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On Some Rogers–Ramanujan Type Continued Fraction Identities

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In this paper we establish in a simple, unified way two continued fraction expansions for the ratios of the basic hypergeometric function ${}_2\Phi_1(a,b;c;x)$ with its contiguous functions. Further, as special cases of these identities, we generate a number of continued fraction expansions analogous to the identities of Ramanujan and Eisenstein. As an interesting special case, the famous Rogers–Ramanujan continued fraction also follows.

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1. Introduction

Ramanujan has made some significant contributions to the theory of continued fraction expansions. The most beautiful and penetrating continued fraction expansions of Ramanujan can be found in Chapter 12 of his second notebook [10] which is almost entirely devoted to the study of continued fraction expansions. Further, continued fractions can be found in Chapter 16 and unorganised portions of second notebook and in the "lost" notebook [11]. The *famous Rogers–Ramanujan continued fraction identity* can be stated as

$$(1)\qquad \frac{(q;q^5)_\infty\,(q^4;q^5)_\infty}{(q^2;q^5)_\infty\,(q^3;q^5)_\infty} = \frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n}} = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+\ldots}\,\frac{q^n}{1+\ldots},$$

where

$$(a)_\infty := (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1$$

and

$$(a)_n := (a;q)_n = \frac{(a)_\infty}{(aq^n)_\infty}, \quad n \text{ is any integer.}$$

The continued fraction (1) and several of its evaluations have been communicated by Ramanujan [9, p. xxviii] in his second letter to Hardy. Ramanujan also mentions in his letters to Hardy that (1) and several other continued fractions are particular cases of a more general theorem; but he does not give this theorem. However, eventually in his "lost" notebook he has stated generalizations of these identities.

In [5] S. Bhargava, C. Adiga and D. D. Somashekara have established several continued fraction identities which contain as special cases several continued fraction identities stated in the "lost" notebook of Ramanujan and proved earlier by different methods by G. E. Andrews [1],[2], Bhargava and A diga [4], M. D. Hirschhorn [7] and K. G. Ramanathan [8].

The present work is a sequel to these. In this paper we obtain in a simple, unified way a continued fraction expansion for ${}_2\Phi_1(a,b;c;xq)/{}_2\Phi_1(a,b;c;x)$ and ${}_2\Phi_1(a,bq;cq;xq)/{}_2\Phi_1(a,b;c;xq)$, where

$$(2) \quad {}_2\Phi_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(q)_n(c)_n} x^n, \qquad |q|<1, \quad |x|<1, \quad (a)_n = (a;q)_n,$$

is the *basic hypergeometric function*. We also obtain several identities analogous to Ramanajan's and Eisenstein's as special cases.

For our convenience we introduce the function

$$(3) \qquad h(a,b,c,x) = (c)_\infty (x)_\infty \, {}_2\Phi_1(a,b;c;x)$$

and note that

$$(4) \qquad h(a,b,c,x) = h(b,a,c,x).$$

Then, the well-known *E. Heine's transformation* for ${}_2\Phi_1(a,b;c;x)$ takes the form:

$$(5) \quad h(abx/c, b, bx, c/b) = h(a,b,c,x), \qquad |q|<1, \quad |x|<1, \quad |c/b|<1.$$

2. Main theorems

We first state some functional relations satisfied by the auxiliary function h, which are useful to prove our main theorems.

$$(6) \qquad \begin{aligned} h(a,b,c,x) &= (1-x)h(a,b,c,xq) \\ &\quad + x(1-a)(1-b)h(aq,bq,cq,x), \end{aligned}$$

$$h(a,b,c,x) = h(aq,bq,cq,x/q) + (abx-c)h(aq,bq,cq,x), \tag{7}$$

$$\begin{aligned} h(a,bq,cq,x/q) &= (1+c-bx-ax/q)h(a,bq,cq,x) \\ &\quad + (1-x)(abx-c)h(a,bq,cq,xq), \end{aligned} \tag{8}$$

$$\begin{aligned} h(a,b,c,xq) &= (1-c+ax-bxq)h(a,bq,cq,xq) \\ &\quad + x(a-cq)(bq-1)h(a,bq^2,cq^2,xq), \end{aligned} \tag{9}$$

$$\begin{aligned} h(a,b,c,x) &= (1-c)h(a,bq,cq,x) \\ &\quad + x(c-b)(1-a)h(aq,bq,cq^2,x), \end{aligned} \tag{10}$$

$$\begin{aligned} h(a,b,c,x) &= (1-c)h(aq,b,cq,x) \\ &\quad + x(c-a)(1-b)h(aq,bq,cq^2,x). \end{aligned} \tag{11}$$

These functional relations follow easily from

$$\begin{aligned} &h(a,b,c,x) - h(aq,b,c,x) = ax(b-1)h(aq,bq,cq,x), \\ &h(a,b,c,x) - (1-x)h(a/q,b,c,xq) = x(1-b)h(a,bq,cq,x), \\ &h(a,b,c,x) - h(a/q,bq,c,x) = x(a/q-b)h(a,bq,cq,x), \\ &h(a,b,c,x) - h(a,bq,cq,x/q) = (ax/q-c)h(a,bq,cq,x), \\ &h(a,b,c,x) - h(aq,b,c,x/q) = (x/q)(b-c)h(aq,b,cq,x), \end{aligned}$$

by symmetry relation (4) and the Heine's transformation (5). For details of the proof one may refer [5].

Theorem 1. *The following continued fraction expansion for a ratio of two basic hypergeometric functions* ${}_2\Phi_1$ *(2) holds:*

$$\frac{{}_2\Phi_1(a,b;c;xq)}{{}_2\Phi_1(a,b;c;x)} = \frac{(1-x)h(a,b,c,xq)}{h(a,b,c,x)}$$

$$= \frac{1-x}{1-x+}\,\frac{A_0}{1+}\,\frac{B_0}{C_0+}\,\frac{D_0}{1-xq+\ldots}\,\frac{A_n}{1+}\,\frac{B_n}{C_n+}\,\frac{D_n}{1-xq^{n+1}+\ldots},$$

where $|q|<1$, $|x|<1$ *and*

$$\begin{aligned} A_n &= xq^n(1-aq^n)(1-bq^n), \\ B_n &= abxq^{3n+1} - cq^n, \\ C_n &= 1 + cq^n - bxq^{2n+1} - axq^{2n+1}, \\ D_n &= (1-xq^{n+1})(abxq^{3n+2} - cq^n), \quad n = 0,1,2,\ldots,. \end{aligned}$$

Proof. Changing a to aq^n, b to bq^n, c to cq^n and x to xq^n, we can write (6) as:

$$\mathbf{P_n} \equiv \frac{h(aq^n, bq^n, cq^n, xq^n)}{h(aq^n, bq^n, cq^n, xq^{n+1})} \tag{12}$$
$$= (1 - xq^n) + \frac{A_n}{\dfrac{h(aq^n, bq^n, cq^n, xq^{n+1})}{h(aq^{n+1}, bq^{n+1}, cq^{n+1}, xq^n)}}.$$

Changing a to aq^n, b to bq^n, c to cq^n and x to xq^{n+1}, (7) can be written as:

$$\frac{h(aq^n, bq^n, cq^n, xq^{n+1})}{h(aq^{n+1}, bq^{n+1}, cq^{n+1}, xq^n)} \tag{13}$$
$$= 1 + \frac{B_n}{\dfrac{h(aq^{n+1}, bq^{n+1}, cq^{n+1}, xq^n)}{h(aq^{n+1}, bq^{n+1}, cq^{n+1}, xq^{n+1})}}.$$

Analogously, by changing a to aq^{n+1}, b to bq^n, c to cq^n and x to xq^{n+1}, (8) turns into

$$\frac{h(aq^{n+1}, bq^{n+1}, cq^{n+1}, xq^n)}{h(aq^{n+1}, bq^{n+1}, cq^{n+1}, xq^{n+1})} = C_n + \frac{D_n}{\mathbf{P_{n+1}}}. \tag{14}$$

Now, from (12), (13) and (14) we have

$$\mathbf{P_n} = (1 - xq^n) + \frac{A_n}{1+}\,\frac{B_n}{C_n+}\,\frac{D_n}{\mathbf{P_{n+1}}}. \tag{15}$$

Iterating (15) with $n = 0, 1, 2, \ldots$, taking reciprocals and then multiplying both sides by $(1 - x)$, we get the required result. ■

Theorem 2. *The following continued fraction expansion for a ratio of two functions* ${}_2\Phi_1$ *(2) follows:*

$$\frac{{}_2\Phi_1(a, bq; cq; xq)}{{}_2\Phi_1(a, b; c; xq)} = \frac{(1 - c)h(a, bq, cq, xq)}{h(a, b, c, xq)}$$

$$= \frac{1 - c}{E_0+}\,\frac{F_0}{1 - cq+}\,\frac{G_0}{1 - cq^2+}\,\frac{H_0}{E_1 + \ldots}\,\frac{F_n}{1 - cq^{3n+1}+}\,\frac{G_n}{1 - cq^{3n+2}+}\,\frac{H_n}{E_{n+1} + \ldots},$$

where $|q| < 1$, $\quad |x| < 1$ *and*

$$E_n = 1 - cq^{3n} + axq^n - bxq^{2n+1},$$
$$F_n = xq^n(a - cq^{2n+1})(bq^{2n+1} - 1),$$
$$G_n = xq^{2n+2}(cq^n - b)(1 - aq^n),$$
$$H_n = xq^{n+1}(cq^{2n+2} - a)(1 - bq^{2n+2}), \quad n = 0, 1, 2, \ldots.$$

Proof. Changing a to aq^n, b to bq^{2n}, c to cq^{3n}, (9) can be written as:

$$\begin{aligned}\mathbf{Q_n} &\equiv \frac{h(aq^n, bq^{2n}, cq^{3n}, xq)}{h(aq^n, bq^{2n+1}, cq^{3n+1}, xq)} \\ &= E_n + \cfrac{F_n}{\cfrac{h(aq^n, bq^{2n+1}, cq^{3n+1}, xq)}{h(aq^n, bq^{2n+2}, cq^{3n+2}, xq)}}.\end{aligned} \tag{16}$$

Changing a to aq^n, b to bq^{2n+1}, c to cq^{3n+1} and x to xq, (10) turns into

$$\begin{aligned}&\frac{h(aq^n, bq^{2n+1}, cq^{3n+1}, xq)}{h(aq^n, bq^{2n+2}, cq^{3n+2}, xq)} \\ &= (1 - cq^{3n+1}) + \cfrac{G_n}{\cfrac{h(aq^n, bq^{2n+2}, cq^{3n+2}, xq)}{h(aq^{n+1}, bq^{2n+2}, cq^{3n+3}, xq)}}.\end{aligned} \tag{17}$$

Changing a to aq^n, b to bq^{2n+2}, c to cq^{3n+2} and x to xq, (11) can be written as

$$\frac{h(aq^n, bq^{2n+2}, cq^{3n+2}, xq)}{h(aq^{n+1}, bq^{2n+2}, cq^{3n+3}, xq)} = (1 - cq^{3n+2}) + \frac{H_n}{\mathbf{Q_{n+1}}}. \tag{18}$$

Now, from (16), (17) and (18) we have

$$\mathbf{Q_n} = E_n + \frac{F_n}{(1 - cq^{3n+1})+}\,\frac{G_n}{(1 - cq^{3n+2})+}\,\frac{H_n}{\mathbf{Q_{n+1}}}. \tag{19}$$

Iterating (19) with $n = 0, 1, 2, \ldots$, taking reciprocal and then multiplying both sides by $(1 - c)$, we get the required result. ■

3. Special cases

Changing a to $-\lambda/a$, b to c, c to $-bq$, x to $-aq/c$ and then letting c to ∞ in Theorem 1, we have the continued fraction:

$$\begin{aligned}&\frac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-\lambda/a)_n (aq)^n}{(q)_n(-bq)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-\lambda/a)_n (a)^n}{(q)_n(-bq)_n}} \\ &= \frac{1}{1+}\,\frac{aq + \lambda q}{1+}\,\frac{bq + \lambda q^2}{1 - bq + aq^2+}\,\frac{bq + \lambda q^3}{1 + \ldots} \\ &\frac{aq^{2n+1} + \lambda q^{3n+1}}{1+}\,\frac{bq^{n+1} + \lambda q^{3n+2}}{1 - bq^{n+1} + aq^{2n+2}+}\,\frac{bq^{n+1} + \lambda q^{3n+3}}{1 + \ldots}.\end{aligned} \tag{20}$$

This is *analogous to an identity of Ramanujan found in his "lost" notebook.* Using the fact that

$$(-bq)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-\lambda/a)_n a^n}{(q)_n(-bq)_n} = (-aq)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-\lambda/b)_n b^n}{(q)_n(-aq)_n}$$

and then putting $a = 0$, $b = -\lambda$ in (20) we get the expansion

$$\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}\lambda^n = \frac{1}{1+}\,\frac{\lambda q}{1+}\,\frac{\lambda(q^2-q)}{1+\lambda q+}\,\frac{\lambda(q^3-q)}{1+\ldots} \frac{\lambda q^{3n+1}}{1+}\,\frac{\lambda(q^{3n+2}-q^{n+1})}{1+\lambda q^{n+1}+}\,\frac{\lambda(q^{3n+3}-q^{n+1})}{1+\ldots}. \tag{21}$$

This is analogous to entry (13) in Chapter 16 of Ramanujan's second notebook. Putting $\lambda = 1$ in (21) we get

$$\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2} =$$

$$\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2-q}{1+q+}\,\frac{q^3-q}{1+\ldots}\,\frac{q^{3n+1}}{1+}\,\frac{q^{3n+2}-q^{n+1}}{1+q^{n+1}+}\,\frac{q^{3n+3}-q^{n+1}}{1+\ldots},$$

which is *analogous to an identity* of Eisenstein [6]. Changing x to x/ab and then letting a and b to ∞ in Theorem 1, we get

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2}x^n}{(q)_n(c)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n(n-1)}x^n}{(q)_n(c)_n}} = \frac{1}{1+}\,\frac{x}{1+}\,\frac{xq-c}{1+c+}\,\frac{xq^2-c}{1+\ldots} \frac{xq^{3n}}{1+}\,\frac{xq^{3n+1}-cq^n}{1+cq^n+}\,\frac{xq^{3n+2}-cq^n}{1+\ldots}. \tag{22}$$

This interesting identity seems to be new and gives rise to many identities including some of Ramanujan. For example, putting $c = 0$ and $x = aq$ in (22) we get

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n(n+1)}a^n}{(q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2}a^n}{(q)_n}} = \frac{1}{1+}\,\frac{aq}{1+}\,\frac{aq^2}{1+\ldots}\,\frac{aq^n}{1+\ldots}, \tag{23}$$

which is corollary to entry (15) in Chapter 16 of Ramanujan's second notebook. This identity was established first by L. J. Rogers [12] and then later by G. N. Watson [13]. The special case of (23) with $a = 1$ is the *Rogers–Ramanujan continued fraction* (1). Putting $x = q$, $c = -q$ in (22) we obtain

$$\frac{\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n(-q)_n}}{\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n(-q)_n}} = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2+q}{1-q+}\,\frac{q^3+q}{1+\ldots}\,\frac{q^{3n+1}}{1+}\,\frac{q^{3n+2}+q^{n+1}}{1-q^{n+1}+}\,\frac{q^{3n+3}+q^{n+1}}{1+\ldots},$$

which is analogous to an identity of Ramanujan found in his "lost" notebook. Changing x to q/x^2, c to 0 then q to q^4 in (22) and multiplying both sides of the resulting identity by q/x, we get

$$\frac{q\sum_{n=0}^{\infty} \frac{q^{4n(n+1)}x^{-2n}}{(q^4;q^4)_n}}{x\sum_{n=0}^{\infty} \frac{q^{4n^2}x^{-2n}}{(q^4;q^4)_n}} = \frac{q}{x+}\,\frac{q^4}{x+}\,\frac{q^8}{x+\ldots}\,\frac{q^{4n}}{x+\ldots}, \tag{24}$$

which can be found in [3, eqn. (7.2)].

The following result was stated by Ramanujan in the unorganised portions of his second notebook [10, p.289] which involves a modest generalization of the Rogers–Ramanujan continued fraction:

$$1 - \frac{qx}{1+}\,\frac{q^2}{1-}\,\frac{q^3x}{1+}\,\frac{q^4}{1-}\,\frac{q^5x}{1+\ldots} = \frac{q}{x+}\,\frac{q^4}{x+}\,\frac{q^8}{x+}\,\frac{q^{12}}{x+\ldots} \text{ nearly.}$$

Here one continued fraction is approximated by another continued fraction. D. Zagier [14] has used the continued fraction identity (24) while giving the proper interpretation of this result.

Changing a to $-\lambda/a$, b to c, c to $-bq$, x to $-aq/c$ and then letting c to

∞ in Theorem 2, we get

$$\frac{\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-\lambda/a)_n(aq^2)^n}{(q)_n(-bq^2)_n}}{\sum_{n=0}^{\infty} \frac{q^{n(n+1)2}(-\lambda/a)_n(aq)^n}{(q)_n(-bq)_n}}$$
$$= \frac{1+bq}{1+bq+aq^2+} \frac{\lambda q^2 - abq^4}{1+bq^2+} \frac{aq^3+\lambda q^3}{1+bq^3+} \frac{\lambda q^4 - abq^7}{1+bq^4+aq^4+\dots}$$
$$\frac{\lambda q^{3n+2} - abq^{5n+4}}{1+bq^{3n+2}+} \frac{aq^{2n+3}+\lambda q^{3n+3}}{1+bq^{3n+3}+} \frac{\lambda q^{3n+4} - abq^{5n+7}}{1+bq^{3n+4}+aq^{2n+4}+\dots}.$$

This is *analogous to an identity of Ramanujan found in his "lost" notebook.* Changing x to x/ab and letting a and b to ∞ in Theorem 2, we obtain

$$\frac{\sum_{n=0}^{\infty} \frac{q^{n(n+1)}x^n}{(q)_n(cq)_n}}{\sum_{n=0}^{\infty} \frac{q^{n^2}x^n}{(q)_n(c)_n}} = \frac{1-c}{1-c+} \frac{xq}{1-cq+} \frac{xq^2}{1-cq^2+\dots} \frac{xq^n}{1-cq^n+\dots},$$

which is same as entry (15) in Chapter 16 of Ramanujan's second notebook.

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