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Submanifolds of Some Almost Contact Manifolds with B-Metric with Codimension Two, I

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In this paper, we study submanifolds of almost contact manifolds with B-metric of two types with respect to the normal spaces. Examples of such submanifolds are constructed.

1. Introduction

Geometry of almost contact manifolds with B-metric can be considered as a natural extension of geometry of almost complex manifolds with B-metric to the odd dimensional case. A classification of almost contact manifolds with B-metric $(M, \varphi, \xi, \eta, g)$ with respect to the covariant derivative of the fundamental tensor φ of type (1.1) is given in [4].

In this paper we study submanifolds of almost contact manifolds with B-metric $(M, \varphi, \xi, \eta, g)$ with 2-dimensional normal spaces. We consider two types of submanifolds with respect to the normal section at a point of the submanifold.

2. Preliminaries

Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact manifold with B- metric g, i.e. (φ, ξ, η) is an almost contact structure [1] and g is a metric on M [4] such that

$$\varphi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \tag{2.1}$$

where I denotes the identity transformation,

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.2}$$

for arbitrary vector fields X, Y on M. We denote by χM the Lie algebra of C^∞ - vector fields on M.

The associated with g metric $\tilde{g}$ [4] on the manifold is given by

$$\tilde{g}(X, Y) = g(X, \varphi Y) + \eta(X)\eta(Y), \quad X, Y \in \chi M. \tag{2.3}$$

Both metrics g and $\tilde{g}$ are indefinite of signature $(n+1, n)$ [4].

Further X, Y, Z, W will stand for arbitrary differentiable vector fields on M and x, y, z, w - for arbitrary vectors in the tangential space $T_pM, p \in M$.

Let $\nabla(\tilde{\nabla})$ be the Levi-Civita connection of the metric $g(\tilde{g})$. The tensor field F of type (0,3) on M, defined by $F(x, y, z) = g((\nabla_x \varphi)y, z)$ [4] has the following properties:

$$F(x, y, z) = F(x, z, y), \tag{2.4}$$

$$F(x, \varphi y, \varphi z) = F(x, y, z) - \eta(y)F(x, \xi, z) - \eta(z)F(x, y, \xi),$$

for all vectors x, y, z in T_pM.

The following 1-forms are associated with F :

$$\Theta(x) = g^{ij}F(e_i, e_j, x), \; \Theta^*(x) = g^{ij}F(e_i, \varphi e_j, x), \tag{2.5}$$

$$w(x) = F(\xi, \xi, x),$$

where $x \in T_pM$, $\{e_i, \xi\}$, $(i = 1, \ldots, 2n)$ is a basis of T_pM and (g^{ij}) is the inverse matrix of (g_{ij}) [4].

A classification of the almost contact manifolds with B-metric with respect to the tensor F is given in [4], where are defined eleven basic classes $\mathcal{F}_i (i = 1, 2, \ldots, 11)$ of almost contact manifolds with B-metric. The class $\mathcal{F}_0$ is defined by the condition $F(x, y, z) = 0$. This special class belongs to everyone of the basic classes.

Let R $(\tilde{R})$ be the curvature tensor field of type (1.3) of the Levi-Civita connection $\nabla(\tilde{\nabla})$ of $g(\tilde{g})$, i.e.

$$R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X, Y, Z \in \chi M. \tag{2.6}$$

$$\tilde{R}(X, Y, Z) = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z, \quad X, Y, Z \in \chi M. \tag{2.7}$$

The corresponding to $R(\tilde{R})$ tensor field of type (0,4) is given by

$$R(X,Y,Z,W) = g(R(X,Y,Z),W); \tag{2.8}$$

$$\tilde{R}(X,Y,Z,W) = \tilde{g}(\tilde{R}(X,Y,Z),W). \tag{2.9}$$

If α is a section in the tangential space of (M,φ,ξ,η,g) with a basis $\{x,y\}$ in general we can define analogously as in [2] the following curvatures for the section α :

$$K(\alpha;p) = K(x,y) = \frac{R(x,y,y,x)}{\pi_1(x,y,y,x)} \tag{2.10}$$

for every nondegenerate section α with respect to g. This is the usual Riemannian sectional curvature.

$$\tilde{K}(\alpha;p) = \tilde{K}(x,y) = \frac{\tilde{R}(x,y,y,x)}{\pi_1(x,y,y,x)} \tag{2.11}$$

for every nondegenerate section α with respect to g.

Let (M,φ,ξ,η,g) be an $\mathcal{F}_0$ - manifold, i.e. $\nabla\varphi = \nabla\eta = \nabla\xi = 0$. Then from [4] we have $\tilde{\nabla} = \nabla$ and from (2.9) it follows that $\tilde{R}(X,Y,Z,W) = R(X,Y,Z,\varphi W) = R(X,Y,\varphi Z,W)$. It is easy to verify that R and $\tilde{R}$ are Kaehler tensors ([2], [6]).

The contact distribution D on M : $p \in M \rightarrow D_p = \{x \in T_pM \ / \ \eta(x) = 0\}$ is involutive and hence through every $x \in M$ there passes a unique maximal integral manifold N and the tangential space at each point of N coincides with hT_pM [1]. Taking into account (2.1) it follows that hT_pM is a $2n$-dimensional real vector space with a complex structure φ and B-metric g. If $M \in \mathcal{F}_0$, i.e. $\nabla\varphi = 0$, then N is a Kaehler manifold with B-metric.

Let α be a 2-dimensional section in T_pM. A section α is said to be totally real if $\varphi\alpha\perp\alpha$ and the curvatures $K(\alpha)$ and $\tilde{K}(\alpha)$ of α are said to be totally real sectional curvatures [2].

Since R is a Kaehler tensor on $(M,\varphi,\xi,\eta,g) \in \mathcal{F}_0$, we have $hR(X,Y,Z,W) = R(hX,hY,hZ,hW) = R(X,Y,Z,W)$. Taking into account Theorem 1. from [2] for an $\mathcal{F}_0$ - manifold the following assertion, analogous to Theorem 1 ([2]) is valid.

Theorem 2.1. *Let (M,φ,ξ,η,g) ($\dim M \geq 5$) be an $\mathcal{F}_0$ - manifold. M is of constant totally real sectional curvatures ν and $\tilde{\nu}$, i.e,*

$$K(\alpha;p) = \nu(p), \qquad \tilde{K}(\alpha;p) = \tilde{\nu}(p),$$

whenever α is a nondegenerate totally real orthogonal to ξ section in T_pM, $p \in M$, iff

$$R(X,Y,Z,W) = \nu(\pi_1(\varphi X,\varphi Y,\varphi Z,\varphi W) - \pi_2(X,Y,Z,W))$$

$$+\tilde{\nu}\pi_3(\varphi X,\varphi Y,\varphi Z,\varphi W).$$

Both functions ν and $\tilde{\nu}$ are constant if M is connected and $\dim M \geq 7$.

The tensors π_1, π_2, π_3 are given by:

$$\begin{aligned} \pi_1(x,y,z,w) &= g(y,z)g(x,w) - g(x,z)g(y,w); \\ \pi_2(x,y,z,w) &= g(y,\varphi z)g(x,\varphi w) - g(x,\varphi z)g(y,\varphi w); \end{aligned} \tag{2.12}$$

$$\begin{aligned} \pi_3(x,y,z,w) = & -g(y,z)g(x,\varphi w) + g(x,z)g(y,\varphi w) \\ & -g(x,w)g(y,\varphi z) + g(y,w)g(x,\varphi z). \end{aligned}$$

3. Submanifolds of an almost contact manifold with B-metric with 2-dimensional holomorphic normal spaces

Holomorphic hypersurfaces of Kaehler manifolds with Norden metric (B-metric) are studied in [3]. In this section we consider analogous submanifolds of $\mathcal{F}_0$ - manifolds. Since for an almost contact manifold (M,φ,ξ,η,g) with B-metric hT_pM is a $2n$-dimensional real vector space with a complex stucture α and B-metric g, then in our considerations we can use the algebraic results from [3].

Let $(\overline{M},\overline{\varphi},\overline{\xi},\overline{\eta},g)(\dim \overline{M} = 2n+3)$ be an almost contact manifold with B-metric g and let M be a submanifold of codimension 2 of $\overline{M}$. We assume that there exists a normal holomorphic section $\alpha = \{N,\overline{\varphi}N\}(\overline{\varphi}\alpha = \alpha)$ defined globally over the submanifold M such that:

$$g(N,N) = -g(\overline{\varphi}N,\overline{\varphi}N) = 1, \qquad g(N,\overline{\varphi}N) = 0. \tag{3.1}$$

In fact, if Z and $\overline{\varphi}Z$ are vector fields normal to M and the section spanned by to $\{Z,\overline{\varphi}Z\}$ is holomorphic it follows $\overline{\eta}(Z) = 0$. Now, if $g(Z,Z) = 1$, using (2.2) we obtain $g(\overline{\varphi}Z,\overline{\varphi}Z) = -g(Z,Z) = -1$. If $g(Z,\overline{\varphi}Z) = sht$, then $N = \frac{1}{cht}(ch\frac{t}{2}Z + sh\frac{t}{2}\overline{\varphi}Z)$ and $\overline{\varphi}N = \frac{1}{cht}(-sh\frac{t}{2}Z + ch\frac{t}{2}\overline{\varphi}Z)$ satisfy (3.1).

Since $\overline{\xi}$ is a tangential vector field on M we set

$$\overline{\xi} = \xi, \quad \xi \in \chi M. \tag{3.2}$$

From the condition (3.1) it follows that in $T_p\overline{M}$, $p \in M$ there exists an adapted basis of the type $\{e_1,e_2,\ldots,e_n,\overline{\varphi}N,\overline{\varphi}e_1,\ldots,\overline{\varphi}e_n,N,\overline{\xi}\}$. Then we have

$T_p\overline{M} = T_pM \oplus \{N, \overline{\varphi}N)$ and $\overline{\varphi}(T_pM) \subset T_pM$. Let $X \in \chi M$ and consequently $\overline{\varphi}X \in T_pM$. We put

$$\overline{\varphi}X = \varphi X \text{ and } \overline{\eta}(X) = \eta(X), \tag{3.3}$$

where φ and η are the restrictions of $\overline{\varphi}$ and $\overline{\eta}$ on T_pM. Taking into account (3.2) and (3.3) it follows that (φ, ξ, η) is an almost contact structure on the submanifold M and g is B-metric on M. Hence, the submanifold M is an almost contact manifold with B-metric. As in [3], such a submanifold M we shall call a *holomorphic hypersurface* of $\overline{M}$.

Let $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, g)(\dim \overline{M} = 2n+3)$ be an $\mathcal{F}_0$-manifold and $\overline{\nabla}$ is the Levi-Civita connection of g. If ∇ is the induced Levi-Civita connection on the holomorphic hypersurface M of $\overline{M}$, then the Gauss and Weingarten formulas are

$$\overline{\nabla}_X Y = \nabla_X Y + g(X, AY)N - g(\varphi X, AY)\overline{\varphi}N, \quad X, Y \in \chi M;$$

$$\overline{\nabla}_X N = -AX, \quad X \in \chi M, \tag{3.4}$$

where A is the second fundamental tensor with respect to N and $A_{\overline{\varphi}N} = \varphi \circ A = A \circ \varphi$, $A\xi = 0$, and for the second fundamental form σ we have

$$\sigma(X, Y) = g(X, AY)N - g(\varphi X, AY)\overline{\varphi}N, \ \sigma(X, \xi) = 0, X, Y \in \chi M.$$

It is easily to check, that $\nabla\varphi = \nabla\eta = \nabla\xi = 0$. Thus,every holomorphic hypersurface $(M, \varphi, \xi, \eta, g)$ of $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, g)$ is also an $\mathcal{F}_0$-manifold.

From now in this section $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, g)(\dim \overline{M} = 2n+3)$ will be an $\mathcal{F}_0$-manifold and $(M, \varphi, \xi, \eta, g)(\dim M = 2n+1)$ will be a holomorphic hypersurface of $\overline{M}$.

Let H be the mean curvature vector on M. i.e. $H = \frac{1}{2n+1} tr\sigma$. Since $\sigma(\xi, \xi) = 0$, then $H = \frac{1}{2n} tr\sigma$.

As in [3], the holomorphic hypersurface M we shall call *holomorphically umbilic (h-umbilic)* if at every point p in M

$$\begin{aligned} \sigma(X, Y) &= -\tfrac{tr\sigma}{2n} g(\varphi X, \varphi Y) - \tfrac{tr(\varphi\circ\sigma)}{2n} g(X, \varphi Y) \\ &= -H g(\varphi X, \varphi Y) - \varphi H g(X, \varphi Y), \ X.Y \in \chi M. \end{aligned} \tag{3.5}$$

Taking into account Theorem 2.1., we can conclude that for the h-umbilic holomorphic hypersurfaces of $\overline{M} \in \mathcal{F}_0$ the following theorem, analogous to Theorem 1. from [3], is valid.

Theorem 3.1 *Let $(\overline{M},\overline{\varphi},\overline{\xi},\overline{\eta},g)(\dim\overline{M}=2n+3\geq 9)$ be of constant totally real sectional curvatures $\overline{\nu}$ and $\overline{\tilde{\nu}}$. If M is h-umbilic, then M is of constant totally real sectional curvatures ν and $\tilde{\nu}$, such that:*

$$\nu=\overline{\nu}+g(H,H),\qquad \tilde{\nu}=\overline{\tilde{\nu}}+\tilde{g}(H,H).$$

Example 3.1. Let $\overline{M}=(\overline{\mathbb{R}}^{2n+3},\overline{\varphi},\overline{\xi},\overline{\eta},g)$ [4]. Identifying the point $z=(u^1,\dots,u^{n+1},v^1,\dots,v^{n+1},t)$ in $\overline{M}$ with its position vector Z, we define in an analogical way, as in [3] a submanifold $S^{2n+1}(z_0;a,b)$ by the equalities:

$$g(\overline{\varphi}(Z-Z_0),\overline{\varphi}(Z-Z_0))=a, \tag{3.6}$$

$$g(Z-Z_0,\overline{\varphi}(Z-Z_0))=b,\quad a,b\in\mathbb{R},\quad (a,b)\neq(0,0). \tag{3.7}$$

S^{2n+1} is a $(2n+1)$- dimensional submanifold of $\overline{M}$ and $\overline{\varphi^2}(Z-Z_0)$, $\overline{\varphi}(Z-Z_0)$ are normal to T_zS^{2n+1}. The condition $(a,b)\neq(0,0)$ implies the rank of g on T_zS^{2n+1} is $(2n+1)$ and furthermore $\varphi T_zS^{2n+1}\subset T_zS^{2n+1}$, i.e. S^{2n+1} is a holomorphic hypersurface of $\overline{M}$, which we shall call *an h-sphere* with parameters a,b.

4. Submanifolds of an almost contact manifold with B-metric with 2-dimensional normal ξ-spaces

In this section we consider submanifolds of almost contact manifold $(\overline{M},\overline{\varphi},\overline{\xi},\overline{\eta},g)\in\mathcal{F}_0$ with B-metric with 2-dimensional normal spaces, such that at every point of the submanifold M, the normal section is a $\overline{\xi}$-section, i.e. $\overline{\xi}$ belongs to the normal section.

Let $(\overline{M},\overline{\varphi},\overline{\xi},\overline{\eta},g)(\dim\overline{M}=2n+3)$ be an almost contact manifold with B-metric g and let M be a submanifold of codimension 2 of $\overline{M}$. We assume that there exists a normal $\overline{\xi}$-section $\alpha=\{N_1,N_2\}$ defined globally over the submanifold M such that:

$$g(N_1,N_1)=-g(N_2,N_2)=1,\qquad g(N_1,N_2)=0. \tag{4.1}$$

Then we obtain the following decomposition for $\overline{\xi},\overline{\varphi}X,\overline{\varphi}N_1,\overline{\varphi}N_2$ with respect to $\{N_1,N_2\}$ and T_pM.

$$\overline{\xi}=aN_1+bN_2; \tag{4.2}$$

$$\overline{\varphi}X=\varphi X+\tfrac{b}{a}\eta^1(X)N_1+\eta^1(X)N_2,\qquad X\in\chi M; \tag{4.3}$$

$$\overline{\varphi}N_1=-\tfrac{b}{a}\overline{\varphi}N_2; \tag{4.4}$$

$$\overline{\varphi}N_2=-\xi_1+bN_1+aN_2; \tag{4.5}$$

where φ denotes a tensor field of type (1.1), ξ_1- vector field, η^1 is a 1-form, $a\neq$

0 and $b \neq 0$ are functions on M. We denote the restriction of g on M by the same letter. The equality (4.4) immediately follows from (4.2) and $\overline{\varphi}\,\overline{\xi} = 0$. Using (4.1), $g(\overline{\xi}, \overline{\xi}) = 1$ and (4.2) we find

$$a^2 - b^2 = 1, \tag{4.6}$$

and (4.3) implies

$$\eta^1(X) = g(X, \xi_1), \quad X \in \chi M. \tag{4.7}$$

After direct computations of (4.2), (4.3) and (4.5) by using (2.1) for $\overline{\varphi}, \overline{\xi}, \overline{\eta}$ and g we get:

$$\varphi^2 = -I + \tfrac{1}{a^2}\eta^1 \otimes \xi_1; \tag{4.8}$$

$$\varphi\xi_1 = -\tfrac{1}{a}\xi_1; \tag{4.9}$$

$$\eta^1(\varphi X) = -\tfrac{1}{a}\eta^1(X), \quad X \in \chi M; \tag{4.10}$$

$$g(\xi_1, \xi_1) = 1 + a^2; \tag{4.11}$$

$$g(\varphi X, \varphi Y) = -g(X, Y) + \frac{1}{a^2}\eta^1(X)\eta^1(Y), \quad X, Y \in \chi M. \tag{4.12}$$

Now we define a vector field ξ, a 1-form η and a tensor field ϕ of type (1.1) on M by

$$\xi = \frac{1}{\sqrt{a^2+1}}\xi_1; \tag{4.13}$$

$$\eta(X) = \frac{1}{\sqrt{a^2+1}}\eta^1(X) \quad X \in \chi M; \tag{4.14}$$

$$\phi X = \varphi X + \frac{1}{a}\eta(X)\xi, \quad X \in \chi M. \tag{4.15}$$

Taking into account the equalities (4.7)÷ (4.15) we obtain

$$\phi^2 = -I + \eta \otimes \xi,\ g(\xi, \xi) = 1,\ \eta(\xi) = 1,\ \eta(\phi X) = 0 \text{ for } X \in \chi M.$$

Hence, (ϕ, ξ, η) is an almost contact structure on M and from (4.12), (4.15) we have

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \quad X, Y \in \chi M, \text{ i.e.}$$

the restriction of g on M is B-metric. Thus, the submanifold (M, ϕ, ξ, η, g) (dim M $= 2n + 1$) of $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, g)$ (dim $\overline{M} = 2n + 3$) is an almost contact manifold with B-metric.

Denoting by $\overline{\nabla}$ and ∇ the Levi-Civita connections of the metric g in $\overline{M}$ and M respectively, the formulas of Gauss and Weingarten are

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + \sigma(X,Y), \quad X,Y \in \chi M; \\ \overline{\nabla}_X N_1 &= -A_{N_1}X + D_X N_1, \quad X \in \chi M; \\ \overline{\nabla}_X, N_2 &= -A_{N_2}X + D_X N_2, \quad X \in \chi M, \end{aligned}$$

where σ is the second fundamental form on M, A_{N_1} is the second fundamental tensor with respect to N_1, AN_2- with respect to N_2 and D is the normal connection on M. Having in mind the properties of $\overline{\nabla}$ and (4.1), from the formulas of Gauss and Weingarten we compute

$$\begin{aligned} \sigma(X,Y) &= g(A_{N_1}X,Y)N_1 \equiv g(A_{N_2}X,Y)N_2 \\ &= g(X,A_{N_1}Y)N_1 - g(X,A_{N_2}Y)N_2, \quad X,Y \in \chi M; \end{aligned}$$

$$\begin{aligned} D_X N_1 &= \alpha(X)N_2, \quad X \in \chi M; \\ D_X N_2 &= \alpha(X)N_1, \quad X \in \chi M, \end{aligned}$$

where α is a 1-form on M.

From now on, in this section $(\overline{M},\overline{\varphi},\overline{\xi},\overline{\eta},g)$ $(\dim \overline{M} = 2n+3)$ will be an $\mathcal{F}_0$ -manifold and $(M,\phi,\xi,\eta,g)(\dim M = 2n+1)$ will be a submanifold of $\overline{M}$ with normal $\overline{\xi}$-spaces.

If $\overline{M} \in \mathcal{F}_0$, i.e. $\overline{\nabla}\overline{\varphi} = \overline{\nabla}\,\overline{\eta} = \overline{\nabla}\,\overline{\xi} = \overline{\nabla}g = 0$, then the formulas of Gauss and Weingarten become

$$\begin{aligned} &\overline{\nabla}_X Y = \nabla_X Y - g(AX,Y)(\tfrac{b}{a}N_1 + N_2), \quad X,Y \in \chi M; \\ (4.16) \quad &\overline{\nabla}_X N_1 = \tfrac{b}{a}AX + \alpha(X)N_2, \quad X \in \chi M; \\ &\overline{\nabla}_X N_2 = AX + \alpha(X)N_1, \quad X \in \chi M, \end{aligned}$$

where $\alpha(X) = -\frac{1}{b}(X \circ a) = -\frac{1}{b}da(X)$, $\quad AX = A_{N_2}X = -\frac{a}{b}A_{N_1}X$.

From $(\overline{\nabla}_X\overline{\varphi})N_1 = (\overline{\nabla}_X\overline{\varphi})N_2 = 0$ we find

$$g(AX,\xi) = -\frac{b\alpha(X)}{2\sqrt{a^2+1}} = \eta(AX), \quad X \in \chi M. \tag{4.17}$$

Since $g(AX,\xi) = g(X,A\xi)$ from (4.17) it follows

$$A\xi = -\frac{b}{2\sqrt{a^2+1}}P, \tag{4.18}$$

where P is a vector field, corresponding to the 1-form $\alpha(X)$, i.e. $\alpha(X) = g(P,X)$.

Let $\overline{R}$ and R be the curvature tensors of $\overline{M}$ and M respectively. Then for the equations of Gauss and Codazzi we have

$$\begin{aligned}\overline{R}(x,y,z,w) &= R(x,y,z,w)+\tfrac{1}{a^2}\pi_1(Ax,Ay,z,w);\\ (\overline{R}(x,y,z))^{\perp} &= \tfrac{b}{a}\{[g((\nabla_y A)x,z)-g((\nabla_x A)y,z)]\\ &+\tfrac{b}{a}[\alpha(y)g(Ax,z)-\alpha(x)g(Ay,z)]\}N_1\\ &+\{g((\nabla_y A)x,z)-g((\nabla_x A)y,z)\\ &+\tfrac{b}{a}[\alpha(y)g(Ax,z)-\alpha(x)g(Ay,z)]\}N_2,\end{aligned}$$

for arbitrary $x,y,z,w \in T_pM,\ p\in M$.

Using $\overline{\nabla}\,\overline{\varphi}=0$ and (4.15) we calculate

$$(\nabla_X\phi)y = \frac{1}{\sqrt{a^2+1}}\{\eta(y)Ax+g(Ax,y)\xi\} + \frac{1}{a.\sqrt{a^2+1}}\{\eta(y)\phi(Ax)+g(Ax,\phi y)\xi\} + \frac{b}{a^2+1}\alpha(x)\eta(y)\xi,\quad x,y\in T_pM. \tag{4.19}$$

From (4.19) we obtain the following assertion:

Theorem 4.1. *Let M be a submanifold of the $\mathcal{F}_0$-manifold $\overline{M}$. Then*

$$\begin{aligned}F(x,y,z) &= \tfrac{1}{\sqrt{a^2+1}}\{g(Ax,z)\eta(y)+g(Ax,y)\eta(z)\}\\ &+\tfrac{1}{a.\sqrt{a^2+1}}\{g(Ax,\phi z)\eta(y)+g(Ax,\phi y)\eta(z)\}\\ &+\tfrac{b}{a^2+1}\alpha(x)\eta(y)\eta(z),\end{aligned} \tag{4.20}$$

for arbitrary vectors x,y,z in T_pM.

Because of Proposition 3.4., Proposition 3.5. ant Theorem 3.10 from [4], an almost contact manifold with B-metric g $M\in\mathcal{F}_4\oplus\mathcal{F}_5\oplus\mathcal{F}_6\oplus\mathcal{F}_7\oplus\mathcal{F}_8\oplus\mathcal{F}_9\oplus\mathcal{F}_{11}$ has the following characterization condition

$$F(x,y,z)=\eta(y)F(x,y,\xi)+\eta(z)F(x,y,\xi). \tag{4.21}$$

According to (4.20),

$$F(x,y,\xi)=\frac{1}{\sqrt{a^2+1}}g(Ax,y)+\frac{1}{a\sqrt{a^2+1}}g(Ax,\phi y)+\frac{b}{2(a^2+1)}\alpha(x)\eta(y).$$

Finally, taking into account the last equality and (4.20), we get (4.21). Thus it follows the next

Proposition 4.2. *Let M be a submanifold of the $\mathcal{F}_0$-manifold $\overline{M}$. Then* $M \in \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_7 \oplus \mathcal{F}_8 \oplus \mathcal{F}_9 \oplus \mathcal{F}_{11}$.

Proposition 4.3. *Let M be a submanifold of the $\mathcal{F}_0$-manifold $\overline{M}$. $M \in \mathcal{F}_0$ iff*

$$Ax = -\frac{b}{2\sqrt{a^2+1}}\alpha(x)\xi = \frac{da(x)}{2\sqrt{a^2+1}}\xi, \quad x \in T_pM. \tag{4.22}$$

Proof. Let $M \in \mathcal{F}_0$ and consequently $F(x,y,z) = 0$. By a simple calculation we obtain (4.22).

Conversely, if $Ax = -\frac{b}{2\sqrt{a^2+1}}\alpha(x)\xi$ and substituting Ax in (4.20), we have $F(x,y,z) = 0$, i.e. $M \in \mathcal{F}_0$. ■

Example 4.1. Let $\overline{M} = (\mathbb{R}^{2n+3}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, g)$ [4]. Identifying the point $z = (u^1, \dots, u^{n+1}, v^1, \dots, v^{n+1}, t)$ in $\overline{M}$ with its position vector Z, we define a submanifold M by the equalities:

$$\begin{aligned} &\overline{\eta}(Z) = 0; \\ &g(\overline{\varphi}Z, Z) = 0; \\ &g(Z,Z) > 0. \end{aligned} \tag{4.23}$$

At every point $z \in M$ we can put $g(Z,Z) = ch^2 t$. M is a $(2n+1)$-dimensional submanifold of $\overline{M}$ and $\overline{\xi}, \overline{\varphi}Z$ are normal to T_zM^{2n+1}. We choose the unit normal vector fields $N_1 = \overline{\xi}$ and $N_2 = \frac{1}{cht}\overline{\varphi}Z$. It is clear, that $g(N_1, N_1) = g(\overline{\xi}, \overline{\xi}) = 1, g(N_2, N_2) = \frac{1}{ch^2 t}g(\overline{\varphi}Z, \overline{\varphi}Z) = -\frac{1}{ch^2 t}g(Z,Z) = -1$. The equalities (4.23) imply $Z \perp \overline{\xi}$ and $Z \perp \overline{\varphi}Z$. Then $Z \in T_zM$.

We define the structure vector field ξ on M by

$$\xi = \overline{\varphi}N_2 = \frac{1}{cht}\overline{\varphi}^2 Z = -\frac{1}{cht}Z. \tag{4.24}$$

For an arbitrary vector x in T_zM we consider the vector $\overline{\varphi}x$. Denoting the orthogonal projection of $\overline{\varphi}x$ into T_zM by ϕx, we have the unique decomposition

$$\overline{\varphi}x = \phi x - \eta(x)N_2, \tag{4.25}$$

where η is a 1-form in T_zM.

From (4.24) and (4.25) it follows that

$$\begin{aligned}\phi^2 x &= -x + \eta(x)\xi, \\ \eta(\phi x) &= 0, \quad \phi\xi = 0. \quad \eta(\xi) = 1, \quad g(\xi, x) = \eta(x), \quad x \in T_z M;\end{aligned} \tag{4.26}$$

$$g(\phi x, \phi y) = -g(x, y) + \eta(x)\eta(y), \quad x, y, \in T_z M.$$

Taking into account (4.26), we can conclude that $M, \phi, \xi, \eta, g)$ is an almost contact manifold with B-metric.

Denoting by $\overline{\nabla}$ and ∇ the Levi-Civita connections of the metric g in $\overline{M}$ and M, respectively, the formulas of Gauss and Weingarten are

$$\begin{aligned}\overline{\nabla}_X Y &= \nabla_X Y - g(A_{N_2} X, Y) N_2, \quad X, Y \in \chi M; \\ \overline{\nabla}_X N_2 &= -A_{N_2} X, \quad X \in \chi M\end{aligned} \tag{4.27}$$

Since $\overline{\nabla}$ is flat, then $\overline{\nabla}_X Z = X$, Z being the position vector field and X being an arbitrary vector field on M. Using (4.27) and the definition of N_2, we get

$$A_{N_2} X = -\tfrac{1}{cht}\phi X, \quad X \in \chi M; \tag{4.28}$$

$$\eta(X) = -sht(X \circ t), \quad X \in \chi M, \quad t \neq 0. \tag{4.29}$$

Then the formulas (4.27) become

$$\begin{aligned}\overline{\nabla}_X Y &= \nabla_X Y + \tfrac{1}{cht} g(\phi X, Y) N_2, \quad X, Y \in \chi M; \\ \overline{\nabla}_X N_2 &= \tfrac{1}{cht}\phi X, \quad X \in \chi M.\end{aligned} \tag{4.30}$$

Having in mind (4.25) and (4.30) we compute

$$(\nabla_X \phi) = \frac{1}{cht}\{\eta(y)\phi x + (x, \phi y)\xi\}, \quad x, y, \in T_z M.$$

Hence, $F(x, y, z) = \frac{1}{cht}\{\eta(y) g(\phi x, z) + \eta(z) g(x, \phi y)\}$, $x, y, z \in T_z M$. From the last equality and from (2.5) we have $\Theta(\xi) = 0$, $-\frac{\Theta^*(\xi)}{2n} = \frac{1}{cht}$. Then $F(x, y, z) = -\frac{\Theta^*(\xi)}{2n}\{\eta(y) g(\phi x, z) + \eta(z) g(\phi x, y)\}$.

According to [4] $M \in \mathcal{F}_5$ iff $F(x, y, z) = -\frac{\Theta^*(\xi)}{2n}\{\eta(y) g(\phi x, z) + \eta(z) g(\phi x, y)\}$ and consequently the submanifold (M, ϕ, ξ, η, g) is an almost contact manifold with B-metric in the class $\mathcal{F}_5$.

R e m a r k. An example of an almost contact manifold with B-metric in the class $\mathcal{F}_5$ is given in [4]. The example 3. from [4] is an example for a manifold, which is a real hypersurface of a complex Riemannian manifold with B-metric. The almost contact structure (ϕ, ξ, η) in the Example 4.1. is constructed in a similar way as in [4].

Now, from (2.6),(4.30) and $\overline{R} = 0$ for the curvature tensor R of the submanifold M we find

$$(4.31) \qquad R(x,y,z,w) = -\frac{1}{ch^2 t}\pi_2(x,y,z,w), \quad x,y,z,w \in T_z M.$$

Moreover for $\tilde{R}$ we get

$$(4.32) \quad \tilde{R}(x,y,z,w) = -\frac{1}{ch^2 t}\{\pi_2(x,y,z,w) + \pi_3(x,y,z,w) + \pi_5(x,y,z,w)\},$$

$x,y,z,w \in T_z M$, where $\pi_5(x,y,z,w) = g(y,\phi z)\eta(x)\eta(w) - g(x,\phi z)\eta(y)\eta(w) + g(x,\phi w)\eta(y)\eta(z) - g(y,\phi w)\eta(x)\eta(z)$.

Theorem 4.4. *If* (M,ϕ,ξ,η,g) $(\dim M = 2n+1)$ *is a submanifold of the flat* $\mathcal{F}_0$*-manifold* $\overline{M} = (\mathbb{R}^{2n+3}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, g)$, *defined by the equalities(4.23), then* M *has vanishing totally real and* ξ*-sectional curvatures* K *and* $\tilde{K}$ *and pointwise constant holomorphic sectional curvatures* $K(\alpha;p) = -\frac{1}{ch^2 t} = \tilde{K}(\alpha;p)$.

P r o o f. Let $\alpha = \{x,y\}$, $x,y \in T_p M$, $p \in M$ is an arbitrary totally real section. From (4.31) and (4.32) it follows that $R(x,y,y,x) = \tilde{R}(x,y,y,x) = 0$. Then the formulas (2.10) and (2.11) imply immediately $K(\alpha) = \tilde{K}(\alpha) = 0$. Now, let α is a ξ-section in $T_p M$. Hence, $R(x,\xi,\xi,x) = \tilde{R}(x,\xi,\xi,x) = 0$. i.e. $K(\alpha) = \tilde{K}(\alpha) = 0$. In the case, when α is a holomorphic section i.e. $\alpha = \{\phi x, \phi^2 x\}$, $x \in T_p M$ we have

$$R(\phi x,\phi^2 x,\phi^2 x,\phi x) = \tilde{R}(\phi x,\phi^2 x,\phi^2 x,\phi x) = -\frac{1}{ch^2 t}\pi_1(\phi x,\phi^2 x,\phi^2 x,\phi x)$$

and $K(\alpha;p) = -\frac{1}{ch^2 t} = \tilde{K}(\alpha;p)$. Then it follows, that the holomorphic sections are of one and the same pointwise constant sectional curvatures. ∎

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