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Convergence of Mixing Transforms of Random Sequences

Sarat Kumar Acharya [1], *Chaitanya Kumar Tripathy* [2]

Presented by P. Kenderov

The paper deals with the a.s. convergence of martingale transforms of random sequences both under local regularity and global regularity conditions.

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1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\{\mathcal{F}_n,\ n \geq 1\}$, an increasing sequence of sub σ–fields of $\mathcal{F}$ and $\{X_n,\ n \geq 1\}$, a sequence of real valued random variables adapted to $\{\mathcal{F}_n,\ n \geq 1\}$. Throughout this paper the following notations and definitions are used.

Given a sequence of random variables $\{X_i,\ i \geq 1\}$, let $\mathcal{F}_{ij}$ be the σ–field generated by $\{X_k,\ i \leq k \leq j\}$ for all $1 \leq i \leq j < \infty$. The sequence $\{X_i,\ i \geq 1\}$ is said to be $\star$-mixing (star–mixing) if there exists an integer M, a function Φ with $\Phi(m) \to 0$ as $m \to \infty$ and $A \in \mathcal{F}_{1n}$, $B \in \mathcal{F}_{m+n,\infty}$ such that

$$(1) \qquad |P(A \cap B) - P(A)P(B)| \leq \Phi(m)P(A)P(B)$$

for all $m \geq M$ and all $n \geq 1$.

As in Stout [3], we define a mixing transform as follows.

Let $\{X_i,\ \mathcal{F}_i,\ i \geq 1\}$ be a $\star$–mixing sequence and ν_i be $\mathcal{F}_{i-1}$ measurable for each $i \geq 1$. Then $\{T_n,\ n \geq 1\}$ defined by $T_n = \sum_{i=1}^{n} \nu_i X_i$, is called a mixing transform and $\{\nu_i,\ i \geq 1\}$ is called the transform sequence.

A $\star$–mixing sequence $\{X_i,\ \mathcal{F}_i,\ i \geq 1\}$ is said to be regular mixing sequence if

$$\infty > E\left[|X_i| \mid \mathcal{F}_{i-1}\right] \geq \delta E^{1/2}\left[X_i^2 \mid \mathcal{F}_{i-1}\right] \quad \text{a.s.}$$

for all $i \geq 1$ and some $\delta > 0$.

A regular mixing sequence $\{X_i, \mathcal{F}_i, i \geq 1\}$ satisfying $E[X_i^2 \mid \mathcal{F}_{i-1}] = 1$ a.s. for each $i \geq 1$ is referred to as a normed regular mixing sequence.

The convergence of $S_n = \sum_{i=1}^{n} \nu_i Y_i$, where $\{Y_i, \mathcal{F}_i, i \geq 1\}$ is a martingale difference sequence and $\{\nu_i's, i \geq 1\}$ are $\mathcal{F}_{i-1}$ measurable, has been studied by many authors (cf. Stout [3]). In this paper some of their results are extended to $\star$-mixing stochastic sequence. The main technique adopted here is that the concept of $\star$-mixing has been fitted conveniently in to the martingale framework. In Theorem 2.1 we prove the convergence of $\sum_{i=1}^{\infty} \nu_i X_i$ using the local regularity conditions and in Theorem 3.1 we prove the same result using the global regularity conditions.

2. Convergence of mixing transforms satisfying local regularity conditions

Theorem 2.1. *Let (i) $\{\nu_i, i \geq 1\}$ be a transforming sequence for which $\sum_{i=1}^{\infty} \nu_i^2 < \infty$ and (ii) $\{X_i, i \geq 1\}$ be a $\star$-mixing sequence such that*

(a) $E(X_i) = 0$, (b) $E(X_i^2 \mid \xi_{i-1}) = 1$ and $|X_i| \leq K < \infty$.

Then $\sum_{i=1}^{\infty} \nu_i X_i$ converges a.s.

We need the following lemma to prove the theorem.

Lemma 2.1. *Let $\{Y_i, i \geq 1\}$ be $\star$-mixing with respect to a function Φ and an integer M. Suppose $E|Y_i| < \infty$ for each $i \geq 1$. Then*

$$\left|E(Y_{n+M} \mid \mathcal{F}') - E(Y_{n+M})\right| \leq \Phi(m) E|Y_{n+M}| \quad a.s.$$

for each σ-field $\mathcal{F}' \subset \mathcal{F}_{1n}$, each $n \geq 1$ and each $m \geq M$.

The proof of Lemma 2.1 is given in Stout [3], p. 138.

Proof of Theorem 2.1. Fix $\varepsilon > 0$. Referring to Lemma 2.1, choose M sufficiently large so that foe each $n \geq 1$ and $0 \leq k \leq M-1$,

$$\left|E\left[X_{nM+k} \mid X_{(n-1)M+k}, X_{(n-2)M+k}, \ldots, X_{M+k}\right] - E\left[X_{nM+k}\right]\right| \leq \varepsilon K.$$

Since $E(X_i) = 0$ for each i,

$$\left|E\left[X_{nM+k} \mid X_{(n-1)M+k}, X_{(n-2)M+k}, \ldots, X_{M+k}\right]\right| \leq \varepsilon K. \tag{2}$$

Fix $0 \leq k \leq M$. Let $\xi_n = \mathcal{B}\left(X_{nM+k}, X_{(n-1)M+k}, \ldots, X_{M+k}\right)$ for each $n \geq 1$ and $\xi_0 = \{\Phi, \Omega\}$.

Now, $\{X_{nM+k} - E(X_{nM+k} \mid \xi_{n-1}), n \geq 2\}$ is a martingale difference sequence. Let us denoted by Z_{nM+k} its general term. Then

$$\begin{aligned} E(Z^2_{nM+k} \mid \xi_{n-1} &= E\left[\{X_{nM+k} - E(X_{nM+k} \mid \xi_{n-1})\}^2 \mid \xi_{n-1}\right] = \\ &= E(X^2_{nM+k} \mid \xi_{n-1}) = 1. \quad \text{[By condition ii(b)].} \end{aligned}$$

Again $|Z_{nM+k}| \leq K + \varepsilon K = K'$ (say).

Hence all the conditions of Theorem 2.10.2(i) in Stout [3] are satisfied and so $\{Z_{nM+k}, n \geq 2\}$ is a normed regular MZ. Now by Theorem 2.10.4 of Stout [3],

$$\sum_{n=2}^{\infty} \nu_{nM+k} Z_{nM+k} \qquad \text{converges iff} \qquad \sum_{n=2}^{\infty} \nu^2_{nM+k} < \infty.$$

But since $\varepsilon > 0$ is arbitrarily chosen, then

$$\begin{aligned} \sum_{n=2}^{\infty} \nu_{nM+k} Z_{nM+k} &= \sum_{n=2}^{\infty} \nu_{nM+k} [X_{nM+k} - E(X_{nM+k} \mid \xi_{n-1})] = \\ &= \sum_{n=2}^{\infty} \nu_{nM+k} X_{nM+k}. \end{aligned}$$

Therefore, $\sum_{n=2}^{\infty} \nu_{nM+k} X_{nM+k}$ converges if $\sum_{n=2}^{\infty} \nu^2_{nM+k} < \infty$ and hence the result follows. ■

3. Global regularity conditions

Theorem 3.1 *Let (i) $\{\nu_i, i \geq 1\}$ be a transforming sequence such that $\sup_{n\geq 1} |\nu_n| < \infty$ a.s. and (ii) $\{X_i, i \geq 1\}$ be a $\star$-mixing sequence for which $EX_i = 0$ and $E|X_i| < \infty$ for each i. Then $\sum_{i=1}^{\infty} \nu_i X_i$ converges a.s.*

Proof. Fix $\varepsilon > 0$. Referring to Lemma 2.1, choose M sufficiently large so that

$$\left| E\left[X_{nM+k} \mid X_{(n-1)M+k}, X_{(n-2)M+k}, \ldots, X_{M+k}\right] - E\left[X_{nM+k}\right]\right| \leq \varepsilon K,$$

for each $n \geq 2$ and each $0 \leq k \leq M - 1$.

Since $EX_i = 0$ for each i,

$$E\left[X_{nM+k} \mid X_{(n-1)M+k}, X_{(n-2)M+k}, \ldots, X_{M+k}\right] \leq \varepsilon K. \tag{3}$$

Fix $0 \leq k \leq M$. Let, $\xi_n = \mathcal{B}(X_{nM+k}, X_{(n-1)M+k}, \ldots, X_{M+k})$ for $n \geq 2$ and $\xi_0 = \{\Phi, \Omega\}$.

To prove the theorem it is sufficient to show that $\sum_{n=2}^{N} \nu_{nM+k} X_{nM+k}$ converges a.s. Now

$$Z_{nM+k} = \{X_{nM+k} - E(X_{nM+k} \mid \xi_{n-1}), \xi_n, n \geq 2\}$$

is a margingale difference sequence. Hence in view of relation (3) it is sufficient to show that $\sum_{n=2}^{N} \nu_{nM+k} Z_{nM+k}$ converges a.s.

In view of the fact that $E|X_i| \leq \infty$ for each i, we have

$$E\left|\sum_{n=2}^{N} Z_{nM+k}\right| \leq \sum_{n=2}^{N} E|Z_{nm+k}| < \infty.$$

So by Theorem 2.9.2 (due to Austin, 1966) of Stout [3], $\sum_{n=2}^{\infty} Z_{nM+k}^2 < \infty$ a.s. and hence

$$S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left\{\sum_{n=2}^{N} Z_{nM+k}^2\right\}^{1/2} < \infty \quad \text{a.s.}$$

Without loss of generality, we can assume that $|\nu_n| \leq 1$ a.s. for each n. Hence

$$\sum_{n=2}^{\infty} \nu_{nM+k}^2 Z_{nM+k}^2 \leq \sum_{n=2}^{\infty} Z_{nM+k}^2 < \infty \quad \text{a.s.}$$

For $K > 0$ put

$t = t_k = \inf\{n : X_{nM+k} \geq K\}$, $Y_L = I_{[t\geq L]} \nu_{LM+k} Z_{LM+k}$ and $q_n = \sum_{L=2}^{n} Y_L$, $n \geq 2$.

Now $E(q_n \mid \xi_{n-1}) = E(q_{n-1} \mid \xi_{n-1}) + E(Y_n \mid \xi_{n-1}) = q_{n-1}$ i.e. $\{q_n, \xi_n, n \geq 2\}$ is a martingale.

Again $\sum_{L=2}^{\infty} Y_L^2 = \sum_{L=2}^{\infty} I_{[t\geq L]} \nu_{LM+k}^2 Z_{LM+k}^2 < \infty$.

Since

$$|Y_L| = \left|I_{[t\geq L]} \nu_{LM+k} Z_{LM+k}\right| \leq I_{[t\geq L]} |Z_{LM+k}| =$$

$$= I_{[t\geq L]} |X_{LM+k} - E(X_{LM+k} \mid \xi_{L-1})| \leq K + \varepsilon K = K',$$

we have $E \sup_L |Y_L| < \infty$. So $\sum_{L=2}^{\infty} Y_L$ converges a.s. and hence $\sum_{L=2}^{\infty} \nu_{LM+k} Z_{LM+k}$ convergence a.s. on the set $[t = \infty]$.

Since $\lim_{k\to\infty} P[t_k = \infty] = 1$, the series $\sum_{L=2}^{\infty} \nu_{LM+k} Z_{LM+k}$ converges a.s. This completes the proof of Theorem 3.1. ∎

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[1] *Department of Statistics*
Sambalpur University
Jyotivihar – 768019, Sambalpur (Orissa)
INDIA

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[2] *G M College, Sambalpur University*
Sambalpur (Orissa)
INDIA