\gamma}$	0	0	0
W_{12}	$2iF_{\gamma\beta\alpha} = N_{\alpha\beta\gamma}$	0	0	0

Table 2.

In the last three columns the components of the Lee form ϑ on each of the basic classes are given. The class $W_1 \oplus W_2 \oplus W_3 \oplus W_9 : \vartheta \neq 0$ is closed with respect to contact conformal transformations and is an analogous class to the class CK (Table 1.) in the set of almost Hermitian manifolds.

4. Integrability conditions for almost contact structures

Further we shall consider real analytical manifold M^{2n+1} with real analytical almost contact structure (φ, ξ, η).

If we put $[Z_A, Z_B] = C^S_{AB} Z_S$, then have

$$(4.1)\qquad \begin{pmatrix} [Z_\alpha, & Z_\beta] \\ [Z_\alpha\ , & Z_{\overline{\beta}}] \\ [Z_\alpha\ , & Z_\circ] \end{pmatrix} = \begin{pmatrix} C_{\alpha\beta} & -\frac{1}{4}N^{\overline{\sigma}}_{\alpha\beta} & -\frac{1}{2}N^{\circ}_{\alpha\beta} \\ C^{\sigma}_{\alpha\overline{\beta}} & C^{\overline{\sigma}}_{\alpha\overline{\beta}} & -2\eta_{\alpha\overline{\beta}} \\ C^{\sigma}_{\alpha\circ} & -\frac{1}{2}N^{\overline{\sigma}}_{\alpha\circ} & -N^{\circ}_{\alpha\sigma} \end{pmatrix} \begin{pmatrix} Z_\sigma \\ Z_{\overline{\sigma}} \\ Z_\circ \end{pmatrix}.$$

(4.1) implies that the class of *normal* almost contact metric maninfolds $W_2 \oplus W_3 \oplus W_4 \oplus W_5 \oplus W_9 \oplus W_{10}$, such that $N \equiv 0$ has the following geometrical description: the space of vector fields D^{10} is Lie-ideal of the space $D^{10} \times \{\xi\} =$ span $\{Z_\alpha, Z_\circ\}, \alpha \in I$. Then $N \equiv 0$ can be regarded as a weakly integrability condition: there exists open covering $\{U_p, p \in M\}$ of M, such that

$$D^{10}U_p = \text{ span } \{Z_\alpha = \partial/\xi^\alpha, \alpha \in I\},\ D^{01}U_p = \text{ span } \{Z_{\overline{\beta}}, \overline{\beta} \in \overline{I}\}$$

where $\xi^\alpha = \xi^\alpha(x^i, t), \alpha \in I$ are $\mathbb{C}$-valued functionally independent functions of real arguments, satisfying the system $Z_{\overline{\beta}}\xi^\alpha = 0$, but $U_p, \{\xi^\alpha, \xi^{\overline{\beta}}\})$ is not local coordinate system: $[Z_\alpha, Z_{\overline{\beta}}] = -2\eta_{\alpha\overline{\beta}}(\partial/\partial t)$. The condition $N \equiv 0$ implies (M^{2n+1}, D^{10}) is $C - R$ manifold [8]. Examples of such manifolds are given in [8].

We shall precise the results of Yano and Ishihara [6] in the case of real analytical almost contact structure (φ, ξ, η) on $2n+1$-dimensional differentiable manifold M.

Theorem 4.1. *Let $M(\varphi, \xi, \eta)$ be almost contact manifold. The following conditions are equivalent:*

i) the contact distribution D is involutive;

ii) $N^\circ_{\alpha\beta} = 0$; $\eta_{\alpha\overline{\beta}} = 0$;

iii) η is Lie-parallel on D: $(\mathcal{L}_u\eta) \equiv 0$, $u, v \in D$.

Proof. It is well known that$D$ is involutive iff $hd\eta \equiv 0$. The relations for the essential components of $hd\eta$:

$$2\eta_{\alpha\overline{\beta}} = (\mathcal{L}_{Z_\alpha}\eta)Z_{\overline{\beta}}, \quad 2\eta_{\alpha\beta} = N^\circ_{\alpha\beta} = 2(\mathcal{L}_{Z_\alpha}\eta)Z_\beta$$

proves the theorem. ■

Definition 4.1. ([6]) The structure (φ, ξ, η) (resp. the almost contact manifold) is said to be *partially integrable*, if the contact distribution D is involutive and the restriction $\varphi_{|_D}$ is integrable almost complex structure for arbitrarry integral manifold $\mathcal{D}$ of D.

Theorem 4.2. *Let $M(\varphi, \xi, \eta)$ be almost contact manifold. The following conditions are equivalent:*

i) (φ, ξ, η) is partially integrable;

ii) $N^\circ_{\alpha\beta} = N^{\overline{\sigma}}_{\alpha\beta} = \eta_{\alpha\overline{\beta}} = 0$;

iii) η is Lie-parallel on D and φ is Lie-parallel on $D^{10}(D^{01})$:

$$(\mathcal{L}_u\eta)v \equiv 0, \; u, v \in D^{10}_p \; (D^{01}_p).$$

Proof. Let (φ, ξ, η) be partially integrable. Then D is involutive and Theorem 4.1. implies $N^\circ_{\alpha\beta} = \eta_{\alpha\overline{\beta}} = 0$. Since $\varphi_{|_D}$ is integrable, then $[\varphi_{|_D}, \varphi_{|_D}] = 0$. The relation $[\varphi, \varphi](hx, hy) = N(hx, hy)$, [6], and Theorem 2.1. imply $N^{\overline{\sigma}}_{\alpha\beta} \equiv 0$. The implication i)$\Longrightarrow$ ii) is proved. The inverse is clear.

The equivalence of ii) and iii) follows from Theorem 4.1. and the decomposition

$$(\mathcal{L}_{Z_\alpha}\varphi)Z_\beta = -\frac{i}{2}(N^{\overline{\sigma}}_{\alpha\beta}Z_{\overline{\sigma}} + N^\circ_{\alpha\beta}Z_\circ).$$

■

From Corollary 2.1. and Theorem 4.2. we get

Corollary 4.1. *Let $M^{2n+1}(\varphi,\xi,\eta)$ be partially integrable almost contact manifold. Then M carries local coordinate covering $\{(U;\{x^\alpha, y^\alpha = x^{n+\alpha}, t\}),\ \alpha I\}$, such that for arbitrary integral manifold $\mathcal{D}$ of D there exists immersion $i:\mathcal{D}\longrightarrow M$, so that for the local coordinate covering*

$$\{(U_{|calD};\{x^\alpha, y^\alpha = x^{n+\alpha}, t = \text{const}\}),\ \alpha\in I\}$$

$$\overset{1-1}{\longleftrightarrow}\{(U_{|\mathcal{D}};\{z^\alpha = x^\alpha + iy^\alpha, t = \text{const}\}),\ \alpha\in I\}$$

adapted to $\{(U;\{x^\alpha, y^\alpha = x^{n+\alpha}, t\}),\ \alpha\in I\}$, it follows that $Z_{\alpha_{|U_{|\mathcal{D}}}}\equiv\partial/\partial z^\alpha$, i.e. the first two rows of the matrix in (4.1) vanish: $C^\sigma_{\alpha\beta} = C^{\overline{\sigma}}_{\alpha\overline{\beta}} = -\overline{C^{\prime\sigma}_{\beta\overline{\alpha}}}\equiv 0$, simply we have the implication "$N^\circ_{\alpha\beta} = N^{\overline{\circ}}_{\alpha\beta} = \eta_{\alpha\overline{\beta}}\equiv 0$" $\Longrightarrow$ "$C^{\prime\sigma}_{\alpha\beta} = C^{\overline{\sigma}}_{\alpha\overline{\beta}} - \overline{C_{\beta\{\overline{\alpha}}}^{\sigma}\equiv 0$" in the above sense.

R e m a r k 1. The existence of a such local coordinate covering can be introduced analogousely to Section 2, by using the method of Frölicher [2]. If $M^{2n+1}(\varphi,\xi,\eta)$ is real analytical almost contact manifold and $\{U;\{x^a,t\}),\ a\in I\cup\overline{I}\}$ is local coordinate system, then

1) $\xi\neq 0$, always locally can be chosen as $\partial/\partial t$ (vector field rectifying theorem);

2) the components of φ and η with respect to $(U;\{x^a,t\}),\ a\in I\cup I, \varphi(\partial/\partial x^a) = \varphi^b_a\partial/\partial x^b + \varphi^\circ_a\partial/\partial t,\ \eta = dt + \eta_a dx^a$, satisfy:

$$\varphi^b_a\varphi^c_b = -\delta^c_b,\ \varphi^\circ_b\varphi^b_a = \eta_a,\ \varphi^\circ_a = -\varphi^b_a\eta_b.$$

3) the null-space of the set of forms $\mathcal{J} - \{\eta, \omega^\circ = (\varphi^c_b + i\delta^b_c)dx\}$ is $\mathcal{NJ} = \operatorname{span}\{v_b = (\varphi^c_b - i\delta^)_b(\partial/\partial x^c - \eta_c\partial\backslash\partial t)\}, b, c\in 1\div 2n$ and coincide with the contact distribution D.

4) $\mathcal{J}$ is a differential ideal iff

$$\left|\begin{array}{l} d\eta(v_b, v_g) = 0 \\ \\ d\omega^0(v_p, v_q) = 0\end{array}\right. \Longleftrightarrow \left|\begin{array}{l}(\varphi^d_b - i\delta^d_b)(\varphi^a_g - i\delta^a_g)[\frac{\partial\eta_a}{\partial x_d} - \frac{\partial\eta_d}{\partial x_a} + \frac{\partial\eta_d}{\partial t}\eta_a - \frac{\partial\eta_a}{\partial t}\eta_d] = 0 \\ \\ (\varphi^r_p - i\delta^r_p)(\varphi^b_q - i\delta^b_q)[\frac{\partial\varphi^c_b}{\partial x_r} - \frac{\partial\varphi^c_r}{\partial x_b} + \frac{\partial\varphi^c_r}{\partial t}\eta_b - \frac{\partial\varphi^c_b}{\partial t}\eta_r] = 0.\end{array}\right.$$

These are the Frobenius conditions for the totally integrabitity of the system

$$\left|\begin{array}{l}\eta = 0\\ \omega^\circ = 0\end{array}\right.$$

and are equivalent to the vanishing of $[\varphi,\varphi]$ and the horizontal part of $d\eta - hd\eta$. It's remarkable to note, that the components of η and φ depend on t. Then there exists smooth transformation

$$\psi : U \longrightarrow V$$

$$(x^\alpha, x^{n+\alpha}, t) \longrightarrow (u^\beta(x^\alpha, x^{n+\alpha}, t)v^\beta(x^\alpha, x^{n+\alpha}, t), q(x^\alpha, x^{n+\alpha}, t)),\ \alpha,\beta \in I,$$

such that for the functions $z^\beta = u^\beta + iv^\beta$ and q, the system

$$\left| \begin{array}{l} dz^\beta = 0,\ \beta \in I \\ dq = 0 \end{array} \right. \quad \text{is equivalent to the system} \quad \left| \begin{array}{l} \eta = 0 \\ \omega^\circ = 0 \end{array} \right.$$

5) The functions $u^\beta(x^\alpha, x^{n+\alpha}, t), v^\beta(x^\alpha, x^{n+\alpha}, t), q(x^\alpha, x^{n+\alpha}, t)$ are solutions of the homogenious system of lineary partial differential equations

$$\left| \begin{array}{l} dz^\beta(v_p) = 0,\ \beta \in I \\ \\ dq(v_p) = 0 \end{array} \right. \Longleftrightarrow \left| \begin{array}{l} \varphi^r_p.\frac{\partial u^\beta}{\partial x_r} - \varphi^r_p \eta_r \frac{\partial u^\beta}{\partial t} = -\frac{\partial v^\beta}{\partial x^p} + \eta_p \frac{\partial v^\beta}{\partial t} \\ \varphi^r_p.\frac{\partial v^\beta}{\partial x_r} - \varphi^r_p \eta_r \frac{\partial v^\beta}{\partial t} = \frac{\partial u^\beta}{\partial x^p} - \eta_p \frac{\partial u^\beta}{\partial t},\ \beta \in I \\ \frac{\partial q}{\partial x^p} - \eta_p.\frac{\partial q}{\partial t} = 0. \end{array} \right.$$

We get the following

Theorem 4.3. *Let $M(\varphi,\xi,\eta)$ be almost contact manifold. The following conditions are equivalent:*

i) $N^\circ_{alphao} = N^{\overline{\sigma}}_{\alpha\circ} \equiv 0$;

ii) φ (and consequently η) is Lie-parallel with respect to xi :

$$\mathcal{L}_\xi\varphi \equiv 0,\ (\mathcal{L}_\xi\eta \equiv 0) >$$

Proof. It follows from the decomposition $(\mathcal{L}_\xi\varphi)Z_\alpha = i(N^{\overline{\sigma}}_{\alpha\circ}Z_{\overline{\sigma}} + N^\circ_{\alpha\circ}Z_\circ)$ and the equality $(\mathcal{L}_\xi\eta)Z_\alpha = -N^\circ_{\alpha\circ}$. ■

Theorems 4.1. and 4.3. imply

Theorem 4.4. *Let $M(\varphi,\xi,\eta)$ be almost contact manifold with involutive contact distribution. The stricture tensor field φ (and coseqquently the form η) are Lie-parallel with respect to ξ iff for arbitrarry integral manifold $|calD$ of D immersed in $Mi : \mathcal{D} \longrightarrow M$ the components φ^j_i of φ (and coseqquently η_i of η) with respect to local coordinate system $(U_{|\mathcal{D}}; \{x^\alpha, y^\alpha = x^{n+\alpha}, t =$ const $\})$, $\alpha \in I$, adapted to $(U; \{x^a, t\})$, $a \in I \cup \overline{I}$, does not depend on t.*

Proof. Let $\mathcal{L}_\xi\varphi \equiv 0, (\mathcal{L}_\xi\eta \equiv 0)$ be valid and adapted local coordinate system for immersion of $\mathcal{D}$ in $M - (U_{|\mathcal{D}}; \{x^\alpha, y^\alpha = x^{n+\alpha}, t = \text{ const }\})$, $\alpha \in I$ be choosen. Then $\xi = \partial/\partial t$ and $(\mathcal{L}_\xi\varphi)(\partial/\partial x^i) = \mathcal{L}_\xi(\varphi_i^j\partial/\partial x^j) = (\partial\varphi_i^j/\partial t)\partial/\partial x^j$ imply $\partial\varphi_i^j/\partial t = 0$. Analogousely $\partial\eta_i/\partial t = 0$. The inverse is clear.

Moreover, on such an adapted coordinate system as in Theorem 4.4 we have $\mathcal{L}_{(\partial/\partial x^i)}\varphi)\xi = 0$ and so on the adapted local complex coordinate system $(U_{|\mathcal{D}}; \{z^\alpha = x^\alpha + iy^\alpha, t = \text{ const }\})$, $\alpha \in I\ (\mathcal{L}_{Z_\alpha}\varphi)\xi = 0$ is valid too.

Then the decomposition

$$(\mathcal{L}_{Z_\alpha}\varphi)\xi = -i(\frac{1}{2}N^{\overline{\sigma}}_{\alpha\circ}Z_{\overline{\sigma}} + C^\sigma_{\alpha 0}Z_\sigma)$$

implies $C^\sigma_{\alpha 0} \equiv 0$ when $N^\circ_{\alpha\beta} = N^\circ_{\alpha\circ} = N^{\overline{\sigma}}_{\alpha\circ} = \eta_{\alpha\overline{\beta}} = 0$. ∎

Thus, analogousely to Corollary 4.1., we get in the simple form, the following

Corollary 4.2. *Let $M(\varphi,\xi,\eta)$ be almost contact manifold. Then the implication $N^\circ_{\alpha\beta} = N^\circ_{\alpha\circ} = N^{\overline{\sigma}}_{\alpha\circ} = \eta_{\alpha\overline{\beta}} = 0 \Longrightarrow C^\sigma_{\alpha 0} \equiv$ is valid, i.e. the matrix in (4.1) is with vanishing last row and column.*

Definition 4.2. ([6]) The structure (φ,ξ,η) (resp. the almost contact manifold) is said to be *integrable* iff (φ,ξ,η) is partially integrable and the components φ_i^j of φ with respect to adapted local coordinate system $(U_{|\mathcal{D}}; \{x^\alpha, y^\alpha = x^{n+\alpha}, t = \text{ const }\})$, $\alpha \in I$, for any integral manifold $\mathcal{D}$ of D, immersed in M, does not depend on t.

Theorem 4.5. *Let $M(\varphi,\xi,eta)$ be almost contact manifold. The following conditions are equivalent:*

i) (φ,ξ,eta) is integrable;

ii) $N \equiv 0,\ d\eta \equiv 0$.

Proof. All the essential components of N are $N^{\overline{\sigma}}_{\alpha\beta}, N^\circ_{\alpha\beta}, N^\circ_{\alpha\circ}, N^{\overline{\sigma}}_{\alpha\circ}$. The essential components of η are $\eta_{\alpha\beta}, \eta_{\alpha\overline{\beta}}, \eta_{\alpha\circ}$ and we have

$$N^\circ_{\alpha\beta} = \eta_{\alpha\beta},\ N^\circ_{\alpha\circ} = \eta_{\alpha\circ}.$$

Then Theorems 4.1, 4.2, 4.3 imply the equivalence of i) and ii). ∎

Analogousely to Corollaries 4.1, 4.2, taking into account Remark 1, from (4.1) we get in the simple form the following

Corollary 4.3. *Let $M(\varphi,\xi,\eta)$ be almost contact manifold. Then the following implication is valid:$N \equiv 0,\ d\eta \equiv 0 \Longrightarrow$ (4.1) has vanishing matrix, i.e. $Z_A = \partial/\partial z^A$, $A \in I \cup \overline{I} \cup I_\circ$, $z^\circ = t$, $Z_\circ = \partial/\partial t = \xi$.*

Finally by using Table 2. of Theorem 3.1. we get

Theorem 4.6

i) The class of partially integrable almost contact manifolds is $W_1 \oplus W_3 \oplus W_5 \oplus W_6 \oplus W_8 \oplus W_9 \oplus W_{10}$;

ii) The class of integrable almost contact manifolds is

$$W_3 \oplus W_5 \oplus W_9 \oplus W_{10}.$$

5. Structurial theorems for almost contact metric manifolds

In this section we describe the twelve basic classes of almost contact metric manifold with respect to their locally product structure of an almost Hermitian manifold and the real line.

The following lemmas are valid.

Lemma 5.1. ([7]) *Let* $M(\varphi, \xi, \eta, g)$ *be almost contact metric manifold, such that* ξ *is Killing vector field. Then the integral curves of* ξ *are geodesics and geodeseics, which are initially orthogonal to* ξ, *remain orthogonal to* ξ.

Lemma 5.2. ([7]) *Let* $M(\varphi, \xi, \eta, g)$ *be almost contact metric manifold such that* ξ *is Killing vector field and* η *is closed. Then* M *is locally the product of an almost Hermitian manifold and the real line* $\mathbb{R}$.

From Table 2, we get the following

Theorem 5.1. *Let* $M^{2n+1}(\varphi, \xi, \eta, g)$ *be almost contact metric manifold in the class* $W_8 \oplus W_9 \oplus W_{10} \oplus W_{11} \oplus W_{12}$. *Then locally* $M^{2n+1} = \mathcal{D}^{2n} \times \mathbb{R}$, *where* $\mathcal{D}^{2n} \in \mathcal{H}^{2n}$ *is an almost Hermitian manifold.*

Further we state

Theorem 5.2. *Let* $M^{2n+1}(\varphi, \xi, \eta, g)$ *be partially integrable almost contact metric manifold and let the following conditions be valid:*

1) η *is Lie-parallel with respect to* ξ : $cal L_\xi \eta = 0$;

2) $\mathcal{L}_\xi \varphi$ *is* g*-antisymmetric on* D^{10} : $g((\mathcal{L}_\xi)y, x)$, $x, y \in D^{10}$;

3) the vertical part $vd\Phi$ *of the differential of the fundamental form is of "pure" type:* $d\Phi(\varphi x, \varphi y, \xi) = -d\Phi(hx, hy, \xi)$, $x, y \in D_p$.

P r o o f. From partially integrability of M it follows that $\{(D_p, \varphi_p)\}$ is involutive Hermitian distribution and the components $\eta_{\alpha\overline{\beta}}$ and $\eta_{\alpha\beta}$ of $d\eta$ vanish. The following expressions are valid [4]:

$$(\mathcal{L}_\xi \eta) Z_\alpha = -N_{o\alpha o} = 2\eta_{\alpha o} = -(\mathcal{L}_\xi g)_{\alpha o},$$

$$g((\mathcal{L}_\xi \varphi) Z_\alpha, Z_\beta) + g((\mathcal{L}_\xi \varphi) Z_\beta, Z_\alpha) = -i(N_{o\alpha\beta} + N_{o\beta\alpha}) = -2i(\mathcal{L}_\xi g)_{\alpha\beta},$$

$$(\mathcal{L}_\xi g))_{\alpha\overline{\beta}} = 3i\Phi_{\alpha\overline{\beta}o}.$$

Then the conditions 1), 2), 3) imply ξ is Killing vector field and η is closed. From Lemma 5.2 it follows that locally $M^{2n+1} = \mathcal{D}^{2n} \times \mathbb{R}, \mathcal{D}^{2n} \in \mathcal{AH}^{2n}$ is integral manifold of D. The integrability of $\varphi_{|D}$ implies $(\mathcal{D}^{2n}, \varphi) \in \mathcal{H}^{2n}$ is an Hermitian manifold, which ends the proof. ■

Consequence 5.1. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be integrable almost contact metric manifold and let the vertical part $vd\Phi$ of the differential of the fundamental form be of "pure" type: $d\Phi(\varphi x, \varphi y, \xi) = -d\Phi(hx, hy, \xi)$, $x, y \in D_p$. Then M is locally the product of an Hermitian manifold and the real line.*

From Table 2, we get the following

Theorem 5.3. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be almost contact metric manifold in the class $W_8 \oplus W_9 \oplus W_10$. Then locally $M^{2n+1} = \mathcal{D}^{2n} x\mathbb{R}$, where $\mathcal{D}^{2n} \in \mathcal{H}^{2n}$ is an Hermitian manifold.*

Theorems 5.1 and 5.3 imply the locally inclusions $W_8 \oplus W_9 \oplus W_{10} \oplus W_{11} \oplus W_{12} \subseteq AH^{2n} x\mathbb{R}$, $W_8 \oplus W_9 \oplus W_{10} \subseteq H^{2n} \times \mathbb{R}$ respectively. It is well known that on any manifold in $AH^{2n} x\mathbb{R}$ naturally arises a canonical almost contact metric structure [8]. Then by using Tables 1,2 one can get the following structurial theorem.

Theorem 5.4. *The following relations are locally valid:*

$$W_8 \oplus W_9 \oplus W_{10} \oplus W_{11} \oplus W_{12} = AH^{2n} x\mathbb{R}, W_8 \oplus W_9 \oplus W_{10} = H^{2n} x\mathbb{R};$$

$$W_8 = \mathcal{K}^{2n} x\mathbb{R}; \quad W_8 \oplus W_9 = \mathcal{CK}^{2n} x\mathbb{R}; \quad W_8 \oplus W_{10} = \mathcal{SH}^{2n} x\mathbb{R};$$

$$W_8 \oplus W_{11} = NK^{2n} x\mathbb{R}; \quad W_8 \oplus W_{12} = \mathcal{AK}^{2n} x\mathbb{R}$$

We consider the operator $A = \triangle\xi$ and let A^* be the $g-$conjugated operator to $A : g(Ax, y) = g(x, A^*y)$ on an almost contact metric manifold $M(\varphi, \xi, \eta, g)$.

Definition 5.1. The contact distribution D for an almost contact metric manifold $M(\varphi, \xi, \eta, g)$ is said to be:

i) *totally umbilic* iff $A = \lambda\varphi + \mu h, \lambda, \mu \neq 0$smooth functions on M;

ii) *h-totally umbilic* iff $a = \mu h$;

iii) φ - *totally umbilic* iff $A = \lambda\varphi$;

iv) *totally geodesics* iff $A = 0$.

The descriptions of the basic classes with respect to the operator A, formulated in the following theorem, are well known [9].

Theorem 5.5. *$M^{2n+1}(\varphi,\xi,\eta,g)$ is an almost contact metric manifold in the class:*
i) $W_1 iff G(Ax,y) = -(\eta \otimes \omega \circ \varphi)(x,y), A^* = -(\omega \circ \varphi) \otimes \xi$;
ii) W_2 iff $A = -A^* = \varphi \circ A \circ \varphi^{-1}$;
iii) W_3 iff $A = A^* = \varphi \circ A \circ \varphi^{-1}$;
iv) W_4 iff $A = -A^* = \varphi \circ A \circ \varphi^{-1}$;
v) W_5 iff $A = A^* = \varphi \circ A \circ \varphi^{-1}$;
vi) W_6 iff $A = A^* = \varphi \circ A \circ \varphi^{-1}$;
vii) W_7 iff $A = -A^* = \varphi \circ A \circ \varphi^{-1}$;
viii) $W_i, i = 8,...,12$ iff $A = 0$.

The following Theorem follows directly from Theorem 5.5.

Theorem 5.6. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be almost contact metric manifold. The contact discribution is:*
i) totally umbilical iff $M \in W_2 \oplus W_3, M \notin W_2 M \notin W_3$;
ii) h-totally umbilical iff $M \in W_2$;
iii) φ - totally umbilical iff $M \in W_3$;
iv) totally geodesics iff $M \in W_8 \oplus W_9 \oplus W_{10} \oplus W_{11} \oplus W_{12}$.

Finally, Theorems 5.4 and 5.6 imply

Theorem 5.7. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be almost contact metric manifold in the class $M \in W_8 \oplus W_9 \oplus W_{10} \oplus W_{11} \oplus W_{12}$. Then there exists almost Hermitian manifold $\mathcal{D}^{2n}$, which is totally geodesics hypersurface immersed in M^{2n+1}, and locally $M^{2n+1} = \mathcal{D}^{2n} x \mathbb{R}$.*

References

[1] A. G r a y, L. H e r v e l l a, The sixteen classes of almost Hermitian manifolds and their linear invaiants. *Ann. di Mat. Pura et Appl.* (IV) **CXXIII** (1980), 35-58.

[2] A. F r ö l i c h e r, Zur Differentialgeometrie der komplexen Strukturen. *Math. Ann.* **129** (1955), 50-95.

[3] V. A l e x i e v, G. G a n c h e v, On the classification of almost contact metric manifolds. *Math. and Educ. in Math. (Proc. of the XV Spring Conf. of UBM. Sunny Beach* (1986), 155-161.

[4] M. H r i s t o v, Curvature identities for almost contact metric manifolds of Hermitian type. *SERDICA (Bulg. Math. Publ.)* **19** (1993), 273-286.

[5] S. D i m i e w, Fonctions presque holomorphes, complex analysis. *Banach Center Publ.* **11**, Warsaw, 1983.

[6] K. Y a n o, S. I s h i h a r a, On integrability conditions of a structure f, satisfying $f^3 + f = 0$. *Quart. J. Math.* **15** (1964), 217-222.

[7] D. B l a i r, L. V a n h a c k e, Symmetries and φ-symmetric spaces. *Tohoku Math.* **39** (1987), 373-383.

[8] D. B l a i r, Contact manifolds in Riemanian geometry. *Lecture Notes in Math.* **509**, Springer-Verlag, Berlin, 1976.

[9] G. G a n c h e v, V. A l e x i e v, On some classes of almost contact metric manifols. *Math. and Educ. in Math. (Proc. of the XV Spring Conf. of UBM, Sunny Beach)* (1986), 186-191.

Department of Mathematics
Vasil Levsky, Higher Military School
5000 Veliko Tarnovo
BULGARIA

Received 09.06.1995