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## On a Class of Linear Approximating Operators

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In this paper we introduce a new class of linear approximating operators  $(L_{nr})_{n \geq 1}$ ,  $r = 0, 1, 2, \dots$ , for the functions  $f \in C^r[0, 1]$ . In order to construct them we use Taylor's polynomial of  $r$  degree and a classical class of linear positive operators generated by a probabilistic method. Also, we study approximation degree with the modulus of continuity of first and second order.  $(L_{nr}f)_{n \geq 1}$  include as a particular case the generalized Bernstein polynomials defined by G.H. Kirov in [5].

### 1. Introduction

Any problem concerning the consideration of positive linear approximating operators, on the space of continuous real-valued functions on an interval of the real axis can naturally be interpreted as a problem of probability theory. There are several papers dedicated to this subject.

Let  $(X_n)_{n \geq 1}$  be a real sequence of random variables having the following distribution:  $P(X_n = x_{nk}) = p_{nk}(x)$  where  $x_{nk} \in [0, 1]$ ,  $0 \leq x \leq 1$  represents a parameter,  $k \in I \subseteq \mathbb{N}$ .

Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a continuous function. We consider an operator

$$L_n : C[0, 1] \rightarrow C[0, 1], \quad f \rightarrow L_n f,$$

defined by the following formula:

$$(1) \quad (L_n f)(x) = \sum_{k \in I} p_{nk}(x) f(x_{nk}).$$

This type of operator has been considered in remarkable papers by D.D. Stancu [7],[8] and W. Feller [2] and borrowed by many other authors either to get through a probabilistic way to some classical linear positive operators

(Bernstein, Mirakyan-Favard-Szász, Baskakov, Meyer-König and Zeller) or to get some important generalizations of these operators.

In our previous paper [1] we approached a similar formula for functions of two variables. It is evident that the operator defined by the formula (1) is a positive linear operator and according to the definition of a distribution we have

$$(2) \quad (L_n e_0)(x) = \sum_{k \in I} p_{nk}(x) = 1.$$

We denote  $e_j : [0, 1] \rightarrow \mathbb{R}$ ,  $e_j(x) = x^j$ ,  $j = 0, 1, 2, \dots$

Starting from (1), the purpose of this paper is to form a new sequence of linear operators. We will also evaluate the order of approximation by means of the modulus of continuity of first and second order. As a particular case, a class of the Bernstein polynomials introduced in [5] are to be found.

## 2. Construction of the operators

By  $C^r[0, 1]$ ,  $r = 0, 1, 2, \dots$ , we denote the set of all functions  $f : [0, 1] \rightarrow \mathbb{R}$  with a continuous derivative of order  $r$  on  $[0, 1]$ .

Let  $f \in C^r[0, 1]$  and  $(T_r f)(x_{nk}; \bullet)$  be Taylor's polynomial of  $r$  degree associated to the function  $f$  on the point  $x_{nk}$ :

$$(3) \quad (T_r f)(x_{nk}; x) = \sum_{i=0}^r \frac{f^{(i)}(x_{nk})}{i!} (x - x_{nk})^i.$$

We define the operators:

$$L_{nr} : C^r[0, 1] \rightarrow C[0, 1], \quad f \rightarrow L_{nr} f$$

where

$$(4) \quad (L_{nr} f)(x) = \sum_{k \in I} (T_r f)(x_{nk}; x) p_{nk}(x).$$

It is easy to verify that these operators are linear and choosing  $r = 0$ , we obtain the operators presented in (1) ( $L_{n0} f = L_n f$ ).

Further on, we consider  $r \geq 1$  integer.

## 3. Approximation degree

First, we need Taylor's modified formula with the rest under integral form:

$$(5) \quad f(x) = (T_r f)(x_{nk}; x)$$

$$+ \frac{(x - x_{nk})^r}{(r - 1)!} \int_0^1 (1 - t)^{r-1} (f^{(r)}(x_{nk} + t(x - x_{nk})) - f^{(r)}(x_{nk})) dt.$$

Second, we need the modulus of continuity of function  $f$  defined by:

$$(6) \quad \omega_1(f; \delta) = \sup\{|f(x) - f(y)| : x, y \in [0, 1], |x - y| \leq \delta\}, \quad 0 < \delta \leq 1.$$

From relations (2),(5),(6) successively applied, for any  $x \in [0, 1]$  we obtain:

$$\begin{aligned} |f(x) - (L_{nr}f)(x)| &= \left| \sum_{k \in I} f(x) p_{nk}(x) - \sum_{k \in I} \sum_{i=0}^r \frac{f^{(i)}(x_{nk})}{i!} (x - x_{nk})^i p_{nk}(x) \right| \\ &= \left| \sum_{k \in I} \frac{(x - x_{nk})^r}{(r - 1)!} p_{nk}(x) \int_0^1 (1 - t)^{r-1} (f^{(r)}(x_{nk} + t(x - x_{nk})) - f^{(r)}(x_{nk})) dt \right| \\ &\leq \sum_{k \in I} \frac{|x - x_{nk}|^r}{(r - 1)!} p_{nk}(x) \int_0^1 (1 - t)^{r-1} \omega_1(f^{(r)}; t|x - x_{nk}|) dt. \end{aligned}$$

Using the following known property of the modulus of continuity

$$\omega_1(f; \lambda \delta) \leq (\lambda + 1) \omega_1(f; \delta), \quad \lambda > 0,$$

with  $\lambda = t|x - x_{nk}|n^{1/2}$  and  $\delta = n^{-1/2}$ , we can write further:

$$\begin{aligned} &|f(x) - (L_{nr}f)(x)| \\ &\leq \sum_{k \in I} \frac{|x - x_{nk}|^r}{(r - 1)!} p_{nk}(x) \int_0^1 (1 - t)^{r-1} (t|x - x_{nk}|n^{1/2} + 1) \omega_1(f^{(r)}; n^{-1/2}) dt \\ &= \sum_{k \in I} \frac{|x - x_{nk}|^r}{(r - 1)!} p_{nk}(x) \omega_1(f^{(r)}; n^{-1/2}) \left\{ |x - x_{nk}|n^{1/2} \int_0^1 t(1 - t)^{r-1} dt \right. \\ &+ \left. \int_0^1 (1 - t)^{r-1} dt \right\} = \sum_{k \in I} \frac{|x - x_{nk}|^r}{(r + 1)!} p_{nk}(x) \omega_1(f^{(r)}; n^{-1/2}) (|x - x_{nk}|\sqrt{n} + r + 1) \\ &\leq \frac{\sqrt{n} + r + 1}{(r + 1)!} \sum_{k \in I} |x - x_{nk}|^r p_{nk}(x) \omega_1\left(f^{(r)}; \frac{1}{\sqrt{n}}\right), \end{aligned}$$

because  $|x - x_{nk}| \leq 1$ .

Applying the Cauchy-Schwarz inequality and taking into account the identity (2), we get:

$$(7) \quad \sum_{k \in I} |x - x_{nk}|^r p_{nk}(x) \leq \left( \sum_{k \in I} (x - x_{nk})^{2r} p_{nk}(x) \right)^{1/2}.$$

**Definition.** ([3], p. 189) The expectation of the variable  $(Y - a)^k$  is called the moment of the  $k$ -th order about  $a$  of the random variable  $Y$ :

$$\nu_k(Y; a) = E(Y - a)^k.$$

Consequently (7) can be rewritten:

$$\sum_{k \in I} |x - x_{nk}|^r p_{nk}(x) \leq \nu_{2r}^{1/2}(X_n; x)$$

and we find

$$|f(x) - (L_{nr}f)(x)| \leq \frac{\sqrt{n} + r + 1}{(r + 1)!} \nu_{2r}^{1/2}(X_n; x) \omega_1\left(f^{(r)}; \frac{1}{\sqrt{n}}\right).$$

Now we can state the following proposition:

**Theorem.** Let be the sequences  $(L_n)_{n \geq 1}$  and  $(L_{nr})_{n \geq 1}$  defined in (1) respectively in (4). If the following constants  $\lambda_r > 0$  and  $\alpha \geq \frac{1}{2}$  exist with the property:

$$\nu_{2r}(X_n; x) \leq \lambda_r n^{-2\alpha},$$

for any natural number  $n$ , then the inequality

$$(8) \quad |f(x) - (L_{nr}f)(x)| \leq \frac{\lambda_r^{1/2}(\sqrt{n} + r + 1)}{(r + 1)! n^\alpha} \omega_1\left(f^{(r)}; \frac{1}{\sqrt{n}}\right)$$

holds.

**R e m a r k.** The above result implies:

$$(9) \quad \|f - L_{nr}f\| = \mathbf{0} \left( n^{\frac{1}{2} - \alpha} \omega_1\left(f^{(r)}; \frac{1}{\sqrt{n}}\right) \right),$$

where  $\|\bullet\|$  stands for the uniform norm.

It is known that besides  $\omega_1$ , the modulus of smoothness  $\omega_2$  is frequently used in quantitative approximation. This  $\omega_2$  is defined, for  $f \in C[a, b]$ , by

$$\omega_2(f; \delta) = \sup\{|f(x-h) - 2f(x) + f(x+h)| : x, x \pm h \in [a, b], 0 < h \leq \delta\}, 0 < \delta \leq b - a.$$

H.H. Gonska [4] established the following inequality:

$$\omega_1(f; \delta) \leq \left(3 + \frac{2(b-a)}{\delta}\right) \omega_2(f; \delta) + \frac{6\delta}{b-a} \|f\|,$$

for all  $f \in C[a, b]$  and  $0 < \delta \leq b - a$ .

Our result (8) becomes:

$$|f(x) - (L_{nr}f)(x)| \leq c_r(n)\omega_2\left(f^{(r)}; \frac{1}{\sqrt{n}}\right) + d_r(n)\|f^{(r)}\|,$$

where:

$$c_r(n) = \frac{\lambda_r^{1/2}}{(r+1)!} (2n + (2r+5)\sqrt{n} + 3(r+1)) n^{-\alpha}$$

and

$$d_r(n) = \frac{6\lambda_r^{1/2}}{(r+1)!} \left(1 + \frac{r+1}{\sqrt{n}}\right) n^{-\alpha}.$$

#### 4. Example

We choose  $I = \{0, 1, 2, \dots, n\}$ ,  $x_{nk} = \frac{k}{n}$ ,  $p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ . In this case  $L_n f$  coincides with the classical Bernstein polynomial  $B_n f$  and  $L_{nr} f$  becomes the generalized Bernstein polynomial of the  $(n, r)$ -th order for a function  $f \in C^r[0, 1]$ , introduced and studied in the recent paper [5]. For any  $x \in [0, 1]$  we have:

$$\sum_{k \in I} (x - x_{nk})^{2r} p_{nk} = n^{-2r} \sum_{k=0}^n (k - nx)^{2r} \binom{n}{k} x^k (1-x)^{n-k} \leq A n^{-r},$$

where  $A$  is a constant depending only on  $r$  (see e.g. [6], p.248).

If we take  $\lambda_r = A$  and  $\alpha = \frac{r}{2} \geq \frac{1}{2}$  the condition of the theorem given in Section 3 is satisfied and we have:

$$|f(x) - (B_n f)(x)| \leq \frac{A^{1/2}(\sqrt{n} + r + 1)}{(r+1)!n^{r/2}} \omega_1\left(f^{(r)}; \frac{1}{\sqrt{n}}\right).$$

Consequently we can write

$$\|f - B_n f\| = O\left(n^{\frac{1-r}{2}} \omega_1\left(f^{(r)}; \frac{1}{\sqrt{n}}\right)\right).$$

This result is similar with the one obtained by G.H. Kirov in the paper [5] published in this journal.

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