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On Uniqueness of Meromorphic Functions

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We prove a uniqueness theorem of meromorphic functions improving a result of Ozawa.

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1. Introduction, definitions and main result

Let f, g be two nonconstans mermorphic functions defined in the open complex plane. If f, g have the same a-points with the same multiplicities we say that f, g share the value CM (counting multiplicities) and if only the locations (not necessarily the multiplicities) of the a-points of f, g are same we say that f, g share the value IM (ignoring multiplicities).

In the paper we do not explain the standard notations and definitions of Nevanlinna's theory of meromorphic functions. Let E denote the exceptional set of finite linear measure that arises in the second fundamental theorem.

Ozawa [3] proved the following result.

Theorem A. *If f, g are entire functions of finite orders sharing* $0, 1$ CM *and* $\delta(0; f) > \frac{1}{2}$, *then* $f \cdot g \equiv 1$ *unless* $f \equiv g$.

Withdrawing the order restriction in Theorem A, Ueda [4] proved the following more general theorem.

Theorem B. *If f, g share* $0, 1, \infty$ CM *and*

$$\limsup_{r\to\infty} \frac{N(r,0;f) + N(r,f)}{T(r,f)} < \frac{1}{2},$$

then either $f \cdot g \equiv 1$, *or* $f \equiv g$.

In [5] Yi further improved Theorem A to the following result.

Theorem C. *If f, g share $0, 1, \infty$ CM and $\overline{N}(r, 0; f) + \overline{N}(r, f) < (\lambda + o(1))T(r, f)$ $(r \to \infty, r \notin E)$, where $\lambda < \frac{1}{2}$, then $f \cdot g \equiv 1$ unless $f \equiv g$.*

Yi [6] also gave another improvement of Theorem A, stated below with using the following notation.

Notation 1. For a complex number a, finite or infinite, we denote by $N(r, a; f, = 1)$ the counting function of simple zeros of $f - a$.

Theorem D. ([6]) *If f, g share $0, 1, \infty$ CM and $N(r, 0; f = 1) + N(r, \infty; f, = 1) < (\lambda + o(1)) \max\{T(r, f), T(r, g)\}(r \to \infty, r \notin E)$, where $\lambda < \frac{1}{2}$, then either $f \cdot g \equiv 1$ or $f \equiv g$.*

We see in above theorems that attempts are made to relax the hypotheses of Theorem A on the nature of the functions, the order of the functions and on the deficiency of a function at the origin. So far, the author knows no attempts made till now to relax the hypotheses of Theorem A on the nature of sharing the values. In the paper we make such an attempt. Before stating our result we give some more notations.

Notation 2. We denote by $\overline{N}(r, a; f, \geq p)$ the counting function of distinct zeros of $f - a$ of multiplicities not less that p.

Notation 3 ([1]). By $N_k(r, a; f)$ we denote the counting function of the zeros of $f - a$, where a zero of multiplicity p is counted p times, if $p \leq k$ and k times, if $p > k$. Clearly, $\overline{N}(r, a; f) = N_1(r, a; f) \leq N_k(r, a; f) \leq N(r, a; f)$ for $k = 1, 2, 3, \ldots$.

Notation 4. Let $f_1, f_2, \ldots, f_n$ be meromorphic functions. We denote by $S(r; f_1, f_2, \ldots, f_n)$ a real valued function of the nonnegative real variable r such that

$$S(r; f_1, f_2, \ldots, f_n) = o\{\sum_{i=1}^{n} T(r, f_i)\} \text{ as } r \to \infty (r \notin E).$$

In the paper we prove the following result.

Theorem. *Let*

(i) f, g share 0 IM and $1, \infty$ CM;

(ii) f, g have the same set of simple zeros;

(iii) $N_2(r, 0; f) + \overline{N}(r, f) \leq \{\lambda + o(1)\} \cdot T(r, f)$ as $r \to \infty (r \notin E)$, where $\lambda < \frac{1}{2}$. Then either: (a) $f \equiv 1$, or (b) $f \cdot g \equiv 1$. Further, if

(iv) f has at least one zero or pole, the case (b) does not arise.

The example $f = 3\exp(z) - 9\exp(2z)$, $g = \frac{1}{3}\exp(-z) - \frac{1}{9}\exp(-2z)$ shows that condition (iii) is the best possible. Also, the functions $f = exp(z), g = exp(-z)$ show that condition (iv) is necessary for non-occurrence of case (b).

2. Lemmas

First we give some lemmas necessary for the proof of the main theorem.

Lemma 1. *If f, g share $0, 1, \infty$ IM, then $T(r,f) = O(T(r,g))$ and $T(r,g) = O(T(r,f))$ for $r \notin E$.*

Proof. By the second fundamental theorem we get

$$T(r,f) \leq \overline{N}(r,0;f) + \overline{N}(r,1;f) + \overline{N}(r,f) + S(r,f),$$

i.e.,

$$\{1 + o(1)\}T(r,f) \leq \overline{N}(r,0;g) + \overline{N}(r,1;g) + \overline{N}(r,g)$$

$$\leq 3T(r,g) \quad \text{for } r \notin E,$$

wich shows that $T(r,f) = O(T(r,g))$ for $r \notin E$. Similarly the other result can be proved. This proves the lemma. ■

Lemma 2. *If c_1, c_2, c_3 are nonzero constants and $c_1 f + c_2 g \equiv c_3$, then $T(r,f) \leq \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,f) + S(r,f)$ for $r \notin E$.*

Proof. By the second fundamental theorem we get

$$T(r,f) \leq \overline{N}(r,0;f) + \overline{N}(r, c_3/c_1, f) + \overline{N}(r,f) + S(r,f)$$

$$= \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,f) + S(r;f)$$

and this proves the lemma. ■

Lemma 3. *If f, g share zero IM and they have the same set of simple zeros then $N_2(r,0;g) = N_2(r,0;f)$.*

Proof. Now,

$$N_2(r,0;g) = N(r,0;g,=1) + 2\overline{N}(r,0;g,\geq 2)$$

$$= N(r,0;f,=1) + 2\overline{N}(r,0;f,\geq 2) = N_2(r,0;f).$$

This proves the lemma. ■

Lemma 4. *Let f_1, f_2, f_3 be nonconstant meromorphic functions and D be their Wronskian determinant. Then*

$$\sum_{i=1}^{3} N(r,0,f_i) - N(r,0,D) \leq \sum_{i=1}^{3} N_2(r,0;f_i).$$

Proof. Since if z_0 is a zero of f_i of multiplicity $p(> 2)$, it is a zero of D of multiplicity $p-2$, it follows that

$$\sum_{i=1}^{3} N(r,0,f_i) - N(r,0;D) \leq \sum_{i=1}^{3} N_2(r,0,f_i).$$

This proves the lemma. ∎

Lemma 5. *Let f_1, f_2, f_3 be nonconstsnt meromorphic functions such that $f_1+f_2+f_3 \equiv 1$. If D is the Wronskian determinant of f_1, f_2, f_3, then*

$$N(r,f_i) + N(r,D) - \sum_{l=1}^{3} N(r,f_l) \leq 2\{\overline{N}(r,f_j) + \overline{N}(r,f_k)\},$$

where $j \neq k$; $j,k \in \{1,2,3\} \setminus \{i\}$ and $i \in \{1,2,3\}$.

Proof. For the sake of definiteness, we choose $i = 1$ and the other two cases are similar. First we note that

$$D = \begin{vmatrix} 1 & f_2 & f_3 \\ 0 & f_2 & f_3 \\ 0 & f_2 & f_3 \end{vmatrix} = f'_2 \cdot f''_3 - f''_2 \cdot f'_3$$

and thus so the poles of D occur only at the poles of f_2 and f_3. If z_0 is a pole of f_2, f_3 of multiplicities q_2, q_3 respectively, then it is a pole of D of multiplicity $q_2 + q_3 + 3$ and if z_0 is a pole of $f_j(j = 2,3)$ of multiplicity p_j, then it is a pole of D of multiplicity at most $p_j + 2$. Therefore, $N(r,D) \leq N(r,f_2) + N(r,f_3) + 2\overline{N}(r,f_2) + 2\overline{N}(r,f_3)$ and the lemma is proved. ∎

Lemma 6. *Let $f_1, f_2, \ldots, f_n$ be linearly independent meromorphic functions satisfying $\sum_{i=1}^{n} f_i \equiv 1$. Then for $j = 1,2,\ldots,n$ we get*

$$T(r,f_j) \leq \sum_{i=1}^{n} N(r,0;f_i) + N(r,f_j) + N(r,D) - \sum_{i=1}^{n} N(r,f_i) - N(r,0,D)$$
$$+ S(r,f_1,f_2,\ldots,f_n) \quad as\ r \to \infty (r \notin E),$$

where D is the Wronskian determinant of $f_1, f_2, \ldots f_n$.

3. Proof of the theorem

We put $h = \dfrac{f-1}{g-1}$. Then $h \not\equiv 0$ and since f, g share $1, \infty$ CM, it follows that $N(r,0,h) + N(r,h) = S(r,f,g)$. Also we put

(1) $\quad f_1 = f, f_2 = h$ and $f_3 = -gh$ so that $f_1 + f_2 + f_3 \equiv 1.$

If f_1, f_2, f_3 are linearly independent, by Lemma 6 and Lemma 1 we get because $T(r,h) \le T(r,f) + T(r,g) + O(1)$,

$$T(r,f_1) \le \sum_{i=1}^{3} N(r,0;f_i) + N(r,f_1) + N(r,D) - \sum_{i=1}^{3} N(r,f_i)$$

$$-N(r,0,D) + S(r,f).$$

Now by Lemma 4 and Lemma 5 we obtain

$$T(r,f) \le \sum_{i=1}^{3} N_2(r,0;f_i) + 2\overline{N}(r,h) + 2\overline{N}(r;gh) + S(r,f).$$

Finally by Lemma 1 and Lemma 3 we get because $\overline{N}(r,g) = \overline{N}(r,f)$

$$\begin{aligned} T(r,f) \quad &\le N_2(r,0;f) + 2N(r,0;h) + N_2(r,o;g) \\ &+4\overline{N}(r,h) + 2\overline{N}(r,g) + S(r,f) \\ &= 2N_2(r,0;f) + 2\overline{N}(r,f) + S(r,f) \end{aligned}$$

and so by the condition (iii) we see that

$$T(r,f) \le \{2\lambda + o(1)\} \cdot T(r,f) \text{ as } r \to \infty \ (r \notin E)$$

which implies a contradiction because $\lambda < 1/2$. Therefore f_1, f_2, f_3 are linearly dependent and so there exist constants c_1, c_2, c_3, not all zero such that

$$(2) \qquad c_1 f_1 + c_2 f_2 + c_3 f_3 \equiv 0.$$

If $c_1 = o$, from (2) we get $h(c_2 - c_3 g) \equiv 0$ and then g becomes a constant, which is not the case. Thus, So $c_1 \ne 0$. Now eliminating f_1 from (1) and (2) we get

$$(3) \qquad cf_2 + df_3 \equiv 1,$$

where $c = 1 - \dfrac{c_2}{c_1}, d = 1 - \dfrac{c_3}{c_1}$. Clearly c, d can not be simultaneously zero.

We consider the following cases.

Case I. Let $c.d \ne 0$. Then from (3) we get $\dfrac{1}{h} + dg \equiv c$, and by Lemma 1 and Lemma 2 we obtain

$$(4) \qquad \begin{aligned} T(r,g) \quad &\le \overline{N}(r,h) + \overline{N}(r,0;g) + \overline{N}(r,g) + S(r;g) \\ &= \overline{N}(r,0;f) + \overline{N}(r,f) + S(r,f) \text{ as } r \to \infty \ (r \notin E). \end{aligned}$$

Again by the second fundamental theorem we get

$$\begin{aligned} T(r,f) &\le \overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}(r,f)+S(r;f) \\ &= \overline{N}(r,0;f)+\overline{N}(r,1;g)+\overline{N}(r,f)+S(r;f) \\ &\le \overline{N}(r,0;f)+\overline{N}(r,f)+T(r,g)+S(r;f) \end{aligned}$$

and this gives by (4) that

$$T(r,f) \le 2\overline{N}(r,0;f)+2\overline{N}(r,f)+S(r;f).$$

In view of the condition (iii) we get $T(r,f) \le \{2\lambda+o(1)\}\cdot T(r,f)$ as $r \to \infty (r, \notin E)$ which is a contradiction because $\lambda < 1/2$. Hence the case $c.d \neq 0$ cannot arise.

Case II. Let $c.d = 0$.

Subcase (i). Let $d = 0$. Then from (3) we get $cf - g \equiv c - 1$. If $c \neq 1$, we obtain from Lemma 2 that

$$\begin{aligned} T(r,f) &\le \overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,f)+S(r,f) \\ &\le 2\overline{N}(r,0;f)+2\overline{N}(r,f)+S(r,f). \end{aligned}$$

This gives in view of the condition (iii) that $T(r,f) \le \{2\lambda+o(1)\}\cdot T(r,f)$ as $r \to \infty\,(r \notin E)$, which is a contradiction because $\lambda < 1/2$. Therefore $c = 1$ and $f \equiv g$.

Subcase (ii). Let $c = 0$. Then from (3) we get $df - \frac{1}{g} = d - 1$. If $d \neq 1$, we obtain from Lemma 2

$$\begin{aligned} T(r,f) &\le \overline{N}(r,g)+\overline{N}(r,0;f)+\overline{N}(r,f)+S(r,f) \\ &\le 2\overline{N}(r;0;f)+2\overline{N}(r,f)+S(r,f). \end{aligned}$$

So in view of the condition (iii) that $T(r,f) \le \{2\lambda+o(1)\}\cdot T(r,f)$ as $r \to \infty\,(r \notin E)$, and this implies a contradiction because $\lambda < 1/2$. Therefore $d = 1$ and so $f \cdot g \equiv 1$.

Further, if f has at least a zero or a pole at z_0, say, then z_0 is respectively a zero or a pole of g and this is impossible if $f \cdot g \equiv 1$. So if f has at least a zero or a pole, the case $f \cdot g \equiv 1$ cannot arise. This proves the theorem.

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