Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

---

Mathematica Balkanica

New Series Vol.11, 1997, Fasc. 3-4

Principal Ideals of Crossed Products

Kiril H. Kolikov

Presented by P. Kenderov

Let $K^\sigma_\rho G$ be a crossed product [1] of the group G and the field K with a system of factors $\rho = \{\rho(g,h) \in K^*|g,h \in G\}$ and a mapping $\sigma : G \mapsto \mathrm{Aut}K$, where K-basis $\{\bar{g}|g \in G\}$ satisfies the requirements $\bar{g}\alpha = \alpha^{\sigma g}\bar{g}$, $\bar{g}\bar{h} = \rho(g,h)\overline{g,h}$ for all $g,h \in G$ and $\alpha \in K$.

If $G_{ker} = \{g \in G|\sigma g = 1\}$ and $F = F(G) = \{g \in G|\rho(g,h) = \rho(h,g) = 1$ for all $h \in G\}$, then G_{ker} is a normal subgroup of G [1] and F is a subgroup of G [2].

In the case when G is a nilpotent group, Fisher and Segal [3] investigated the group rings which are principal left and principal right ideal rings. Passman [4] has finished these investigations for any group G in the next theorem.

Theorem (Passman) *Let KG be the group ring of he group G over the field K. Then the following requirements are equivalent.*

1. KG is a right principal ideal ring.

2. KG is right Noetherian and its augmentation ideal $A(KG)$ is principal as a right ideal.

3. If char $K = 0$, *then the group G is finite or finite-by-infinite cyclic. If* char $K = p > 0$, *then the group G is finite p'-by-cyclic p, or finite p'-by-infinite cyclic.*

Here char K is the characteristic of the field K, and p'-group is a group which does not contain elements of order of p.

In the work presented here the Theorem of Passman is generalizied. It is described here the kernel of the mapping σ in the case when the skew product $K^\sigma_\rho G$ is a right principal ideal ring.

We note that if $\rho \neq 1$ and $\sigma \neq 1$, then $K^\sigma_\rho G$ could also be principal left and principal right ideal rings which is proved by the next result.

Theorem 1. *Let K be a field, the group G be finite-by-cyclic and let the order of the finite group not to be divisible by the characteristic of the field K. Then the crossed product $K^\sigma_\rho G$ is a principal ideal ring.*

Proof. Let N be a finite normal subgroup of the group $G =< N, s >$ and the factor group $G/N =< s >$ be infinite cyclic. If $K_\rho^\sigma N$ is a crossed product of N and K with a system of factors ρ and a mapping σ both limited over N, then by [5, Lemma 2]

$$K_\rho^\sigma G \cong (K_\rho^\sigma N)_\gamma^\theta (G/N),$$

where θ is a mapping of the group G/N in the group of the automorphisms of the ring $K_\rho^\sigma N$ and γ is a system of factors of G/N over $K_\rho^\sigma N$.

The group $\overline{G/N} =< s >$ is infinite cyclic. Then, if for each integer k we substitute $\bar{s}^k$ by $\overline{s^k}$, then the equality $\overline{s^k}\,\overline{s^l} = \overline{s^{k+l}}$ implies that the system of factors in the basis $\{\bar{s}|s \in G/N\}$ is trivial. Consequently,

$$K_\rho^\sigma G \cong (K_\rho^\sigma N)^\theta (G/N).$$

Then by the corollary of the Theorem 5 [1], the ring $K_\rho^\sigma N$ is semisimple and by [3, Lemma 6], $K_\rho^\sigma G$ is a prinicipal ideal ring.

Let G/N be finite and $|N| \neq 0$ in the field K. Then the group G is a homomorphic image of the group $H =< N, s >$, where H/N is infinite cyclic.

If φ is a homomorphism of H on G, then a crossed product $K_\delta^\theta H$ can be formed. For the purpose we shall define

$$\theta(a) = \sigma(\varphi(a)) \text{ and } \delta(a,b) = \rho(\varphi(a), \varphi(b))$$

for all $a, b \in H$. Obviously θ is a mapping of the group H into the group of the automorphisms of the field K. Then the set

$$\delta = \{\delta(a,b)|a, b \in H\}$$

is a system of factors. Indeed, the equation

$$\delta(a,b)\delta(ab,c) = \delta(b,c)^{\theta(a)}\delta(a,bc) \quad (a,b,c \in H)$$

is fulfilled, as it is equivalent to the correlation

$$\rho(\varphi(a),\varphi(b))\rho(\varphi(a)\varphi(b),\varphi(c)) = \rho^{\sigma(\varphi(a))}(\varphi(b),\varphi(c))\rho(\varphi(a),\varphi(b)\varphi(c)),$$

which is true in $K_\rho^\sigma G$. Consequently $K_\delta^\theta H$ is a crossed product of the group H and field K with a system of factors δ and a mapping θ. But, as we indicated already, $K_\delta^\theta H$ is a principal ideal ring. Furthermore it is easy to prove, that the mapping $K_\delta^\theta H \mapsto K_\rho^\sigma G$ defined as

$$\sum_{a \in H} \alpha_a \bar{a} \mapsto \sum_{a \in H} \alpha_a \overline{\varphi(a)} \quad (\alpha_a \in K)$$

is a ring homomorphism. Consequently $K^{\sigma}_{\rho}G$ is also a principal ideal ring and the theorem is proved. ∎

Let S be a subgroup of the group $F = F(G)$. Some properties of the right ideal

$$A_r(S,G) = \left\{ \sum_{s\in S} (\bar{s} - \bar{1})x_s | x_s \in K^{\sigma}_{\rho}G \right\}$$

of the ring $K^{\sigma}_{\rho}G$ are investigated in [2]. It is proved that the ideal $A_r(S,G)$ is two-sided if and only if S is a normal subgroup of G and $S \leq G_{ker}$.

Furthermore, let $A^{(n)}_r(F,G)$ be the right ideal of the ring $K^{\sigma}_{\rho}G$, which is generated by all elements of the type $(\bar{s}_1 - \bar{1})(\bar{s}_2 - \bar{1})\dots(\bar{s}_n - \bar{1})$, where $s_1, s_2, \dots, s_n \in F$, i.e.

$$A^{(n)}(F,G) = \left\{ \sum_{s_1,\dots,s_n \in F} (\bar{s}_1 - \bar{1})\dots(\bar{s}_n - \bar{1})x_{s_1,\dots,s_n} | x_{s_1,\dots,s_n} \in K^{\sigma}_{\rho}G \right\}.$$

Then the next lemma is true, whose proof is analogous to the proof of Lemma 3.1 [6, p. 84].

Lemma 2. *If $D_n = D_n(F) = \{f \in F | \bar{f} - \bar{1} \in A^{(n)}_r(F,G)\}$, then:*

1. $F = D_1 \supseteq D_2 \supseteq \dots \supseteq D_n \supseteq \dots$ is a descending chain of normal subgroups of F;

2. If (D_m, D_n) is the group generated by all commutators $x^{-1}y^{-1}xy$, with $x \in D_m$, $y \in D_n$, then $(D_m, D_n) \leq D_{m+n}$ for all m and n;

3. If char $K = 0$, *then F/D_n is a torsion free group;*

4. If char $K = p$ *and $g \in D_n$, then $g^p \in D_{np}$.*

Obviously, if $S \subseteq G$, then $A^{(n)}(S,G)$ coincides with the n-degree of the ideal $A(S,G)$.

If R is a ring and $M \subseteq R$, then let $r(M)$ and $l(M)$ denote right and left annihilators respectively of the set M in R.

The proof of the next lemma is analogous to the proof of Lemma 4.3 [4].

Lemma 3. *Let $K^{\sigma}G$ be a skew product of a finite group G and let char $K = p > 0$. If $A_r(G_{ker}, G)$ is a right principal ideal, then G_{ker} is an extension of a p'-subgroup by a p-group.*

P r o o f. If we put $R = K^{\sigma}G$ and $A = A(G_{ker}, G)$, then $A = \alpha R$ for some $\alpha \in K$. Since A is a two-sided ideal and $\alpha \in A$, then $R\alpha \subseteq A$. In accordance with the condition $A = \alpha R$ and [2, Lemma 2], we have

$$l(\alpha) = l(A) = R \sum_{g \in G_{ker}} \bar{g}.$$

Consequently $\dim_k l(\alpha) = |G/G_{ker}|$. Furthermore, the map $\varphi : R \mapsto R\alpha$, where $\varphi(r) = r\alpha$ is an epimorphism with a kernel $l(\alpha)$ and it follows that $\dim_k R\alpha = \dim_k R - \dim_k l(\alpha) = |G| - |G/G_{ker}| = (|G_{ker}| - 1)|G/G_{ker}|$.

Let $\Pi(G/G_{ker})$ be a set of coset representatives of G_{ker} in G. Then the elements $\bar{u}(\bar{h} - \bar{1})(u \in \Pi(G/G_{ker}),\ h \in G_{ker})$ form a basis of the K-module A [see 2]. Consequently

$$\dim_k A = (|G_{ker}| - 1)|G/G_{ker}| = \dim_k R\alpha.$$

This indicates that $A = \alpha R \subseteq R\alpha$ and $R\alpha = \alpha R$. Hence it directly follows, that $A^m = R\alpha^m = \alpha^m R$ for each natural m.

Let N be the last term in the descending chain of normal subgroups $D_k(G_{ker})$. Then $N \triangle G_{ker}$ and we shall prove, that N is a p'-subgroup and G_{ker}/N is a p-group.

It is clear, that the descending chain $A \supseteq A^2 \supseteq A^3 \supseteq \ldots$ is stabilizing and let $A^n = A^{n+1}$. Then $\alpha^n = \alpha^{n+1}s$ for some $s \in R$, i.e. $\alpha^n(\bar{1} - \alpha s) = 0$. By this and by the equation $A^n = R\alpha^n$ we get, that $A^n(\bar{1} - \alpha s) = 0$. But, by the defination N, the left ideal $A_l = A_l(N, G)$ is included in A^n and then $\bar{1} - \alpha s \in r(A_l) = (\sum_{a \in N} \bar{a})R$ [2, Lemma 2]. This indicates, that $\sum_{a \in N} \bar{a} \bar{\in} A$. Consequently $|N| \neq 0$ in K and $p \not| |N|$.

At the end, as $N = D_n(G_{ker})$ for some n and char $K = p$, then by Lemma 2, G_{ker}/N is a p-group. The lemma is proved. ■

We shall define the mapping $* :\ K^\sigma G \mapsto K^\sigma G$, where

$$\left(\sum_{g \in G} \alpha_g \bar{g}\right)^* = \sum_{g \in G} \alpha^{\sigma(g^{-1})} \bar{g}^{-1}.$$

Then the mapping $x \mapsto x^*$ is an antiautomorphism, as for all $x, y \in K^\sigma G$ it is easy to prove, that

$$(x + y)^* = x^* + y^*,\ (xy)^* = y^* x^*,\ x^{**} = x.$$

■

Lemma 4. *Let the skew product $K^\sigma G$ be a right Noetherian prime ring, $G_{ker} \neq < 1 >$ and $A_r(G_{ker}, G)$ be a right principal ideal. Then*

1. If char $K = 0$, *then G_{ker} is an expansion of its normal subgroup by an infinite cyclic group;*

2. If char $K = p$, *then G_{ker} is an expansion of its normal subgroup by a cyclic p-group.*

Proof. Since an antiautomorphism in $K^\sigma G$ exists, then $K^\sigma G$ is right and left Noetherian. Therefore $K^\sigma G$ is a prime and Noetherian ring and it follows from [4, Lemma 4.2(i)] that $A_r^2(G_{ker}, G) = A_r(G_{ker}, G)$. Then $D_2 = D_2(G_{ker}) \neq G_{ker}$ and, by Lemma 2, G_{ker}/D is an abelian group. Furthermore the group G satisfies the condition for maximality of the subgroups and all subgroups of G are finitely generated.

Let char $K = 0$. Then G_{ker}/D_2 is a finitely generated torsion free abelian group and a normal subgroup N of the group G exists, such that G/N is an infinite cyclic group.

Let char $K = p > 0$. Then G/D is a finitely generated abelian p-group and therefore a normal subgroup N exists for which G_{ker}/N is a cyclic p-group.

The lemma is proved. ■

Let $\Delta^+(G)$ be the periodic part of the maximal f.c. subgroup $\Delta(G)$ of the group G. It is known [6, Lemma 1.6, p. 117] that $\Delta^+(G)$ is a characteristic subgroup of G and the group $\Delta(G)/\Delta^+(G)$ is torsion free abelian.

In the next proposition the group G_{ker} is described in the case when $K^\sigma G$ is a right principal ideal ring.

Proposition 5. *Let $K^\sigma G$ be a right principal ideal ring and $\Delta^+(G) \subseteq G_{ker}$. Then*

1. If char $K = 0$, *then G_{ker} is finite or extension of its normal subgroup by an infinite cyclic group;*

2. If char $K = p$, *then G_{ker} is either an extension of a finite p'-group by a p-group, or an expansion of its infinite normal subgroup by a cyclic p-group.*

Proof. Let $K^\sigma G$ be a right principal ideal ring. Then $K^\sigma G$ is a Noetherian ring. Consequently all subgroups of G are finitely generated and by Lemma of Dietzmann [7, p.338] $\Delta^+(G)$ is finite. Therefore if $\bar{G} = G/\Delta^+(G)$, then $\Delta^+(\bar{G}) =< 1 >$.

Keeping in mind the condition $\Delta^+ = \Delta^+(G) \subseteq G_{ker}$ and [2,Lemma4], we have that $A_r(\Delta^+, G)$ is a two-sided ideal. Consequetly by [2, Theorem 1],

$$K^\sigma G/A(\Delta^+, G) \cong K^\theta \bar{G},$$

where $\theta : \bar{G} = G/\Delta^+ \mapsto \text{Aut } K$ is the mapping induced from σ. Furthermore $\Delta^+(\bar{G}) =< 1 >$ implies that G has not nontrivial finite normal subgroups and by Theorem 1.9 [8], $K^\theta \bar{G}$ is a prime ring.

Obviously, $A(\bar{G}_{ker}, \bar{G})$ is the image of the ideal $A(G_{ker}, G)$ by the natural homomorphism

$$K^\sigma G \mapsto K^\sigma G/A(\Delta^+, G)$$

and therefore, $A(\bar{G}_{ker}, \bar{G})$ is a right principal ideal.

Let $\bar{G}_{ker} =< 1 >$. Then $G_{ker} = \Delta^{+}$, i.e. G_{ker} is a finite group and if char $K = p > 0$, the confirmation follows from Lemma 3.

If $\bar{G}_{ker} \neq< 1 >$, then the description of the group G and hence the group G_{ker} are obtained by Lemma 4. The proposition is proved. ■

The right ideal of the crossed product $K_{\rho}^{\sigma}G$, which is a left K-module is called also a right K-ideal.

Proposition 6. *Each right K-ideal of the ring $K_{\rho}^{\sigma}G$ is principal and is generated by an element of the subring $K_{\rho}G_{ker}$ if and only if $K_{\rho}G_{ker}$ is a right principal ideal ring.*

P r o o f. Let each right K-ideal of the ring $K_{\rho}^{\sigma}G$ be generated by an element of $K_{\rho}G_{ker}$ and I be a right ideal of the ring $K_{\rho}G_{ker}$. If $\Pi(G/G_{ker})$ is a set of coset of representatives of G_{ker} in G, then by [1, Theorem 3], the right ideal

$$(1) \qquad J = \left\{ \sum_{u \in \Pi(G/G_{ker})} x_u \bar{u} \middle| \; x_u \in I \right\}$$

of the ring $K_{\rho}^{\sigma}G$ is a K-ideal. Consequently $J = xK_{\rho}^{\sigma}G$ for some $x \in K_{\rho}G_{ker}$. As $I = J \cap K_{\rho}G_{ker}$, then $x \in I$ and $I = xK_{\rho}G_{ker}$. Therefore $K_{\rho}G_{ker}$ is a right principal ideal ring.

Conversely let $K_{\rho}G_{ker}$ be a right principal ideal ring and J be a right K-ideal of the ring $K_{\rho}^{\sigma}G$. Then by [1, Theorem 3] such a right ideal I of the ring $K_{\rho}G_{ker}$ exists, that $J \cap K_{\rho}G_{ker} = I$ for which (1) is fulfilled. Since $I = xK_{\rho}G_{ker}$ for some $x \in K_{\rho}G_{ker}$, then obviously $J = xK_{\rho}^{\sigma}G$. ■

Proposition 7. *If $A(S,G)$ is a right principal ideal of $K_{\rho}^{\sigma}G$, which is generated by an element of the ring $K_{\rho}^{\sigma}G$, then $S \subseteq G_{ker}$.*

P r o o f. Let $A = A_r(S,G) = xK_{\rho}^{\sigma}G$, where $x \in K_{\rho}G_{ker}$. As $\alpha x = x\alpha$ for all $\alpha \in K$, then A is a K-ideal of the ring $K_{\rho}^{\sigma}G$. Therefore, if $\alpha \in K$ and $s \in S$, then $(\bar{s} - \bar{1})\alpha - \alpha^{\sigma s}(\bar{s} - \bar{1}) = \bar{1}(\alpha^{\sigma s} - \alpha) \in A$. Since $\bar{1} \bar{\in} A$, then $\alpha^{\sigma s} = \alpha$ and $S \subseteq G_{ker}$. ■

Corollary 8. *If $A_r(G,G)$ is a right principal ideal, of the skew product $K^{\sigma}G$, which is generated by an element of KG_{ker}, then $G = G_{ker}$.*

The corollary indicates, that if $K^{\sigma}G$ is a right principal ideal ring and $A_r(G,G)$ is generated by an element of KG_{ker}, then $K^{\sigma}G$ is a group ring and we can apply the Theorem of Passman.

Further, from Proposition 6 and the Theorem of Passman it is easy to obtain the following.

Theorem 9. *If $K^\sigma G$ is a skew product, then the following conditions are equivalent.*

1. Each right K-ideal of the ring $K^\sigma G$ is principal and is generated by an element of the subring KG_{ker}.

2. KG_{ker} is a right principal ideal ring.

3. KG_{ker} is a right Noetherian ring and $\Delta(G_{ker}, G)$ is a right principal ideal, which is generated by an element of the subring KG_{ker}.

4. If char $K = 0$, *then G_{ker} is either finite, or an extension of a finite group by an infinite cyclic group.*

If char $K = p > 0$, *then G_{ker} is either an extension of a finite p'-group by a cyclic p-group, or an extension of a finite p'-group by an infinite cyclic group.*

References

[1] A. A. B o v d y, Crossed products of semigroups and rings (In Russian). *Sib. Mat. Zh.* **4** (1963), 481-499.

[2] A. A. B o v d y, K. H. K o l i k o v, Lie nilpotence and ideals of crossed products (In Russian). *Serdica (Bulg. Math. Publ.)* **15** (1989), 275-286.

[3] J. L. F i s h e r, S. K. S e h g a l, Principal ideal group rings. *Comm. in Algebra* **4** (1976), 319-325.

[4] D. S. P a s s m a n, Observations on group rings. *Comm. in Algebra* **5** (1977), 1119-1162.

[5] S. B. M i h o v s k i, *Idempotents and Some Finiteness Conditions in Group Rings and Crossed Products of Groups and Rings* (In Russian). Ph.D. Thesis, Kharckov Univ., 1971.

[6] D. S. P a s s m a n, *The Algebraic Structure of Group Rings.* Interscience, New York, 1977.

[7] A. G. K u r o s h, *The Theory of Groups* (In Russian). 3th Edition, Science, 1967.

[8] S. M o n t g o m e r y, D. S. P a s s m a n, Crossed products over prime ring. *Israel J. Math.* **31** (1978), 224-256.

Plovdiv University "P. Hilendarski"
Faculty of Mathematics
24, Tsar Asen Str.
4000 Plovdiv, BULGARIA

Received: 10.05.1996
Revised: 26.02.1997