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On the Faddeev-Tahtajan Identity [1]

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We explain that from the famous Fay identity for τ-functions, the identity for Wronskians of squared eigenfunctions of a Sturm-Liouville equation follows.

0. Introduction

Let $\psi(x,z)$ and $\psi^*(x,z)$ be solutions of the Sturm-Liouville equation:

$$(\partial_x^2 + q(x))\, y(x,z) \;=\; z^2\, y(x,z). \tag{1}$$

The following relation for products of solutions:

$$\begin{aligned} &W(\psi(x,\mu)\psi^*(x,\mu), \psi(x,\lambda)\psi^*(x,\lambda)) \\ &= -(\mu^2-\lambda^2)^{-1}\,\partial_x\{W(\psi(x,\mu),\psi(x,\lambda))\, W(\psi^*(x,\mu),\psi^*(x,\lambda))\} \end{aligned} \tag{2}$$

(where $\mu,\ \lambda \in \mathbb{C}$ and $W(f,g) := fg' - f'g$ and $'$denotes ∂_x is the Wronskian) is called [Ta-Fa] F a d d e e v-T a h t a j a n identity (shortly FTI).

This relation has a long history. It was used in the theory of inverse spectral problems for Sturm-Liouville equation. Afterwards FTI played an important role in the first years of establishing of soliton theory. In [Fa-Ta] the origin of the identity was interpreted in terms of classical r-matrices.

In this paper we explain the origin of FTI, using the theory of Sato's tau-function. As it will be shown later, the FTI relation follows from the famous Fay identity [Fay] (shortly FI).

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Let $\tau(t)$, $t \equiv (t_1, t_2, t_3, \ldots) \in \mathbb{C}^\infty$, $t_1 \equiv x$ be an arbitrary tau-function, related to the Kadomtzev-Petviashvili (shortly KP) hierarchy [AvM]. For given $z \in \mathbb{C}$ define:

$$\begin{aligned}[z] &:= (z, \tfrac{z^2}{2}, \tfrac{z^3}{3}, \ldots) \in \mathbb{C}^\infty, \\ t + [z] &:= (t_1 + z, t_2 + \tfrac{z^2}{2}, t_3 + \tfrac{z^3}{3}, \ldots) \in \mathbb{C}^\infty.\end{aligned}$$

Then the FI is the relation $(z_0, z_1, z_2, z_3 \in \mathbb{C})$:

$$\begin{aligned}&(z_0 - z_1)(z_2 - z_3)\tau(t + [z_0] + [z_1])\tau(t + [z_2] + [z_3]) + \\ &(z_0 - z_2)(z_3 - z_1)\tau(t + [z_0] + [z_2])\tau(t + [z_3] + [z_1]) + \\ &(z_0 - z_3)(z_1 - z_2)\tau(t + [z_0] + [z_3])\tau(t + [z_1] + [z_2]) = 0.\end{aligned} \tag{3}$$

The FI was firstly obtained [Fay] for theta-functions and in this case was important in the geometric treatement of the soliton equations [Mum]. Later FI was generelized for tau-functions [Shi]. Nowadays the FI is useful in different aspects of studying of tau-(theta-) functions [AvM].

In order to explain the connections between FI and FTI we will restrict our attention only to KdV tau-functions, i.e. to tau-functions, related to the $n = 2$ Gel'fand-Dickey reduction of KP hierarchy (i.e. KdV hierarchy). It is well known [AvM] that these tau-functions are characterized by the conditions:

$$\frac{\partial}{\partial t_{2k}}\tau(t) = 0 \quad , \; k = 1, 2, 3, \ldots$$

which imply for every $z \in \mathbb{C}$:

$$\tau(t - [z]) = \tau(t + [-z]). \tag{4}$$

Using vertex operators $X(t, z)$, $X^*(t, z)$ ($t \in \mathbb{C}^\infty$, $z \in \mathbb{C}$), which act over $\tau(t)$ in the following way:

$$\begin{aligned}X(t, z)\,\tau(t) &:= \exp(\sum_{j=1}^{\infty} t_j z^j)\,\tau(t - [z^{-1}]), \\ X^*(t, z)\,\tau(t) &:= \exp(-\sum_{j=1}^{\infty} t_j z^j)\,\tau(t + [z^{-1}]),\end{aligned}$$

let us define the wave functions [AvM]:

$$\begin{aligned}\psi(t, z) &:= \frac{X(t,z)\tau(t)}{\tau(t)} = \exp(\sum_{j=1}^{\infty} t_j z^j)\,\frac{\tau(t - [z^{-1}])}{\tau(t)}, \\ \psi^*(t, z) &:= \frac{X^*(t,z)\tau(t)}{\tau(t)} = \exp(-\sum_{j=1}^{\infty} t_j z^j)\,\frac{\tau(t + [z^{-1}])}{\tau(t)}.\end{aligned}$$

Denoting $q(t) := 2\partial_x^2 \ln \tau(t)$ $(t \in \mathbb{C}^\infty, t_1 \equiv x)$, it is well known [AvM] that the functions $\psi(t,z)$ and $\psi^*(t,z)$ satisfy the Sturm-Liouville equation (1) with potential $q(t)$ (where $t_1 \equiv x$ and $t_3,\ t_5,\ \ldots$ are parameters).

The main result of the present paper is the following

Theorem 1. *From FI (3) for KdV tau-functions the FTI (2) follows.*

1. Preliminary results

First of all, let us mention some obvious relations for the Wronskians.

Lemma 1.1. *For arbitrary functions we have:*

(i) $W(e^{z_1 x} f,\ e^{z_2 x} g) = e^{(z_1+z_2)x}\{W(f,g) - (z_1 - z_2) fg\}$,
(ii) $W(f_1/g, f_2/g) = W(f_1, f_2)/g^2$,
(iii) $\partial_x(f_1 f_2/g^2) = -\{f_1\, W(f_2, g) + f_2\, W(f_1, g)\}/g^3$,
(iv) $W(f_1 f_2, g_1 g_2) = f_1 g_1\, W(f_2, g_2) + f_2 g_2\, W(f_1, g_1)$.

It is easy to prove [Mi] the relations in the next lemma using the "differential Fay identity" [AvM].

Lemma 1.2. *Let $\tau(t)$ be a KdV tau-function. Then we have* $(\mu, \lambda \in \mathbb{C})$

(i) $W(\tau(t+[\mu^{-1}]), \tau(t+[\lambda^{-1}]))$
$= -(\mu - \lambda)\{\tau(t+[\mu^{-1}])\ \tau(t+[\lambda^{-1}]) - \tau(t)\ \tau(t+[\mu^{-1}]+[\lambda^{-1}])\}$,
(ii) $W(\tau(t-[\mu^{-1}]), \tau(t+[\lambda^{-1}]))$
$= (\mu + \lambda)\{\tau(t-[\mu^{-1}])\ \tau(t+[\lambda^{-1}]) - \tau(t)\ \tau(t-[\mu^{-1}]+[\lambda^{-1}])\}$,
(iii) $W(\tau(t+[\mu^{-1}]), \tau(t-[\lambda^{-1}]))$
$= -(\mu + \lambda)\{\tau(t+[\mu^{-1}])\ \tau(t-[\lambda^{-1}]) - \tau(t)\ \tau(t+[\mu^{-1}]-[\lambda^{-1}])\}$,
(iv) $W(\tau(t-[\mu^{-1}]-[\lambda^{-1}]), \tau(t))$
$= (\mu + \lambda)\ \{\tau(t)\ \tau(t-[\mu^{-1}]-[\lambda^{-1}]) - \tau(t-[\mu^{-1}])\ \tau(t-[\lambda^{-1}])\}$,
(v) $W(\tau(t+[\mu^{-1}]+[\lambda^{-1}]), \tau(t))$
$= -(\mu + \lambda)\{\tau(t)\ \tau(t+[\mu^{-1}]+[\lambda^{-1}]) - \tau(t+[\mu^{-1}])\ \tau(t+[\lambda^{-1}])\}$.

Using the relations of Lemma 1.2 we will explain the Wronskians of the wave functions ψ and ψ^* in terms of tau-function $\tau(t)$.

Lemma 1.3. *Let $\tau(t)$ be a KdV tau-function and ψ, ψ^* be the corresponding wave functions. Then we have* $(\mu, \lambda \in \mathbb{C})$:

(i) $W(\psi(t,\mu), \psi(t,\lambda))$
$= (\mu - \lambda)\ \exp(\sum_{j=1}^{\infty} t_j(\mu^j + \lambda^j))\ \tau(t-[\mu^{-1}]-[\lambda^{-1}])/\tau(t)$,
(ii) $W(\psi^*(t,\mu), \psi^*(t,\lambda))$
$= -(\mu - \lambda)\ \exp(-\sum_{j=1}^{\infty} t_j(\mu^j + \lambda^j))\ \tau(t+[\mu^{-1}]+[\lambda^{-1}])/\tau(t)$.

Proof. Let us denote the functions:

$$\varphi(t,z) := e^{zx}\ \tau(t-[z^{-1}])\ /\tau(t), \quad \varphi^*(t,z) := e^{-zx}\ \tau(t+[z^{-1}])\ /\tau(t).$$

Then we have

$$\psi(t,z) = \exp(\sum_{j=2}^{\infty} t_j z^j)\ \varphi(t,z), \quad \psi^*(t,z) = \exp(-\sum_{j=2}^{\infty} t_j z^j)\ \varphi(t,z),$$

and consequently,

$$W(\psi(t,\mu),\ \psi(t,\lambda)) = \exp(\sum_{j=2}^{\infty} t_j(\mu^j+\lambda^j))\ W(\varphi(t,\mu),\varphi(t,\lambda)), \text{ etc.}$$

Using the relations of Lemma 1.1 and Lemma 1.2, we have

$$\begin{aligned} W(\varphi(t,\mu),\ \varphi(t,\lambda)) &\\ &= W(e^{\mu x}\ \tau(t-[\mu^{-1}])\ /\tau(t),\ e^{\lambda x}\ \tau(t-[\lambda^{-1}])\ /\tau(t)) \\ = e^{(\mu+\lambda)x}\ \{W(\tau(t-[\mu^{-1}]),\ &\tau(t-[\lambda^{-1}]))\ /\tau^2(t) \\ &-(\mu-\lambda)\tau(t-[\mu^{-1}])\ \tau(t-[\lambda^{-1}])\ /\tau^2(t)\} \\ = e^{(\mu+\lambda)x}\ /\tau^2(t)\ \ &\{(\mu-\lambda)(\tau(t-[\mu^{-1}])\tau(t-[\lambda^{-1}]) - \tau(t) \\ &\tau(t-[\mu^{-1}]-[\lambda^{-1}])) - (\mu-\lambda)\tau(t-[\mu^{-1}])\tau(t-[\lambda^{-1}])\} \\ &= (\mu-\lambda)e^{x(\mu+\lambda)}\tau(t-[\mu^{-1}]-[\lambda^{-1}])\ /\tau(t), \end{aligned}$$

and from there follows (i), because we have $(t_1 \equiv x)$

$$e^{x(\mu+\lambda)}\ \exp(\sum_{j=2}^{\infty} t_j(\mu^j+\lambda^j)) \ = \ \exp(\sum_{j=1}^{\infty} t_j(\mu^j+\lambda^j)).$$

It is easy to prove (ii) in the same way. ∎

Lemma 1.4. *Let $\tau(t)$ be a KdV tau-function and $\psi(t,z)$, $\psi^*(t,z)$ be the corresponding wave functions. Then we have $(\mu,\lambda \in \mathbb{C})$*

$$\begin{aligned} &W(\psi(t,\mu)\psi^*(t,\mu), \psi(t,\lambda)\psi^*(t,\lambda)) \\ &= (\mu-\lambda)\ /\tau^3(t)\ \{\tau(t+[\mu^{-1}]+[\lambda^{-1}])\ \tau(t-[\mu^{-1}])\ \tau(t-[\lambda^{-1}]) \\ &\quad -\tau(t-[\mu^{-1}]-[\lambda^{-1}])\ \tau(t+[\mu^{-1}])\ \tau(t+[\lambda^{-1}])\}. \end{aligned}$$

Proof. Using the definition of the wave functions ψ and ψ^* and the relations from Lemma 1.1 and Lemma 1.3 we have

$$
\begin{aligned}
&W(\psi(t,\mu)\psi^*(t,\mu),\ \psi(t,\lambda)\ \psi^*(t,\lambda)) \\
&= W(\tau(t-[\mu^{-1}])\ \tau(t+[\mu^{-1}])/\tau^2(t),\ \tau(t-[\lambda^{-1}])\ \tau(t+[\lambda^{-1}])\ /\tau^2(t)) \\
&= \tfrac{1}{\tau^4(t)}\ W(\tau(t-[\mu^{-1}])\ \tau(t+[\mu^{-1}]),\ \tau(t-[\lambda^{-1}])\ \tau(t+[\lambda^{-1}])) \\
&= \tfrac{1}{\tau^4(t)}\{\tau(t+[\mu^{-1}])\ \tau(t+[\lambda^{-1}])\ W(\tau(t-[\mu^{-1}]),\ \tau(t-[\lambda^{-1}])) \\
&\quad +\tau(t-[\mu^{-1}])\ \tau(t-[\lambda^{-1}])\ W(\tau(t+[\mu^{-1}]),\ \tau(t+[\lambda^{-1}]))\} \\
&= (\mu-\lambda)\ /\tau^3(t)\{\tau(t+[\mu^{-1}]+[\lambda^{-1}])\ \tau(t-[\mu^{-1}])\ \tau(t-[\lambda^{-1}]) \\
&\quad -\tau(t-[\mu^{-1}]-[\lambda^{-1}])\ \tau(t+[\mu^{-1}])\ \tau(t+[\lambda^{-1}])\}.
\end{aligned}
$$

■

2. Proof of the main result

Proof of Theorem 1.

Using the results of Lemma 1.1, Lemma 1.2 and Lemma 1.3 we explain the R.H.S. of (2) in terms of tau-function $\tau(t)$. We have

$$
\begin{aligned}
&-(\mu^2-\lambda^2)^{-1}\ \partial_x\ \{W(\psi(t,\mu),\ \psi(t,\lambda))\ W(\psi^*(t,\mu),\ \psi^*(t,\lambda))\} \\
&= (\mu-\lambda)^2(\mu^2-\lambda^2)^{-1}\ \partial_x\{\tau(t-[\mu^{-1}]-[\lambda^{-1}])\ \tau(t+[\mu^{-1}]+[\lambda^{-1}])\ /\tau^2(t)\} \\
&= -(\mu-\lambda)^2(\mu^2-\lambda^2)^{-1}\ 1/\tau^3(t)\ \{\tau(t-[\mu^{-1}]-[\lambda^{-1}])\ W(\tau(t+[\mu^{-1}]+[\lambda^{-1}] \\
&\quad +\tau(t))+\tau(t+[\mu^{-1}]+[\lambda^{-1}])\ W(\tau(t-[\mu^{-1}]-[\lambda^{-1}]),\ \tau(t))\} \\
&= (\mu-\lambda)\ /\tau^3(t)\ \{\tau(t+[\mu^{-1}]+[\lambda^{-1}])\ \tau(t-[\mu^{-1}])\ \tau(t-[\lambda^{-1}]) \\
&\quad -\tau(t-[\mu^{-1}]-[\lambda^{-1}])\ \tau(t+[\mu^{-1}])\ \tau(t+[\lambda^{-1}])\},
\end{aligned}
$$

i.e. we obtain the expression from Lemma 1.4 of the L.H.S. of the identity (2).

■

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