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Differentiable Functions on Abelian Hilbert-Lie Groups

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A convolution semigroup plays an important role in the theory of probability measure on Lie groups. The basic problem is that one wants to express a semigroup as a Lévy-Khinchine formula. If $(\mu_t)_{t\in\mathbb{R}_+^*}$ is a continuous semigroup of probability measures on an abelian Hilbert-Lie group G, then we define

$$T_{\mu_t}f := \int f_a\mu_t(da) \quad (f \in C_u(G), t > 0).$$

It is appearent that $(T_{\mu_t})_{t\in\mathbb{R}_+^*}$ is a continuously semigroup on the space $C_u(G)$ with the infinitesimal generator N. The generating functional A of this semigroup is defined by $A(f) := (Nf)(e)$. We have the problem of contruction of a subspace $C_{(2)}(G)$ of $C_u(G)$ such that the generating functional A on $C_{(2)}(G)$ exists. This result will be used later to show that the Lévy-Khinchine formula holds for abelian Hilbert-Lie groups.

1. Preliminaries and introduction

$\mathbb{N}$ and $\mathbb{R}$ denote the sets of positive integers and real numbers, respectively. Morever, let $\mathbb{R}_+ := \{r : r \geq 0\}$, $\mathbb{R}_+^* := \{r : r > 0\}$.

Let A be a set and B a subset of A. Then by 1_B we denote the indicator function of B. Let I be a nonvoid set. δ_{ij} is the Kronecker delta $(i, j \in I)$.

By G we denote a topological Hausdorff group with identity e. G is called Polish group, if G is a topological group with a countable basis of its topology and with a complete left invariant metric d which induces the topology.

For every function $f : G \to \mathbb{R}$ and $a \in G$ the functions f^*, $R_a f = f_a$ and $L_a f = {}_a f$ are defined by $f^*(b) = f(b^{-1}), f_a(b) = f(ba)$ and ${}_a f(b) = f(ab)$ for all $b \in G$, respectively. Moreover let $supp(f) = \overline{\{a \in G : f(a) \neq 0\}}$ denote the support of f. By $C_u(G)$ we denote the Banach space of all real-valued bounded left uniformly (or d-uniformly) continuous functions on G furnished with the supremum norm $\|\cdot\|$.

An abelian Hilbert-Lie group is an abelian analytic manifold modeled on a separable Hilbert space, whose group operations are analytic. It is well known that the abelian Hilbert-Lie groups are Polish.

For the exponential mapping $\mathcal{E}xp : T_e \longrightarrow G$ there exists an inverse mapping log from a neighborhood U_e of e onto a neighborhood N_0 of zero in T_e, where T_e is the tangential space in $e \in G$ ([Mai]).

By $\mathcal{B}(G)$ we denote the σ-field of Borel subsets of G. Moreover, $\mathcal{V}(e)$ denotes the system of neighborhoods of the identity e of G which are in $\mathcal{B}(G)$.

$\mathcal{M}(G)$ denotes the vector space of real-valued (signed) measures on $\mathcal{B}(G)$. As is well known, $\mathcal{M}(G)$ is a Banach algebra with respect to convolution $*$ and the norm $\|\cdot\|$ of total variation. $\mathcal{M}_+(G)$ is the set of positive measures in $\mathcal{M}(G)$ and $\mathcal{M}^1(G) = \{\mu \in \mathcal{M}(G) : \mu(G) = 1\}$ is the set of probability measure on G.

Now let $\gamma_X(t) := \mathcal{E}xp(tX)$ for $X \in H$ and $t \in \mathbb{R}^*$.

Definition 1.1. Let $f \in C_u(G)$, $X \in H$ and $a \in G$. f is called *differentiable at $a \in G$ with respect to X* ("$Xf(a)$ exists" for short), if

$$Xf(a) := \lim_{t\to 0} \frac{1}{t}[L_{\gamma_X(t)}f(a) - f(a)] = \lim_{t\to 0} \frac{1}{t}[R_{\gamma_X(t)}f(a) - f(a)]$$

exists. f is called *continuously differentiable*, if $Xf(a)$ exists for all $a \in G$ and $X \in H$, and if the mappings $a \longmapsto Xf(a)$, $X \longmapsto Xf(a)$ are continuous.

Similarly f is called *uniformly differentiable with respect to X* ("Xf exists" for short), if

$$Xf := \|\cdot\| - \lim_{t\to 0} \frac{1}{t}[L_{\gamma_X(t)}f - f]$$

exists.

Derivatives of higher orders are defined inductively.

The following properties of the derivatives are well known for continuously differentiable functions.

Remark 1.2. Let $f, g \in C_u(G)$, $X \in H$ and $a \in G$.

(i) If $Xf(a)$ exists, then the mapping $X \longmapsto Xf(a)$ is linear.

(ii) If $Xf(a)$ and $Xg(a)$ exist, then also $X(f\cdot g)(a)$ exists and $X(f\cdot g)(a) = Xf(a)\cdot g(a) + f(a)\cdot Xg(a)$.

Now let $f \in C_u(G)$ be twice continuously differentiable function. Then the mapping

$$Df(a) : X \longmapsto Xf(a) \qquad (D^2f(a) : (X,Y) \longmapsto XYf(a))$$

is continuous and linear (resp. symmetric, continuous and bilinear) functional on H (resp. $H \times H$) for all $a \in G$. It holds also

$$< Df(a), X >= Xf(a) \qquad \text{and } < D^2f(a)(X), Y >= XYf(a)$$

for all $a \in G$ and $X, Y \in H$.

We define by $C_2(G)$ the space of all twice continuously differentiable functions $f \in C_u(G)$ such that the mapping $a \longmapsto D^2f(a)$ is d-uniformly continuous and $\|Df\| := \sup_{a \in G} \|Df(a)\| < \infty$, $\|D^2f\| := \sup_{a \in G} \|D^2f(a)\| < \infty$. It is easy to see that the space $C_2(G)$ is a Banach space with respect to the norm

$$\|f\|_2 := \|f\| + \|Df\| + \|D^2f\|, \quad f \in C_2(G)$$

and

$$R_aC_2(G) \subset C_2(G))$$

is satisfied for all $a \in G$. (cf. [Er91] However $C_2(G)$ is not dense in $C_u(G)$ (cf. [Sam].)

By $a_i(a) :=< \log(a), X_i >$ $(i \in \mathbb{N})$ we define maps a_i from the canonical neighborhood U_e in $\mathbb{R}$. Now we call the system $(a_i)_{i \in \mathbb{N}}$ of maps from U_e in $\mathbb{R}$ a system of canonical coordinates of G with respect to orthonormal base $(X_i)_{i \in \mathbb{N}}$, if for all $a \in U_e$ the property $a = \mathcal{E}xp(\sum_{i=1}^{\infty} a_i(a)X_i)$ is satisfied.

Lemma 1.3. *Let $f \in C_2(G)$. Then*

(i) $(\sum_{i=1}^{\infty} a_i(a)X_i)f = \sum_{i=1}^{\infty} a_i(a)X_if$ for all $a \in U_e$.

(ii) $(\sum_{i=1}^{\infty} a_i(a)X_i)\left(\left(\sum_{j=1}^{\infty} a_j(c)X_j\right)f\right) = \sum_{i=1,j=1}^{\infty} a_i(a)a_j(c)X_iX_jf$ for all $a, c \in U_e$.

P r o o f. (i) For any $a \in U_e$ there exists a $X \in H$ with $X = \log(a)$. Then we have $X = \sum_{i=1}^{\infty} < X, X_i > X_i = \sum_{i=1}^{\infty} a_i(a)X_i$. Thus

$$\begin{aligned} Xf(e) &= \frac{d}{dt}|_{t=0} f(\gamma_X(t)) =< Df(e), X > \\ &= \sum_{i=1}^{\infty} a_i(a) < Df(e), X_i > \\ &= \sum_{i=1}^{\infty} a_i(a)X_if(e). \end{aligned}$$

Now let $b \in G$ be an arbitrary point. Then is $R_bf \in C_2(G)$, whence the assertion. The proof of (ii) can be proved similarly. ■

In the following we give the Taylor expansion for the functions $f \in C_2(G)$.

Proposition 1.4. *Let $f \in C_2(G)$. Then the Taylor-expansion of the second order for f at $e \in G$ is given by*

$$f(a) = f(e) + \sum_{i=1}^{\infty} a_i(a) X_i f(e) + \frac{1}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i(a) a_j(a) X_i X_j f(\bar{a})$$

for all $a \in U_e$, where $\bar{a}$ is a point of U_e.

Proof. Let $f \in C_2(G)$ and $X \in H$. Then the function $\chi : t \longmapsto f(\gamma_X(t))$ is twice differentiable on $\mathbb{R}$ and therefore admits a Taylor-expansion valid up to second order:

$$\chi(t) = \chi(0) + \chi'(0) \cdot t + \frac{1}{2}\chi''(\bar{t}) \cdot t^2$$

for any $\bar{t} \in [-\mid t \mid, \mid t \mid]$. Since $\chi'(0) = Xf(e)$ and $\chi''(\bar{t}) = XXf(\gamma_X(\bar{t}))$ it follows from by Lemma 1.3 that

$$\begin{aligned} f(\gamma_X(t)) = f(e) \quad &+ \quad \sum_{i=1}^{\infty} < tX, X_i > X_i f(e) \\ &+ \quad \frac{1}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} < tX, X_i >< tX, X_j > X_i X_j f(\gamma_X(\bar{t})) \end{aligned}$$

for some $\bar{t} \in [-\mid t \mid, \mid t \mid]$. This yields the assertion. ■

Remark 1.5. The Taylor-expansion of $f \in C_2(G)$ can be written in a closed form, i.e.

$$f(a) = f(e) + < Df(e), log(a) > + \frac{1}{2} < D^2 f(\bar{a})(log(a)), log(a) >$$

for all $a \in U_e$ and for any $\bar{a}$ in the canonical neighborhood U_e.

2. Convolution semigroups of probability measures

For any probability measure μ on G, we define the operator T_μ on $C_u(G)$ by

$$T_\mu f := \int f_a \mu(da) \qquad \text{(Bochner-Integral)}.$$

It is easy to see that $T_\mu C_u(G) \subset C_u(G)$ and $T_{\mu * \nu} = T_\mu \circ T_\nu$.

A *convolution semigroup* is a family $(\mu_t)_{t \in \mathbb{R}_+^*}$ in $\mathcal{M}^1(G)$ such that $\mu_0 = \varepsilon_e$ and $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \in \mathbb{R}_+^*$.

$(\mu_t)_{t\in\mathbb{R}_+^*}$ is called *continuous*, if $\lim_{t\to 0}\mu_t = \varepsilon_e$ (weakly). It is well known that the convolution semigroup $(\mu_t)_{t\in\mathbb{R}_+^*}$ is continuous iff the corresponding operator semigroup $(T_{\mu_t})_{t\in\mathbb{R}_+^*}$ is (strongly) continuous.

The Hille-Yosida theory establishes a bijection between (strongly) continuous operator semigroups $(T_{\mu_t})_{t\in\mathbb{R}_+^*}$ and their infinitesimal generators. N is defined on its domain $D(N)$ which is dense in $C_u(G)$.

It is clear that N commutes with the left translations, i.e.

$$L_a D(N) \subset D(N) \quad \text{and} \quad L_a \circ N = N \circ L_a \qquad \text{for all } a \in G.$$

A continuous convolution semigroup $(\mu_t)_{t\in\mathbb{R}_+^*}$ in $\mathcal{M}^1(G)$ admits a Lévy measure η i.e η is a σ-finite positive measure on $\mathcal{B}(G)$ such that $\eta(\{e\}) = 0$ and such that

$$\lim_{t\downarrow 0}\frac{1}{t}\int f\,d\mu_t = \int f\,d\eta,$$

for all $f \in C_u(G)$ with $e \notin supp(f)$ (cf. [Sie85]).

Lemma 2.1. *Let $(\mu_t)_{t\in\mathbb{R}_+^*}$ be a continuous convolution semigroup in $\mathcal{M}^1(G)$. Then for every $U \in \mathcal{V}(e)$ is*

$$\sup_{t\in\mathbb{R}_+^*}\frac{1}{t}\mu_t(U^c) < \infty.$$

Proof. Let U and V be two neighborhoods in $e \in G$ with $\overline{V} \subset U$. Since G is a normal group, there exists a function $f \in C_u(G)$ such that

$$0 \le f \le 1, \quad f(V) = \{0\} \quad \text{and} \quad f(U^c) = \{1\}.$$

Then we have $\frac{1}{t}\mu_t(U^c) \le \frac{1}{t}\int f\,d\mu_t$ for all $t \in \mathbb{R}_+^*$. $f \in C_u(G)$ with $e \notin supp(f)$ implies that

$$\lim_{t\downarrow 0}\frac{1}{t}\int f\,d\mu_t = \int f d\eta.$$

Hence the assertion follows. ∎

Let H be a separable Hilbert space with a complete orthonormal system $(X_i)_{i\in\mathbb{N}}$ and G an abelian Hilbert-Lie group on H. Moreover, let

$$H_n := \langle \{X_1, X_2, \cdots, X_n\} \rangle$$

be the space of all linear combinations of $X_1, X_2, \cdots, X_n$ and $H_n^\perp$ the orthogonal complement of H_n in H (for all $n \in \mathbb{N}$). Then $H/H_n^\perp$ and H_n are isomorphic. Clearly

$$G_n := \mathcal{E}xp(H_n^\perp)$$

is a closed subgroup of G for all $n \in \mathbb{N}$. The quotient spaces G/G_n are finite-dimensional Hilbert-Lie groups. Now let p_n be the canonical projection from G onto G/G_n and $\{b_i^n : i = 1, 2, \cdots, n\}$ a system of extended canonical coordinates with respect to $\{X_1, X_2, \cdots, X_n\}$. We now define the functions $d_i^n := b_i^n \circ p_n \in C_2(G)$; then $X_j d_i^n$ exists and

$$X_j d_i^n = X_j(b_i^n \circ p_n) = X_j b_i^n \circ p_n = 0$$

holds for all $j > n$ and $i = 1, 2, \cdots, n$.

Definition 2.2. Let G be an abelian Hilbert-Lie group on H, and $(X_i)_{i \in \mathbb{N}}$ an orthonormal basis in H. For any $n \in \mathbb{N}$ we define

$$\begin{aligned} C_{(2),n}(G) := \{f \in C_2(G) \ : \ & X_i f = 0 \text{ for all } i > n \text{ and} \\ & X_i X_j f = 0 \text{ for all } i > n \text{ or } j > n\}. \end{aligned}$$

Remark 2.3. Let $f \in C_u(G)$ be a uniformly differentiable function with respect to X which satisfies that $X_i f = 0$ for all $i > n$ $(n \in \mathbb{N})$. Let π_n be the orthogonal projection from H onto H_n. Then we have

$$Xf = \pi_n(X)f \qquad \text{for all } X \in H.$$

So, f is continuously differentiable and clearly $(C_{(2),n}(G))_{n \in \mathbb{N}}$ is a monoton increasing sequence of Banach subalgebras of Banach algebra $C_2(G)$.

Further properties of $C_{(2),n}(G)$ ($n \in \mathbb{N}$):
(i) $C_{(2),n}(G)$ are $\|\cdot\|_2$-closed in $C_2(G)$
and
(ii) For any probability measure $\mu \in \mathcal{M}^1(G)$, we have

$$T_\mu C_{(2),n}(G) \subset C_{(2),n}(G) \quad \text{for all } n \in \mathbb{N}.$$

Thus $\overline{C_{(2),n}(G) \cap D(N)}^{\|\cdot\|_2} = C_{(2),n}(G)$. Now consider the subspace

$$C_{(2)}(G) := \bigcup_{n \in \mathbb{N}} C_{(2),n}(G).$$

$C_{(2)}(G)$ is obviously a linear subspace of $C_2(G)$ with $T_\mu C_{(2)}(G) \subset C_{(2)}(G)$ for probability measures $\mu \in \mathcal{M}^1(G)$. Especially $\overline{C_{(2)}(G)}^{\|\cdot\|_2}$ is a Banach space with $T_\mu \overline{C_{(2)}(G)}^{\|\cdot\|_2} \subset \overline{C_{(2)}(G)}^{\|\cdot\|_2}$.

Remark 2.4. For $n \in \mathbb{N}$ let $\{b_i^n : i = 1, 2, \cdots, n\}$ be a system of extended canonical coordinates with respect to $\{X_1, X_2, \cdots, X_n\}$. Then every abelian Hilbert-Lie group has the property that

$$b_i^n \in C_{(2),n}(G) \quad \text{for all } i = 1, 2, \cdots, n, \ n \geq n_0$$

and for any $n_0 \in \mathbb{N}$. In the finite dimensional case we have $n_0 = \dim(G)$.

In what follows we say always that G is abelian.

Since $C_{(2),n}(G) \subset C_{(2),n+1}(G)$, a system $\{b_i^n, b_{n+1}^{n+1} : i = 1,2,\cdots,n\} \subset C_{(2),n+1}(G)$ of canonical coordinates exists with respect to $\{X_1, X_2, \cdots, X_{n+1}\}$. We also have the following proposition:

Proposition 2.5. *There exists a system $(d_n)_{n\in\mathbb{N}}$ of functions in $C_{(2)}(G)$ with*

$$d_i = b_i^{n_0} \quad \textit{for all } i = 1,2,\cdots,n_0$$

and

$$d_n = b_n^n \quad \textit{for all } n > n_0.$$

This system $(d_n)_{n\in\mathbb{N}}$ is called a system of local canonical coordinates with respect to $(X_i)_{i\in\mathbb{N}}$.

We define now for any $n \in \mathbb{N}$ the functions

$$\Phi_n(a) := \sum_{i=1}^{n} d_i(a)^2, \qquad a \in G,$$

where $(d_i)_{i=1,2,\cdots,n}$ is a system of local canonical coordinates with respect to $\{X_1, X_2, \cdots, X_n\}$. Then $\Phi_n \in C_{(2),n}(G)$ and $\Phi_n(a) > 0$ for all $a \in G\backslash\{\Phi_n = 0\}$. Therefore

$$X_i\Phi_n(e) = 0, \quad X_iX_j\Phi_n(e) = 2\delta_{ij}, \quad i,j = 1,2,\cdots,n$$

(cf. [Hey], Lemma 4.1.9 und 4.1.10).

The following lemma is a consequence of the Banach-Steinhaus theorem and the Hille-Yosida theory (cf. [Hey], Lemma 4.1.11).

Lemma 2.6. *For every $f \in C_{(2),n}(G)$ and every $\varepsilon > 0$ there is a $g := g_\varepsilon \in C_{(2),n}(G) \cap D(N)$ such that $\|f - g\|_2 < \varepsilon$, $f(e) = g(e) = 0$, $X_if(e) = X_ig(e) = 0$ and $X_iX_jf(e) = X_iX_jg(e)$ for $i,j = 1,2,\cdots,n$.*

Proposition 2.7. *Let $(\mu_t)_{t\in\mathbb{R}_+^*}$ be a convolution semigroup in $\mathcal{M}^1(G)$ and Φ_n, $(n \in \mathbb{N})$ be as in above. Then the suprema*

$$\sup_{t\in\mathbb{R}_+^*} \frac{1}{t}\int \Phi_n \, d\mu_t$$

are finite for every $n \in \mathbb{N}$.

Proof. Application of Lemma 2.6 to the function $\Phi_n \in C_{(2),n}(G)$ yields the existence of a function $\Psi_n \in C_{(2),n}(G) \cap D(N)$ with the property

$$\|\Phi_n - \Psi_n\|_2 < \varepsilon, \quad \Psi_n(e) = \Phi_n(e) = 0, \quad X_i\Psi_n(e) = X_i\Phi_n(e) = 0$$
$$\text{and} \quad X_iX_j\Psi_n(e) = X_iX_j\Phi_n(e) = 2\delta_{ij}, \qquad i,j = 1,2,\cdots,n.$$

Taylor expansion of $\Psi_n \in C_{(2),n}(G) \cap D(N)$ in a neighborhood W_1 of e with $W_1 \subset U_e$

$$\Psi_n(a) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_i(a)d_j(a)X_iX_j\Psi_n(\bar{a}),$$

for all $a \in W_1$ with $\bar{a} \in W_1$. Since $\|\Phi_n - \Psi_n\|_2 < \varepsilon$ and $X_iX_j\Psi_n(e) = 2\delta_{ij}$, $i,j = 1,2,\cdots,n$ there exists a neighborhood W_2 of e with the properties

$$-\varepsilon \le X_iX_j\Psi_n(a) \le \varepsilon \qquad \text{for all } i,j = 1,2,\cdots,n,\ i \ne j,$$

$$2 - \varepsilon \le X_iX_i\Psi_n(a) \le 2 + \varepsilon \qquad \text{for all } i = 1,2,\cdots,n,$$

whenewer $a \in W_2$. Putting $\delta_n := \delta_n(\varepsilon) := \frac{1}{2}(2 - \varepsilon - \varepsilon(n-1))$ and $W := W_1 \cap W_2$, we obtain

$$\Psi_n(a) \ge \delta_n \cdot \sum_{i=1}^{n} d_i(a)^2 \qquad \text{for all } a \in W.$$

Since $\Psi_n \in C_{(2),n}(G) \cap D(N)$, we obtain $\sup_{t\in\mathbb{R}_+^*} \frac{1}{t}|\int_W \Psi_n \, d\mu_t| < \infty$ from Lemma 2.1. Thus $\sup_{t\in\mathbb{R}_+^*} \frac{1}{t}\int_W \Phi_n \, d\mu_t < \infty$, and since Φ_n is bounded follows from Lemma 2.1 the assertion. ∎

Now let $(d_i)_{i\in\mathbb{N}}$ a system of local canonical coordinates with respect to $(X_i)_{i\in\mathbb{N}}$. By Lemma 2.6 there exist functions $z_i \in C_{(2),n}(G) \cap D(N)$, $(n \in \mathbb{N})$ with the property

$$z_i(e) = d_i(e) = 0, \quad X_jz_i(e) = X_jd_i(e) = \delta_{ij}, \quad i,j = 1,2,\cdots,n.$$

Proposition 2.8. *Let $(\mu_t)_{t\in\mathbb{R}_+^*}$ be a convolution semigroup in $\mathcal{M}^1(G)$. Then the generating functional A of $(\mu_t)_{t\in\mathbb{R}_+^*}$ on $C_{(2)}(G)$ exists, i.e.*

$$C_{(2)}(G) \subset D(A).$$

Proof. Let $f \in C_{(2),n}(G)$ $(n \in \mathbb{N})$ and setting

$$g(a) := f(a) - f(e) - \sum_{i=1}^{n} z_i(a) \cdot X_if(e) \qquad \text{for all } a \in G,$$

where the functions z_i, $i = 1, 2, \cdots, n$ are as in above. Then $g \in C_{(2),n}(G)$ with $g(e) = 0$, $X_j g(e) = X_j f(e) - \sum_{i=1}^n X_j z_i(e) \cdot X_i f(e) = X_j f(e) - \sum_{i=1}^n \delta_{ij} \cdot X_i f(e) = 0$. The Taylor expansion of g in a neighborhood $W \subset U_e$ gives

$$g(a) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} d_i(a) d_j(a) X_i X_j g(\bar{a}), \qquad a \in W.$$

Thus there is a constant $k_1 \in \mathbb{R}_+^*$ such that

$$| g(a) | \leq k_1 \cdot \|g\|_2 \cdot \Phi_n(a) \qquad \text{for all } a \in W.$$

It follows from Lemma 2.7 that

$$\text{(1)} \qquad \sup_{t \in \mathbb{R}_+^*} \left| \frac{1}{t} \int_W g \, d\mu_t \right| \leq k_1 \cdot \|g\|_2 \cdot \sup_{t \in \mathbb{R}_+^*} \int \Phi_n \, d\mu_t < \infty.$$

Clearly, $| \frac{1}{t} \int_{W^c} g \, d\mu_t | \leq \|g\|_2 \cdot \frac{1}{t} \mu_t(W^c)$, and $\sup_{t \in \mathbb{R}_+^*} | \frac{1}{t} \int_{W^c} g \, d\mu_t | < \infty$. Hence, there exists a constant $k_2 \in \mathbb{R}_+^*$ independent of t such that

$$\text{(2)} \qquad \left| \frac{1}{t} \int_{W^c} g \, d\mu_t \right| \leq k_2 \cdot \|g\|_2 \qquad \text{for all } t \in \mathbb{R}_+^*.$$

Adding the inequalities (1) and (2)

$$\left| \frac{1}{t} [T_{\mu_t} f(e) - f(e)] - \frac{1}{t} \sum_{i=1}^{n} X_i f(e) \cdot T_{\mu_t} z_i(e) \right| \leq k_3 \cdot \|f\|_2, \quad \forall t \in \mathbb{R}_+^*.$$

where k_3 is a constant (independent of t). Since $z_i \in D(N)$ and $z_i(e) = 0$, we have $\sup_{t \in \mathbb{R}_+^*} |\frac{1}{t} T_{\mu_t} z_i(e)| < \infty$ for all $i = 1, 2, \cdots, n$.

Hence we obtain a constant $k(n) \in \mathbb{R}_+^*$ depending only on n such that

$$\left| \frac{1}{t} (T_{\mu_t} f(e) - f(e)) \right| \leq k(n) \cdot \|f\|_2$$

for all $t \in \mathbb{R}_+^*$ and $f \in C_{(2),n}(G)$. By the theorem of Banach-Steinhaus the limit

$$\lim_{t \downarrow 0} \frac{1}{t} [T_{\tilde{\mu}_t} f(e) - f(e)]$$

exists for every $f \in C_{(2)}(G)$. ∎

R e m a r k 2.9. Let $(\mu_t)_{t\in\mathbb{R}_+^*}$ be continuous convolution semigroup in $\mathcal{M}^1(G)$. As in the proof of Proposition 2.8, we can find a constant $k(n) \in \mathbb{R}_+^*$ (independent of $a \in G$ and $t \in \mathbb{R}_+^*$) such that

$$\begin{aligned} \left| \frac{1}{t}[T_{\mu_t} f(a) - f(a)] \right| &= \left| \frac{1}{t}[T_{\mu_t}(L_a f)(e) - (L_a f)(e)] \right| \\ &\leq k(n) \cdot \|L_a f\|_2 = k(n) \cdot \|f\|_2 \end{aligned}$$

for all $f \in C_{(2),n}(G)$ and $a \in G$. The Banach-Steinhaus theorem now yields the existence of the limit

$$Nf(a) = \lim_{t\downarrow 0} \frac{1}{t}[T_{\mu_t} f(a) - f(a)]$$

uniformly in $a \in G$. This implies existence of the infinitesimal generator N on $C_{(2)}(G)$.

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