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On the Quadratic Discrepancy of One Class of Two-Dimensional Nets

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In this present paper the author considers one class of two-dimensional nets, constructed in r-adic number system, $r \geq 2$. We calculate exact constant, depending on r in the $O(N^{-1}(\log N)^{1/2})$, giving exact order of magnitude of quadratic discrepancy of the nets from this class.

1. Introduction

Let in the quadrate $E^2 = [0,1]^2$ be given a finite sequence $\sigma_n = \{(x_i, y_i) : 0 \leq i \leq n-1\}$, that we will call a net. For arbitrary rectangle $[0,x) \times [0,y)$ we denote by $A(\sigma_n; x, y)$ the number of points of σ_n, that belong to $[0,x) \times [0,y)$, i.e. $A(\sigma_n; x, y) = \#\{(x_i, y_i) \in \sigma_n : 0 \leq i \leq n-1, \quad x_i < x \text{ and } y_i < y\}$.

The quadratic discrepancy $T(\sigma_n)$ of the net σ_n is defined by the equation

$$T(\sigma_n) = (\int_0^1 \int_0^1 (n^{-1} A(\sigma_n; x, y) - xy)^2 dx dy)^{1/2} = \|n^{-1} A(\sigma_n; (x,y) - xy\|_{L_2[0,1]^2}$$

Roth [1] (see [2, Lemma 2.5]) solves a problem for the best possible order of the quadratic discrepancy of arbitrary net. It is proved that for every net σ_n, composed by n points in E^2 there exists a constant $C > 0$, such as

$$T(\sigma_n) > C n^{-1} (\log n)^{1/2}. \qquad (1)$$

The exactness of this bound is proved by Davenport [3], in sense that for every $n \geq 2$ a net X_n, composed by n points in E^2 exists, such that $T(X_n) = 0(n^{-1}(\log n)^{1/2})$.

Vilenkin [4] and [5] has constructed two-dimensional nets, having small constant in the symbol $O(n^{-1}(\log n)^{1/2})$.

We note a fact that in Vilenkin [4] is given a formula for the quadratic discrepancy $[nT(\sigma_n)]^2$: for arbitrary net $\sigma_n = \{(x_i, y_i) : \quad 0 \leq i \leq n-1\}$, composed by n points in E^2 we have the equation

$$(2) \qquad [nT(\sigma_n)]^2 = \sum_{i,j=0}^{n-1} [1 - \max(x_i, x_j)][1 - \max(y_i, y_j)] + \frac{n^2}{9} - \frac{n}{2} \sum_{i=0}^{n-1} [1 - x_i^2][1 - y_i^2].$$

2. Results

Let $r \geq 2$ be a given integer.

Definition 1. Let i be arbitrary non-negative integer and in r-adic number system has develop in the form

$$i = a_m a_{m-1} \ldots a_1,$$

where $a_j \in \{0, 1, \ldots, r-1\}$ for $j = 1, 2, \ldots, m$ and $a_m \neq 0$. Then in r-adic number system we define

$$p_r(i) = 0, a_1 a_2 \ldots a_m.$$

The sequence $(p_r(i))$ $(i = 0, 1, \ldots)$ is called a sequence of Van der Corput-Halton.

In 1935 Van der Corput [6] considered the sequence $(p_2(i))$ $(i = 0, 1, \ldots)$ in binary system. In arbitrary r-adic number system the sequence $(p_r(i))$ $(i = 0, 1, \ldots)$ is considered by Halton [7].

Using Definition 1, the following two properties of the sequence $(p_r(i))$ $(i = 0, 1, \ldots)$ can be proved.

Property 1. Let $N = r^\nu$, for some non-negative ν. Then the equation

$$\{p_r(i) : \ 0 \leq i \leq N-1\} = \{\frac{j}{N} : \ 0 \leq j \leq N-1\}$$

holds.

Property 2. Let m, n and ν be non-negative integers such that $m \equiv 0 (mod\, r^\nu)$ and $0 \leq n < r^\nu$. Then we have

$$p_r(m+n) = p_r(m) + p_r(n).$$

For every non-negative integer ν we denote $N = r^\nu$ and define the net $\Sigma_{2N}^r = \Sigma_\nu^r$,

$$(3) \qquad \Sigma_{2N}^r = \{(\frac{i}{N}, p_r(i)), (\frac{i}{N}, 1 - p_r(i)) : 0 \le i \le N-1\}.$$

Proinov and Grozdanov [8] proved that the quadratic discrepancy of the net Σ_{2N}^r satisfies the inequality

$$(4) \qquad T(\Sigma_{2N}^r) \le ((r^2-1)/(3\log r))^{1/2} N^{-1} (\log(r-1)N)^{1/2} + \frac{2}{N}.$$

The inequality (1) shows that the best possible order of the quadratic discrepancy of arbitrary two-dimensional net, composed by $2N$ points, is $O(N^{-1}(\log N)^{1/2})$. Hence, the order of upper bound obtained in (4) is exact and $T(\sum_{2N}^r) = O(N^{-1}(\log N)^{1/2})$.

In our paper we develop a geometrical process for study of quadratic discrepancy, that will permit finding the least constant $c(r)$, depending on r in the symbol $O(N^{-1}(\log N)^{1/2})$.

Theorem 1. *Let $N = r^\nu$, for some non-negative integer ν and Σ_{2N}^r be the net, defined by the equation (3). Then we have the equation*

$$T(\Sigma_{2N}^r) = (2N)^{-1}\{\frac{r^4+15r^2-16}{360r^2}\log_r N - \frac{r^2-1}{6r}\frac{\log_r N}{N}$$
$$+\frac{1}{2} - \frac{1}{18N^2} + (1-\frac{1}{N^2})\Theta\ \}^{1/2},$$

where

$$(5) \qquad \Theta = \begin{cases} 0, & \textit{if } r \textit{ is an even number} \\ \frac{1}{8}, & \textit{if } r \textit{ is an odd number.} \end{cases}$$

Let us denote $c(r) = \frac{r^4+15r^2-16}{360r^2}$ and consider the quantity $C = \min_{r\ge 2} c(r)$ It is obvious that $\min_{r\ge2} c(r) = c(2) = \frac{1}{24}$, i.e. $C = \frac{1}{24}$. The last equation shows the smallest constant in symbol $O(N^{-1}(\log N)^{1/2})$, giving exact order of the quadratic discrepancy of the net $\sum_{2N}^r$ is $\frac{1}{24}$ and it is obtained for $r = 2$. Hence, the best is the net $\sum_{2N}^r$ constructed in binary number system.

Corollary 1. *Let $N = 2^\nu$, for some non-negative integer ν and Σ_{2N}^2 be the net, defined by the equation*

$$\Sigma_{2N}^2 = \{(\frac{i}{N}, p_2(i)), (\frac{i}{N}, 1 - p_2(i)) : 0 \le i \le N-1,\},$$

where $(p_2(i))$ $(i = 0, 1, \ldots)$ *is the sequence of Van der Corput. Then we have the equation*

$$T(\Sigma_{2N}^2) = (2N)^{-1}\,\{\frac{1}{24}\log_2 N + \frac{1}{2} - \frac{1}{4N}\log_2 N - \frac{1}{18N^2}\}^{1/2}.$$

3. A formula of a breaking of the quadratic discrepancy

Let $N = r^\nu$, for some non-negative integer ν and we denote $n = \frac{N}{r}$.

We consider the net $\sigma_{2N} = \{(x_i, y_i) : 0 \le i \le 2N - 1\}$, and assume that for $1 \le k \le r$ every rectangle $\Delta_{k,r} = [0,1) \times [\frac{k-1}{r}, \frac{k}{r})$ contains exactly $2n$ points of the net σ_{2N}.

For $1 \le k \le r$ we signify by $(x_i^{(k)}, y_i^{(k)})(0 \le i \le 2N-1)$ the coordinates of the points from the net σ_{2N}, belonging to rectangle $\Delta_{k,r}$ and introduce the nets

$$\sigma_{2n}^{(k)} = \{(x_i^{(k)}, ry_i^{(k)} - (k-1)\}(1 \le k \le r).$$

We have the equation

$$\int_0^1 \int_0^1 (A(\sigma_{2N}; x, y) - 2Nxy)^2 dxdy$$

$$= \sum_{k=1}^{r} \int_0^1 \int_{\frac{k-1}{r}}^{\frac{k}{r}} (A(\sigma_{2N}; x, y) - 2Nxy)^2 dxdy.$$

In the first integral in the last sum we put $t = ry$ and use that for $(x, y) \in \Delta_{l,r}$ we have $t \in [0, 1]$ and

$$A(\sigma_{2N}; x, y) = A(\sigma_{2N}^{(1)}; x, t).$$

Hence, we get the equation

$$\int_0^1 \int_0^{\frac{1}{r}} (A(\sigma_{2N}; x, y) - 2Nxy)^2 dxdy = \frac{1}{r}[2nT(\sigma_{2n}^{(1)})]^2. \tag{6}$$

Let k, $2 \le k \le r$ be fixed integer. In the integrals

$$\int_0^1 \int_{\frac{k-1}{r}}^{\frac{k}{r}} (A(\sigma_{2N}; x, y) - 2Nxy)^2 dxdy \tag{7}$$

we put $t = ry - (k-1)$ and use the obvious equation: for $(x, y) \in \Delta_{k,r}$

$$A(\sigma_{2N}; x, y) = \sum_{j=1}^{k-1} (A(\sigma_{2N}; x, \frac{j}{r}) - A(\sigma_{2N}; x, \frac{j-1}{r}))$$

$$+A(\sigma_{2N};x,y) - A(\sigma_{2N};x,\frac{k-1}{r}).$$

For arbitrary x and y, such $(x,y) \in \Delta_{k,r}$ we have the equation ($n = \frac{N}{r}$)

$$A(\sigma_{2N};x,y) - A(\sigma_{2N};x,\frac{k-1}{r}) = A(\sigma_{2N}^{(k)};x,t),$$

and if $y = \frac{k}{r}$, then $t = 1$ in the last equation, i.e.

$$A(\sigma_{2N};x,\frac{j}{r}) - A(\sigma_{2N};x,\frac{j-1}{r}) = A(\sigma_{2N}^{(j)};x,1).$$

Hence, for $2 \le k \le r$ the integrals (7) satisfy the equations

$$\int_0^1 \int_{\frac{k-1}{r}}^{\frac{k}{r}} (A(\sigma_{2N};x,y) - 2Nxy)^2 dxdy \tag{8}$$

$$= \frac{1}{r}[2nT(\sigma_{2n}^{(k)})]^2 + \frac{1}{r}\int_0^1 [\sum_{j=1}^{k-1}[A(\sigma_{2n}^{(j)};x,1) - 2nx]\,]^2 dx$$

$$+\frac{2}{r}\sum_{j=1}^{k-1}\int_0^1\int_0^1 [A(\sigma_{2n}^{(k)};x,t) - 2nxt\,]$$

$$\times[A(\sigma_{2n}^{(j)};x,1) - 2nx\,]dxdt.$$

From (6), (7) and (8) we get the equation

$$[2NT(\sigma_{2N})]^2 = \frac{1}{r}\sum_{k=1}^{r}[2nT(\sigma_{2n}^{(k)})]^2 \tag{9}$$

$$+\frac{2}{r}\sum_{k=2}^{r}\sum_{j=1}^{k-1}\int_0^1\int_0^1 [A(\sigma_{2n}^{(k)};x,t) - 2nxt\,]$$

$$\times[A(\sigma_{2n}^{(j)};x,1) - 2nx\,]dxdt$$

$$+\frac{1}{r}\sum_{j=1}^{r-1}(r-j)\int_0^1 [A(\sigma_{2n}^{(j)};x,1) - 2nx]^2 dx$$

$$+\frac{2}{r}\sum_{j=1}^{r-2}\sum_{m=j+1}^{r-1}(r-m)\int_0^1 [A(\sigma_{2n}^{(j)};x,1) - 2nx]$$

$$\times[A(\sigma_{2n}^{(m)};x,1) - 2nx]dx.$$

Following [4], this geometrical process will be called a process of breaking of quadratic discrepancy, and the formula (9) - a formula for breaking of quadratic discrepancy.

Obviously, the net Σ_{2N}^r, defined by equation (3) satisfies the term in a process of breaking of the quadratic discrepancy. Using the definition of introduced nets $\sigma_{2n}^{(k)}(\leq k \leq r)$ we obtain that the nets $\Sigma_{2n}^{(k)}$, corresponding to the net Σ_{2N}^r, are in the form

$$\Sigma_{2n}^{(k)} = \{(\frac{i}{n} + \frac{k-1}{rn}, p_r(i)), (\frac{i}{n} + \frac{r-k}{rn}, 1 - p_r(i)) : \\ 0 \leq i \leq n-1, \quad n = r^{\nu-1}\}. \tag{10}$$

We will calculate the integrals in the formula of a breaking of the quadratic discrepancy of the net Σ_{2N}^r. Hold the next lemmas:

Lemma 1. *Let $N = r^\nu$, for some non-negative integer ν. Then we have the equations*

$$\sum_{i=0}^{N-1} p_r(i) = \frac{n-1}{2}, \sum_{i=0}^{N-1} p_r^2(i) = \frac{(N-1)(2N-1)}{6N},$$

$$\sum_{i=0}^{N-1} ip_r(i) = \frac{1}{12}[3N^2 + N(\frac{r^2-1}{r}\log_r N - 6) + 3]$$

$$\sum_{i=0}^{N-1} ip_r^2(i) = \frac{1}{12}[2N^2 + N(\frac{r^2-1}{r}\log_r N - 5) - \\ -\frac{r^2-1}{r}\log_r N + 4 - \frac{1}{N}].$$

The first two equations in the condition of the lemma are corollary from Property 1 of the sequence of Van der Corput-Halton. We will prove the last equation. Let $N = r^\nu$, for some non-negative ν and signify $z_\nu = \sum_{i=0}^{N-1} ip_r^2(i)$. Arbitrary integer i, $0 \leq i \leq N-1$ we present in the form $i = rj + \alpha$, where $0 \leq j \leq n-1$, $0 \leq \alpha \leq r-1$, for $n = \frac{N}{r}$.

Using Properties 1 and 2 of the sequence of Van der Corput-Halton and third equation in the condition of Lemma, we obtain the recurrent relation:

$$z_\nu = \sum_{\alpha=0}^{r-1}\sum_{j=0}^{n-1}(rj+\alpha)(\frac{1}{r}p(j) + \frac{\alpha}{r})^2 = z_{\nu-1} + \frac{r^2-1}{6}n^2$$

$$+\frac{(r-1)(r-4)}{12}n-\frac{r^2-1}{12r}+\frac{(r-1)^2(r+1)}{12r}n\log_r n+\frac{r-1}{12r}\frac{1}{n},$$

with initial value condition $Z_0 = 0$.

From recurrent relation we obtain the equation

$$Z_\nu = \frac{r^2-1}{5}\sum_{s=0}^{\nu-1} r^{2s} + \frac{(r-1)(r-4)}{12}\sum_{s=0}^{\nu-1} r^s - \frac{r^2-1}{12r}\sum_{s=0}^{\nu-1} 1$$

$$+\frac{(r-1)^2(r+1)}{12r}\sum_{s=0}^{\nu-1} sr^s + \frac{r-1}{12r}\sum_{s=0}^{\nu-1} r^{-s}.$$

Finally we obtain

$$\sum_{i=0}^{N-1} ip_r^2(i) = \frac{1}{12}[2N^2 + N(\frac{r^2-1}{r}\log_r N - 5)$$

$$-\frac{r^2-1}{r}\log_r N + 4 - \frac{1}{N}].$$

Let we determine the constant $\eta = \eta(r)$ *as*

$$\eta = \begin{cases} \frac{1}{18}, & \text{if } r \text{ is an even number} \\ \frac{13}{72}, & \text{if } r \text{ is an odd number.} \end{cases}$$

Using the equation (2) and lemma 1 we can be proved the next

Lemma 2. *Let Σ_{2n}^r be the net, defined by the equation (3), i.e. $\Sigma_{2n}^r = \{(\frac{i}{n}, p_r(i)), (\frac{i}{n}, 1-p_r(i)) : 0 \le i \le n-1,\ n = r^{\nu-1}\}$. Then we have the equation*

$$\frac{1}{r}\sum_{k=1}^{r}[2nT(\Sigma_{2n}^{(k)})]^2 - [2nT(\Sigma_{2n}^r)]^2$$

$$= \frac{-25r^2+16+72\Theta}{72r^2} + \frac{(r-1)(r^2-1)}{6r^2n}\log_r n + \frac{\eta(r-1)(r+1)}{r^2n^2},$$

where the $\Theta = \Theta(r)$ is determined by the equation (5).

Lemma 3. *The following equation holds:*

$$\frac{2}{r}\sum_{k=2}^{r}\sum_{j=1}^{k-1}\int_0^1\int_0^1 [A(\Sigma_{2n}^{(k)}; x, t) - 2nxt]\times$$

$$\times[A(\Sigma_{2n}^{(j)}; x, 1) - 2nx]dxdt =$$
$$\frac{-r + 8r - 4 - 24\Theta}{12r62} - \frac{r^2 - 1}{6r^2 n}.$$

Proof. Let us define the function $\chi_+(u)$ by

$$\chi_+(u) = \begin{cases} 1, & \text{if } u > 0 \\ 0, & \text{if } u \leq 0. \end{cases}$$

Then from (10) for $0 \leq x, t \leq 1$ we have the equation

$$A(\Sigma_{2n}^{(k)}; x, t) =$$
$$= \sum_{i=0}^{n-1} \chi_+(x - (\frac{i}{n} + \frac{k-1}{rn}))\chi_+(t - p_r(i)) +$$
$$+ \sum_{i=0}^{n-1} \chi_+(x - (\frac{i}{n} + \frac{r-k}{rn}))\chi_+(t - (1 - p_r(i))).$$

Hence, we obtain the equation

$$(11) \qquad \frac{2}{r}\sum_{k=2}^{r}\sum_{j=1}^{k-1}\int_0^1\int_0^1 [A(\Sigma_{2n}^{(k)}; x, t) - 2nxt]$$
$$\times[A(\Sigma_{2n}^{(j)}; x, 1) - 2nx]dxdt$$
$$= \sum_{m=0}^{n-1} u_m[1 - p_r(m)] + \sum_{m=0}^{n-1} v_m p_r(m) - \frac{r-1}{2r}n + \frac{r-1}{6r^2},$$

where for $0 \leq m \leq n-1$ are introduced the symbols

$$u_m = \frac{2}{r}\sum_{k=2}^{r}\sum_{j=1}^{k-1}\sum_{i=0}^{n-1}[2 - \max(\frac{i}{n} + \frac{j-1}{rn}, \frac{m}{n} + \frac{k-1}{rn})$$
$$- \max(\frac{i}{n} + \frac{r-j}{rn}, \frac{m}{n} + \frac{k-1}{rn})]$$
$$-\frac{2n}{r}\sum_{k=2}^{r}(k-1)[1 - (\frac{m}{n} + \frac{k-1}{rn})^2],$$

$$v_m = \frac{2}{r}\sum_{k=2}^{r}\sum_{j=1}^{k-1}\sum_{i=0}^{k-1}[2 - \max(\frac{i}{n} + \frac{j-1}{rn}, \frac{m}{n} + \frac{r-k}{rn})$$

$$-\max(\frac{i}{n}+\frac{r-j}{rn},\frac{m}{n}+\frac{r-k}{rn})]$$

$$-\frac{2n}{r}\sum_{k=2}^{r}(k-1)[1-(\frac{m}{n}+\frac{r-k}{rn})^2].$$

It is possible to prove that for $0 \leq m \leq n-1$, u_m and v_m satisfy the equations (for Θ see (5))

(12) $$u_m = \frac{r-1}{r} - \frac{r-1}{r}\frac{m}{n} + \frac{-3r^2+4r-8\Theta}{4r^2n},$$

(13) $$v_m = \frac{r-1}{r} - \frac{r-1}{r}\frac{m}{n} + \frac{-5r^2+12r-4-24\Theta}{12r^2n}.$$

From (12) and (13) for (11) we obtain the equation

$$\frac{2}{r}\sum_{k=2}^{r}\sum_{j=1}^{k-1}\int_0^1\int_0^1[A(\Sigma_{2n}^{(k)};x,t)-2nxt]$$

$$\times[A(\Sigma_{2n}^{(j)};x,1)-2nx]dxdt$$

$$=\sum_{m=0}^{n-1}[\frac{r-1}{r}-\frac{r-1}{r}\frac{m}{n}+\frac{-3r^2+4r-8\Theta}{4r^2n}\frac{1}{n}][1-p_r(m)]$$

$$+\sum_{m=0}^{n-1}[\frac{r-1}{r}-\frac{r-1}{r}\frac{m}{n}+\frac{-5r^2+12r-4-24\Theta}{12r^2}\frac{1}{n}]p_r(m)$$

$$-\frac{r-1}{2r}n+\frac{r-1}{6r^2}$$

$$=\sum_{m=0}^{n-1}[\frac{r-1}{r}-\frac{r-1}{r}\frac{m}{n}]-\frac{r-1}{2r}n+\frac{r-1}{6r^2}$$

$$+\frac{-3r^2+4r-8\Theta}{4r^2n}\frac{1}{n}\sum_{i=0}^{n-1}[1-p_r(i)]$$

$$+\frac{-5r^2+12r-4-24\Theta}{12r^2}\frac{1}{n}\sum_{i=0}^{n-1}p_r(i)$$

$$=\frac{-r^2+8r-4-24\Theta}{12r^2}-\frac{r^2-1}{6r^2n}.$$

The lemma is proved. ■

We note that the proof of Lemma 3 is based on the number theoretical equation, exposed in Lemma1.

Lemma 4. *The equation holds:*

$$\frac{1}{r}\sum_{j=1}^{r-1}(r-j)\int_0^1[A(\Sigma_{2n}^{(j)};x,1)-2nx]^2dx$$

$$=\frac{(r-1)(r^2+4+\frac{1-(-1)^r}{2}3)}{12r^2}.$$

Lemma 5. *The the equation holds:*

$$\frac{2}{r}\sum_{j=1}^{r-2}\sum_{m=j+1}^{r-1}(r-m)\int_0^1[A(\Sigma_{2n}^{(j)};x,1)-2nx]$$

$$\times[A(\Sigma_{2n}^{(m)};x,1)-2nx]dx$$

$$=\begin{cases}\frac{(r-2)(r^3-28r^2+144r-72)}{360r^2}, & \textit{if } r \textit{ is an even number}\\[2ex] \frac{(r-1)(r^3-29r^2+171r-279)}{360r^2}, & \textit{if } r \textit{ is an odd number.}\end{cases}$$

Proof of Theorem 1. We will prove the theorem when r is an even number. By the formula of breaking of the quadratic discrepancy (9), Lemmas 2,3,4 and 5 we obtain the recurrent relation for $[2NT(\Sigma_{2N}^r)]^2 (N=rn=r^\nu)$

$$[2NT(\Sigma_{2N}^r)]^2=[2nT(\Sigma_{2N}^r)]^2+\frac{r^4+15r^2-16}{369r^2}+$$

$$+\frac{(r-1)^2(r+1)}{6r^2n}\log_r n-\frac{r^2-1}{6r^2}\frac{1}{n}+\frac{r^2-1}{18r^2n^2},$$

with initial value condition. From (2) for the net $\Sigma_2^r=\{(0,0),(0,1)\}$ we get that $[2T(\Sigma_2^r)]^2=\frac{4}{9}$.

From the recurrent relation we obtain

$$(10)\qquad [2NT(\Sigma_{2N}^r)]^2=\frac{r^4+15r^2-16}{360r^2}\log_r N-\frac{r^2-1}{6r}\frac{\log_r N}{N}$$

$$+\frac{1}{2}-\frac{1}{18N^2}.$$

If r is odd number we show that

$$[2NT(\Sigma^r_{2N})]^2 = \frac{r^4+15r^2-16}{360r^2}\log_r N - \frac{r^2-1}{6r}\frac{\log_r N}{N} + \frac{5}{8} - \frac{13}{72N^2}. \tag{11}$$

From (4),(10) and (11) we get that for every $N = r^\nu$ the quadratic discrepancy of the net Σ^r_{2N}, determined by the equation (2) satisfies the equation

$$T(\Sigma^r_{2N}) = (2N)^{-1}\{\frac{r^4+15r^2-16}{360r^2}\log_r N - \frac{r^2-1}{6r}\frac{\log_r N}{N} + \frac{1}{2} - \frac{1}{18N^2} + (1-\frac{1}{N^2})\Theta\}^1/2.$$

Theorem 1 is proved. ∎

5. Analytic and geometric structure of the net $\Sigma^r_{\nu+1}$

Let us assume that Σ^r_N is the obtained net. We shall give geometrical rules for receiving of the net Σ^r_{rN} from Σ^r_N. For this purpose we study the analytic structure of the net Σ^r_{rN}. We will use the symbols $N_{\nu+1} = r^{\nu+1}$, $N_\nu = r^\nu$ and replace $N_{\nu-1} = N_\nu/r$. Arbitrary integer i, $0 \le i \le N_{\nu+1}-1$ we present in the form $i = r^2 j + k$, where $0 \le j \le N_{\nu-1}-1$ and $0 \le k \le r^2-1$. In addition, arbitrary k, so that $0 \le k \le r^2-1$ we present in the form $k = rs+t$, where $0 \le s \le r-1$, $0 \le t \le r-1$.

Using Property 2 of the sequence of Van der Corput-Halton can be proved that Σ^r_{rN} is a net in the form

$$\Sigma^r_{rN} = \{(\frac{j}{N_{\nu-1}} + \frac{rs+t}{r^2N_{\nu-1}}, \frac{tr+s}{r^2} + \frac{1}{r^2}p_r(j)),$$

$$(\frac{j}{N_{\nu-1}} + \frac{rs+t}{r^2N_{\nu-1}}, 1 - \frac{tr+s}{r^2} - \frac{1}{r^2}p_r(j)) : \quad 0 \le j \le N_{\nu-1}-1,$$

$$0 \le s \le r-1,\ 0 \le t \le r-1\}.$$

We can be proved that the net $\Sigma^r_{N_\nu}$ is a net in the form

$$\Sigma^r_{N_\nu} = \{(\frac{j}{N_{\nu-1}} + \frac{s}{rN_{\nu-1}}, \frac{s}{r} + \frac{1}{r}p_r(j)),$$

$$(\frac{j}{N_{\nu-1}} + \frac{s}{rN_{\nu-1}}, 1 - \frac{s}{r} - \frac{1}{r}p_r(j)) : \quad 0 \le j \le N_{\nu-1}-1,$$

$$0 \leq s \leq r-1,\}.$$

In [4] the concept zone is introduced: a part of quadrate, enclosed between straight lines $x = i/N_{\nu+1}$ and $x = (i+1)/N_{\nu+1}$, for $i = 0, 1, \ldots, N_{\nu+1}-1$ called zone wide $N_{\nu+1}$. We will number the zones from the left to the right as follows: $0, 1, \ldots, N_{\nu+1}-1$. Eeach zone $[\frac{i}{N_{\nu+1}}, \quad \frac{i+1}{N_{\nu+1}}] \times [0,1]$ we divide by r rectangles in the form $[\frac{i+s}{rN_{\nu+1}}, \quad \frac{i+s+1}{rN_{\nu+1}}] \times [0,1]$, where $0 \leq s \leq r-1$. Every rectangle in the last form we will call (i,s) - zone. Every (i,s) - zone we divide by r rectangles in the form $[\frac{i+s}{rN_{\nu+1}}, \frac{i+s+1}{rN_{\nu+1}}] \times [\frac{t}{r}, \frac{t+1}{r}]$, where $0 \leq t \leq r-1$. Every rectangle ofn the last form we will call (i,s,t) - layer of the (i,s) - zone.

The points of the net $\Sigma^r_{N_\nu}$ in the form $(\frac{j}{N_{\nu-1}} + \frac{s}{rN_{\nu-1}}, \frac{s}{r} + \frac{1}{r}p(j))(0 \leq j \leq N_{\nu-1}-1, ;0 \leq s \leq r-1)$ will be called leading points. We will give an algorithm to discover the leading points in the geometritical structure of the net $\Sigma^r_{N_\nu}$.

Let r be an even number and we signify $a = \frac{r}{2} - 1$. If $0 \leq s \leq a$, then for every $j, 0 \leq N_{\nu-1}-1$ the inequality holds $\frac{s}{r} + \frac{1}{r}p(j) < \frac{1}{2}$, and if $a+1 \leq s \leq r-1$, then for every j, $0 \leq j \leq N_{\nu-1}-1$ it holds the inequality $\frac{s}{r} + \frac{1}{r}p(j) \geq \frac{1}{2}$. Hence, if r is an even number, then for every (j,s) - zone, that $0 \leq s \leq a$ the leading points lie in the down half of the quadrate, and in every (j,s) - zone that $a+1 \leq s \leq r-1$ the leading points lie in the upper half of the quadrate.

Let now r be an odd number and we signify $a = [\frac{r}{2}]$. If $0 \leq s \leq a$, then for every $j, 0 \leq j \leq N_{\nu-1}-1$ we have the inequality $\frac{s}{r} + \frac{1}{r}p(j) < \frac{1}{2} + \frac{1}{2r}$, and if $a+1 \leq s \leq r-1$, then for every $j, 0 \leq j \leq N_{\nu-1}-1$ we have the inequality $\frac{s}{r} + \frac{1}{r}p(j) \geq \frac{1}{2} + \frac{1}{2r}$.

Hence, if r is an odd number, then for every (j,s) - zone, that $0 \leq s \leq a$ the leading points lie under the straight $y = \frac{1}{2} + \frac{1}{2r}$, and in every (j,s) - zone that $a+1 \leq s \leq r-1$ the leading points lie over the straight $y = \frac{1}{2} + \frac{1}{2r}$, or on it.

We will give algorithm for receiving the net $\Sigma^r_{N_{\nu+1}}$ from $\Sigma^r_{N_\nu}$.

1. For every j and s, that $0 \leq j \leq N_{\nu-1}-1$ and $s = 0, 1, \ldots, r-1$ in every (j,s) - zone we bend the leading points of $\sum^r_{N_\nu}$ r times to axis $0y$.

2. We shift for $t = 0, 1, \ldots, r-1$ every obtained point in the (j,s,t) - layer of (j,s) - zone.

3. We shift in every fixed (j,s,t) - layer obtained points to the right of the quantity $t/N_{\nu+1}$.

4. Every obtained point we reflect symmetrically, relatively to the straight line $y = \frac{1}{2}$.

We obtain the net $\Sigma^r_{N_{\nu+1}}$.

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