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Fixed Points of Multi-Valued Mappings in Uniform Spaces

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The paper contains fixed point theorems for multi-valued contractive mappings in uniform spaces.

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1. Introduction

The Banach contraction mapping theorem was extended for multi-valued mappings in metric spaces by Sam J. Nadler Jr. [1]. Since then, much work has been done on this subject. It can be seen (cf. [2]-[12]) that the development of the multi-valued fixed point theory follows a natural way outlined by the single-valued theory. That is why our starting point for the study of fixed points of multi-valued mappings in uniform spaces are the results from [13].

The main purpose of the present paper is to introduce generalized contractions in uniform spaces in view of some applications to the existence theory of functional differential inclusions. Two fixed point theorems are proved: for multi-valued mappings with compact values and for mappings with closed and bounded values.

Throughout this paper, by X we mean a sequentially complete T_2-separated uniform space whose uniformity is generated by a saturated family of pseudometrics $\mathcal{A} = \{\rho_a(x,y) : a \in A\}$, A being an index set (cf. [14]). Denote by $CB(X)$ the set of all non-empty bounded closed subsets of X (cf. [15], [16]). For every $M \in CB(X)$ we define as usually $N_\varepsilon(M,a) = \{x \in X : \rho_a(x,y) < \varepsilon$ for some $y \in M\}$, where ε is an arbitrary fixed positive real. The set $CB(X)$ will be regarded as a uniform space endowed with a family of Hausdorff pseudodistances $H_{\mathcal{A}} = \{H_a(M,L) : a \in A\}$ for $M, L \in CB(X)$, where

$H_a(M,L) = \max\{E_a(M,L), E_a(L,M)\}$ and $E_a(M,L)$ is the excess of M over L, i.e. $E_a(M,L) = \sup_{x\in M} \rho_a(x,L)$ and $\rho_a(x,L) = \inf_{y\in L} \rho_a(x,y)$.

We point out that the family $H_{\mathcal{A}}$ really depends on the family $\mathcal{A}$ because even in the metric case X may possess two equivalent metrics which may not generate equivalent Hausdorff metrics for $CB(X)$ (cf. [1], [17]).

The above definition of $H_a(M,L)$ is equivalent to the following one (cf. [15]): $H_a(M,L) = \inf\{\varepsilon > 0 : M \subset N_\varepsilon(L,a)$ and $L \subset N_\varepsilon(M,a)\}$. If the set L is compact and $m_0 \in M$ is arbitrary chosen, then there is $l_0 \in L$ such that $\rho_a(m_0,l_0) \le H_a(M,L)$. Indeed, we have $\rho_a(m_0,L) \le \sup_{m\in M} \rho_a(m,L) = H_a(M,L)$. On the other hand, $\rho_a(m_0,l_0) = \inf_{l\in L} \rho_a(m_0,l)$, for some $l_0 \in L$ because $\rho_a(m_0,\cdot)$ is continuous on the compact set L. Therefore $\rho_a(m_0,l_0) \le H_a(M,L)$. The element l_0 can be depending on $a \in A$. That is why we introduce assumption (N) (see below).

The following example shows that there is no such a point l_0 if L is not a compact set.

Consider the linear space l^∞_{loc} consisting of all real infinite sequences. A countable family of seminorms on l^∞_{loc} which defines a locally convex topology is $\mathcal{A} = \{\|x\|_n : n \in \mathbf{N}\}$, $\|x\|_n = \max\{|x_1|, |x_2|, ..., |x_n|\}$ $(n = 1,2,...)$, $x \in l^\infty_{loc}$, where $x = \{x_1, x_2, ..., x_n, ...\}$.

Consider the sets $M = \{m, l_1, l_2, ..., l_{n-1}, ...\}$ and $L = \{l_1, l_2, ..., l_n, ...\}$: $m = \{\underbrace{2,2,...,2}_{n_0},2,2,...\}$, $l_1 = \{\underbrace{-1,-1,...,-1}_{n_0},1,1,...\}$, $l_2 = \{\underbrace{-\frac{1}{2},-\frac{1}{2},...,-\frac{1}{2}}_{n_0},2,2,...\}, ..., l_k = \{\underbrace{-\frac{1}{k},-\frac{1}{k},...,-\frac{1}{k}}_{n_0},k,k,...\}, ...$, where $n_0 \in \mathbf{N}$ is an arbitrary fixed natural number. The set L is not compact because it is unbounded with respect to the seminorm $\|\cdot\|_{n_0+1}$. Then

$$\|m - l_k\|_{n_0} = \max\{|2+\frac{1}{k}|, |2+\frac{1}{k}|, ..., |2+\frac{1}{k}|\} = 2+\frac{1}{k},\ k \in \mathbf{N}.$$

Since $H_{n_0}(M,L) = \inf\{\|m-l_k\|_{n_0} : k \in \mathbf{N}\} = 2$, there is no $l_k \in L$ such that $\|m-l_k\|_{n_0} \le H_{n_0}(M,L)$.

We denote by $COM(X)$ the totality of all compact subsets of X. We shall use the following assertions

Lemma 1. *If X is sequentially complete with respect to $\mathcal{A}$, so are $CB(X)$ and $COM(X)$ with respect to $H_{\mathcal{A}}$ (cf. [15], [16]).*

Lemma 2. *If $\lim_{n\to\infty} x_n = x_0$ in $(X,\mathcal{A})$, $\lim_{n\to\infty} M_n = M_0$ in $(CB(X), H_{\mathcal{A}})$ or in $(COM(X), H_{\mathcal{A}})$ and $x_n \in M_n$ for every $n \in \mathbf{N}$, then $x_0 \in M_0$ (cf. [3]).*

2. Fixed point theorems

In this section we prove fixed point theorems for mappings with compact images.

Introduce a family of functions $(\Phi) = \{\Phi_a(t) : a \in A\}$ with the properties:
$(\Phi 1)$ $\Phi_a(t)$ is continuous from the right, increasing and $0 < \Phi_a(t) < t$ for $t > 0$ and subadditive

(i. e. $\Phi_a(t_1 + t_2) \leq \Phi_a(t_1) + \Phi_a(t_2)$ for $t_1, t_2 > 0$);

$(\Phi 2)$ for each $a \in A$ there is a function $\overline{\Phi}_a \in (\Phi)$ such that

$$\sup\{\Phi_{j^n(a)}(t) : n = 0, 1, 2...\} \leq \overline{\Phi}_a(t), \text{ where } \sum_{n=0}^{\infty} n\overline{\Phi}_a{}^n(t) < \infty.$$

Here $\overline{\Phi}_a{}^n(t)$ stands for $\overline{\Phi}_a(\overline{\Phi}_a(...\overline{\Phi}_a(t)...))$, while $j : A \to A$ is a mapping of the index set into itself whose iterates are defined as follows: $j^0(a) = a$, $j^k(a) = j(j^{k-1}(a)), k \in \mathbf{N}, a \in A$.

A mapping $T : X \to COM(X)$ is called Φ-contractive if for every $x, y \in X$ is satisfied

$$H_a(Tx, Ty) \leq \Phi_a(\rho_{j(a)}(x, y)) \forall a \in A.$$

We assume: (N) (resp. $(N1)$) For every $M, L \in rangeT$ and $m \in M$ there is $l \in L$ such that $\rho_a(m, l) \leq H_a(M, L)$ (resp. $\rho_a(m, l) \leq H_a(M, L) + \eta_a$, $\eta_a > 0$) and for every $\varepsilon > 0 \, \exists \, l_1 \in L$ such that $\rho_{j(a)}(m, l_1) \leq H_{j(a)}(M, L)$ and $\rho_{j(a)}(l, l_1) < \varepsilon \, \forall a \in A$.

Now we are going to formulate the following

Theorem 1. *Let $T : X \to COM(X)$ be Φ-contractive mapping which satisfies (N). If there exist $x_0 \in X$ and a constant $Q > 0$ such that $\rho_{j^n(a)}(x_0, x_1) \leq Q < \infty \, (n = 0, 1, 2, ...)$ for every $x_1 \in Tx_0$, then T has at least one fixed point in X.*

Proof. Choose $x_1 \in Tx_0$. Since $Tx_0, Tx_1 \in COM(X)$ there is $x_2 \in Tx_1$ such that $\rho_a(x_1, x_2) \leq H_a(Tx_0, Tx_1)$ for arbitrary fixed $a \in A$.

Let x_{n-1} and x_n be already chosen. Since $Tx_{n-1}, Tx_n \in COM(X)$ and $x_n \in Tx_{n-1}$ we can find $x_{n+1} \in Tx_n$ for which $\rho_a(x_n, x_{n+1}) \leq H_a(Tx_{n-1}, Tx_n)$. So we obtained a sequence $\{x_n\}_{n=0}^{\infty}$ of points of X such that $x_{n+1} \in Tx_n$ and $\rho_a(x_n, x_{n+1}) \leq H_a(Tx_{n-1}, Tx_n)$. Consequently, $\rho_a(x_n, x_{n+1}) \leq H_a(Tx_{n-1}, Tx_n) \leq \Phi_a(\rho_{j(a)}(x_{n-1}, x_n))$.

We have to show that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. In view of (N) we can choose $x_n^{(1)} \in Tx_{n-1}$ such that $\rho_{j(a)}(x_{n-1}, x_n^{(1)}) \leq H_{j(a)}(Tx_{n-2}, Tx_{n-1})$ and $\rho_{j(a)}(x_n, x_n^{(1)}) \leq \Phi_{j(a)}(\Phi_{j^2(a)}(...\Phi_{j^{n-1}(a)}(Q)...))$. Then we have

$$\rho_a(x_n, x_{n+1}) \leq \Phi_a(\rho_{j(a)}(x_{n-1}, x_n)) \leq \Phi_a(\rho_{j(a)}(x_{n-1}, x_n^{(1)}) + \rho_{j(a)}(x_n^{(1)}, x_n))$$

$$\leq \Phi_a(\Phi_{j(a)}(\rho_{j^2(a)}(x_{n-2}, x_{n-1}))) + \Phi_a(\Phi_{j(a)}(...\Phi_{j^{n-1}(a)}(Q)...)).$$

Let us choose $x_{n-1}^{(1)} \in Tx_{n-2}$ such that $\rho_{j^2(a)}(x_{n-2}, x_{n-1}^{(1)}) \leq H_{j^2(a)}(Tx_{n-3}, Tx_{n-2})$ and $\rho_{j^2(a)}(x_{n-1}, x_{n-1}^{(1)}) \leq \Phi_{j^2(a)}(\Phi_{j^3(a)}(...\Phi_{j^{n-1}(a)}(Q)...))$.

Therefore,

$$\rho_a(x_n, x_{n+1}) \leq \Phi_a(\Phi_{j(a)}(\rho_{j^2(a)}(x_{n-2}, x_{n-1}))) + \overline{\Phi}_a^n(Q)$$
$$\leq \Phi_a(\Phi_{j(a)}(\rho_{j^2(a)}(x_{n-2}, x_{n-1}^{(1)}) + \rho_{j^2(a)}(x_{n-1}^{(1)}, x_{n-1}))) + \overline{\Phi}_a^n(Q)$$
$$\leq \Phi_a(\Phi_{j(a)}(\Phi_{j^2(a)}(\rho_{j^3(a)}(x_{n-3}, x_{n-2})))) + \Phi_a(\Phi_{j(a)}(\Phi_{j^2(a)}(...\Phi_{j^{n-1}(a)}(Q)...)))$$
$$+\overline{\Phi}_a^n(Q) \leq \Phi_a(\Phi_{j(a)}(\Phi_{j^2(a)}(\rho_{j^3(a)}(x_{n-3}, x_{n-2})))) + 2\overline{\Phi}_a^n(Q).$$

Proceeding in this manner we obtain

$$\rho_a(x_n, x_{n+1}) \leq \Phi_a(\Phi_{j(a)}(...\Phi_{j^{n-1}(a)}(\rho_{j^n(a)}(x_0, x_1))...)) + (n-1)\overline{\Phi}_a^n(Q)$$
$$\leq \overline{\Phi}_a^n(Q) + (n-1)\overline{\Phi}_a^n(Q) = n\overline{\Phi}_a^n(Q),$$

which implies that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Consequently, $\lim_{n\to\infty} x_n = \xi \in X$ and $\lim_{n\to\infty} Tx_n = T\xi$ (cf. Lemma 2). Since $x_{n+1} \in Tx_n$, it follows $\xi \in T\xi$. The element $\xi \in X$ is the desired fixed point of T. Theorem 1 is thus proved. ■

As an immediate consequence of Theorem 1 we obtain a fixed point theorem for Φ-contractive mappings in a metric space (X, ρ). In a metric space (X, ρ) the map j turns into the identity map. Then condition (N) can be formulated in the following way: (N') every point of the set Tx is a limit point and Tx has no isolated points. The other condition of Theorem 1 is also trivially satisfied. The family (Φ) reduces to one function $\Phi(t)$, which satisfies condition $(\Phi 1)$. Then

Theorem 2. *If $T : X \to COM(X)$ is Φ-contractive, satisfies (N') and $\sum_{n=0}^{\infty} n\Phi^n(t) < \infty$, then T has at least one fixed point in X.*

3. Fixed points of mappings with closed and bounded images

Here we replace condition $(\Phi 2)$ by the following one:

$(\Phi 2 - I)$ there is a family (Ψ) of functions $\Psi_a(t) : \mathbf{R}_+ \to \mathbf{R}_+$ with the properties from $(\Phi 1)$ and besides $\Phi_a(t) < \Psi_a(t)$ for $t > 0$ (for instance $\Psi_a(t) = \frac{1}{2}(t + \Phi_a(t))$) and $\sup\{\Psi_{j^n(a)}(t) : n = 0, 1, 2...\} \leq \overline{\Psi}_a(t)$ such that $\sum_{n=0}^{\infty} n\overline{\Psi}_a^n(t) < \infty$.

A mapping $T : X \to CB(X)$ is said to be Φ-contractive if $H_a(Tx, Ty) \leq \Phi_a(\rho_{j(a)}(x, y))$, assuming the functions $\Phi_a(t)$ possess the properties $(\Phi 1), (\Phi 2 - I)$.

Theorem 3. *If $T : X \to CB(X)$ is Φ-contractive, satisfies $(N1)$ and there exist $x_0 \in X$ and $Q > 0$ such that $\rho_{j^n(a)}(x_0, x_1) \leq Q < \infty (n = 0, 1, 2...)$ for every $x_1 \in Tx_0$, then T has a fixed point in X.*

Proof. Choose $x_1 \in Tx_0$ and consider $\rho_{j(a)}(x_0, x_1)$. If there is $x_1 \in Tx_0$ such that $\rho_{j(a)}(x_0, x_1) > 0$ for every $a \in A$, we put $\eta_a^{(1)} = \Psi_a(\rho_{j(a)}(x_0, x_1)) - \Phi_a(\rho_{j(a)}(x_0, x_1)) > 0$. Then we find $x_2 \in Tx_1$ such that $\rho_a(x_1, x_2) \leq H_a(Tx_0, Tx_1 + \Psi_a(\rho_{j(a)}(x_0, x_1)) - \Phi_a(\rho_{j(a)}(x_0, x_1)) \leq \Psi_a(\rho_{j(a)}(x_0, x_1))$.

If for every $x_1 \in Tx_0$ and $a \in A$ follows $\rho_{j(a)}(x_0, x_1) = 0$, then we can choose $x_2 = x_1$. So in both cases we have $\rho_a(x_1, x_2) \leq \Psi_a(\rho_{j(a)}(x_0, x_1))$.

Let x_{n-1} and x_n be already defined.

If $\rho_{j(a)}(x_{n-1}, x_n) > 0, \rho_{j^2(a)}(x_{n-2}, x_{n-1}) > 0, ..., \rho_{j^n(a)}(x_0, x_1) > 0$ for every $a \in A$, we put
$\eta_a^{(n)} = \Psi_a(\rho_{j(a)}(x_{n-1}, x_n)) - \Phi_a(\rho_{j(a)}(x_{n-1}, x_n)) > 0.$

Then there exists $x_{n+1} \in Tx_n$ such that

$$\rho_a(x_n, x_{n+1}) \leq H_a(Tx_{n-1}, Tx_n) + \eta_a^{(n)} \leq \Psi_a(\rho_{j(a)}(x_{n-1}, x_n)).$$

Further, on in view of (N), we choose $x_n^{(1)} \in Tx_{n-1}$ such that

$$\rho_{j(a)}(x_{n-1}, x_n^{(1)}) \leq H_{j(a)}(Tx_{n-2}, Tx_{n-1}) + \Psi_{j(a)}(\rho_{j^2(a)}(x_{n-2}, x_{n-1}))$$
$$- \Phi_{j(a)}(\rho_{j^2(a)}(x_{n-2}, x_{n-1})) \leq \Psi_{j(a)}(\rho_{j^2(a)}(x_{n-2}, x_{n-1}))$$

and

$$\rho_{j(a)}(x_{n-1}, x_n^{(1)}) \leq \Psi_{j(a)}(\Psi_{j^2(a)}(...\Psi_{j^{n-1}(a)}(Q)...)).$$

Therefore,

$$\rho_a(x_n, x_{n+1}) \leq \Psi_a(\rho_{j(a)}(x_{n-1}, x_n)) \leq \Psi_a(\rho_{j(a)}(x_{n-1}, x_n^{(1)})) + \Psi_a(\rho_{j(a)}(x_n^{(1)}, x_n))$$
$$\leq \Psi_a(\Psi_{j(a)}(\rho_{j^2(a)}(x_{n-2}, x_{n-1}))) + \Psi_a(\Psi_{j(a)}(...\Psi_{j^{n-1}(a)}(Q)...))$$
$$\leq \Psi_a(\Psi_{j(a)}(\rho_{j^2(a)}(x_{n-2}, x_{n-1}))) + \overline{\Psi}_a^n(Q).$$

Continuing in this way we conclude that $\rho_a(x_n, x_{n+1}) \leq n\overline{\Psi}_a^n(Q)$.

If at least one of the numbers

$$\rho_{j(a)}(x_{n-1}, x_n), \rho_{j^2(a)}(x_{n-2}, x_{n-1}), ..., \rho_{j^n(a)}(x_0, x_1)$$

is zero then we put $x_{n+1} = x_n$ and then moreover $\rho_a(x_n, x_{n+1}) \le n\overline{\Psi}_a^n(Q)$ is satisfied.

The proof can be completed recalling the usual reasonings. ■

The above theorems can be applied to the initial value problems for functional differential inclusions of neutral type:

$$\begin{cases} \dot{x}(t) \in F(t, x(\Delta_1(t)), ..., x(\Delta_m(t)), \dot{x}(\tau_1(t)), ..., \dot{x}(\tau_n(t))), \ t > 0 \\ x(t) = \varphi(t), \ t \le 0 \\ \dot{x}(t) = \dot{\varphi}(t), \ t \le 0, \end{cases}$$

where the deviations are unbounded and mixed. But this is a subject of a future article.

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